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ON THE STABILITY OF PEXIDER TYPE TRIGONOMETRIC FUNCTIONAL EQUATIONS

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Abstract. The aim of this paper is to study the stability problem for the pexider type trigonometric functional equation $f(x+y)-f(x-y)=2g(x)h(y)$, which is related to the d'Alembert, the Wilson, the sine, and the mixed trigonometric functional equations.

1. Introduction

J. Baker, J. Lawrence and F. Zorzitto in [4] , and Bourgin [5] introduced the following: if f satisfies the inequality $|f(x+y)-f(x)f(y)| \leq \delta$, then either $|f(x)| \leq \max(4, 4\delta)$, or $f(x+y)=f(x)f(y)$. This is frequently referred to as superstability.

In next year, J. Baker [3] proved the superstability of the cosine functional equation (also called the d'Alembert equation)

$$f(x+y)+f(x-y)=2f(x)f(y), \tag{A}$$

which is improved by P. Găvruta [7].

And also the sine functional equation

$$f(x)f(y)=f\Big(\frac{x+y}{2}\Big)^2-f\Big(\frac{x-y}{2}\Big)^2 \tag{S}$$

is investigated by P.W. Cholewa [6].

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The cosine functional equation (A) is generalized to the following functional equations

$$(A_{fg}) \qquad f(x+y)+f(x-y)=2f(x)g(y),$$
$$(A_{gf}) \qquad f(x+y)+f(x-y)=2g(x)f(y),$$
$$(A_{gg}) \qquad f(x+y)+f(x-y)=2g(x)g(y),$$

where the two unknown functions f, g are to be determined. The equation (A_{fg}) introduced by Wilson, is sometimes referred to as the Wilson equation. Their stability have been investigated by Badora, Ger, Kannappan, Kim ([1], [2], [8], [10]) and others.

Motivated by some trigonometric identities (for example, $\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta$), we consider the following trigonometric functional equation

$$(T_{gh}) \qquad f(x+y)-f(x-y)=2g(x)h(y),$$

which has special cases as follow :

$$(T) \qquad f(x+y)-f(x-y)=2f(x)f(y),$$
$$(T_{fg}) \qquad f(x+y)-f(x-y)=2f(x)g(y),$$
$$(T_{gf}) \qquad f(x+y)-f(x-y)=2g(x)f(y),$$
$$(T_{gg}) \qquad f(x+y)-f(x-y)=2g(x)g(y).$$

The aim of this paper is to investigate stability problem for the pexider type trigonometric functional equations (T_{gh}).

As a consequence, we obtain the stability of the above trigonometric type equations (T), (T_{fg}), (T_{gf}), (T_{gg}) as corollaries, and also extend the obtained results to the Banach algebra.

In this paper, let $(G,+)$ be an Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers. Whenever we deal with (S), we need to assume additionally that $(G,+)$ is a uniquely 2-divisible group. We will write then "under 2-divisibility", for short. We may assume that f, g and h are non-zero functions and ε is a nonnegative real constant.

2. Stability of the equation (T_{gh})

In this section, we investigate the stability of the pexider type trigonometric functional equation (T_{gh}) related to the d'Alembert (A), the Wilson types (A_{fg}) and (A_{gf}) the sine (S), and the mixed type functional equations (T_{fg}) and (T_{gf}).

Theorem 1. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2g(x)h(y)| \leq \varepsilon \quad \forall\, x, y \in G. \tag{2.1}$$

If h fails to be bounded, then

(i) g satisfies (S) under 2-divisibility and one of the cases $g(0) = 0$, $f(x) = f(-x)$,

(ii) if, additionally, h satisfies (A) or (T), g and h are solutions of Wilson type :

$$g(x+y) + g(x-y) = 2g(x)h(y). \tag{A_{ggh}}$$

Proof. (i) Let h be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that

$$0 \neq |h(y_n)| \to \infty, \quad \text{as} \quad n \to \infty \tag{2.2}$$

Taking $y = y_n$ in (2.1) we obtain

$$\left|\frac{f(x+y_n) - f(x-y_n)}{2h(y_n)} - g(x)\right| \leq \frac{\varepsilon}{|2h(y_n)|},$$

that is,

$$g(x) = \lim_{n\to\infty} \frac{f(x+y_n) - f(x-y_n)}{2h(y_n)} \quad \forall\ y \in G. \tag{2.3}$$

Using (2.1), we have

$$\begin{aligned}
&\big|f\left(x+(y+y_n)\right) - f\left(x-(y+y_n)\right) - 2g(x)h(y+y_n)\\
&\qquad - f\left(x+(y-y_n)\right) + f\left(x-(y-y_n)\right) + 2g(x)h(y-y_n)\big|\\
&\leq 2\varepsilon
\end{aligned}$$

so that

$$\left|\frac{f((x+y)+y_n)-f\left((x+y)-y_n\right)}{2h(y_n)} + \frac{f((x-y)+y_n)-f\left((x-y)-y_n\right)}{2h(y_n)} - 2g(x)\cdot\frac{h(y+y_n)-h(y-y_n)}{2h(y_n)}\right| \le \frac{\varepsilon}{|h(y_n)|} \tag{2.4}$$

for all $x, y \in G$.

Taking the limit as $n \longrightarrow \infty$ with the use of (2.2) and (2.3), we conclude that, for every $y \in G$, there exists the limit

$$k_1(y) := \lim_{n\to\infty}\frac{h(y+y_n)-h(y-y_n)}{h(y_n)}, \tag{2.5}$$

where the function $k_1 : G \to \mathbb{C}$ obtained in that way has to satisfy the equation

$$g(x+y)+g(x-y) = g(x)k_1(y) \quad \forall x, y \in G. \tag{2.6}$$

Applying the case $g(0) = 0$ in (2.6), it implies that g is an odd function. Keeping this in mind, by means of (2.6), we infer the equality

$$\begin{aligned} g(x+y)^2 - g(x-y)^2 &= [g(x+y)+g(x-y)][g(x+y)-g(x-y)] \\ &= g(x)k_1(y)[g(x+y)-g(x-y)] \\ &= g(x)\big[g(x+2y)-g(x-2y)\big] \\ &= g(x)\big[g(2y+x)+g(2y-x)\big] \\ &= g(x)g(2y)k_1(x). \end{aligned} \tag{2.7}$$

Putting $y = x$ in (2.6) we get the equation

$$g(2x) = g(x)k_1(x), \quad x \in G.$$

This (2.7), in return, leads to the equation

$$g(x+y)^2 - g(x-y)^2 = g(2x)g(2y)$$

valid for all $x, y \in G$ which, in the light of the unique 2-divisibility of G, states nothing else but (S).

Next, in particular case $f(x) = f(-x)$, it is enough to show that $g(0) = 0$. Suppose that this is not the case.

Putting $x = 0$ in (2.1), from the above assumption and a given condition, we obtain the inequality

$$|h(y)| \le \frac{\varepsilon}{2|g(0)|}, \qquad y \in G.$$

This inequality means that h is globally bounded – a contradiction. Thus the claimed $g(0) = 0$ holds.

(ii) if h satisfies (T), the defined limit k_1 of (2.5) states nothing else but $2h$, so (2.6) validates (A_{ggh}).

Finally, an obvious slight change in the steps of the proof applied after (2.3) gives us the inequality

$$\begin{aligned}&\big|f\left(x+(y+y_n)\right)-f\left(x-(y+y_n)\right)-2g(x)h(y+y_n)\\ &\qquad +f\left(x+(-y+y_n)\right)-f\left(x-(-y+y_n)\right)-2g(x)h(-y+y_n)\big|\\ &\le 2\varepsilon\end{aligned}$$

so that

$$\begin{aligned}&\bigg|\frac{f((x+y)+y_n)-f\left((x+y)-y_n\right)}{2h(y_n)}\\ &+\frac{f((x-y)+y_n)-f\left((x-y)-y_n\right)}{2h(y_n)}\\ &-2g(x)\cdot\frac{h(y+y_n)+h(-y+y_n)}{2h(y_n)}\bigg|\\ &\le\frac{\varepsilon}{|h(y_n)|}\end{aligned} \tag{2.8}$$

for all $x, y \in G$. Taking the limit as $n \longrightarrow \infty$ with the use of (2.3), and since h satisfies (A), so (2.8) implies (A_{ggh}). □

THEOREM 2. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality* (2.1) *for all $x, y \in G$.*

If g fails to be bounded, then

(i) h satisfies (S) under 2-divisibility,

(ii) if, additionally, g satisfies (A), g and h are solutions of $h(x+y) - h(x-y) = 2g(x)h(y)$ and $h(x+y) + h(x-y) = 2h(x)g(y)$.

Proof. Let g be unbounded solution of the inequality (2.1). Then, there exists a sequence $\{x_n\}$ in G such that $0 \ne |g(x_n)| \to \infty$ as $n \to \infty$.

Taking $x = x_n$ in the inequality (2.1), dividing both sides by $|2g(x_n)|$ and passing to the limit as $n \to \infty$ we obtain that

$$h(y) = \lim_{n\to\infty} \frac{f(x_n + y) - f(x_n - y)}{2g(x_n)}, \quad x \in G. \tag{2.9}$$

Replacing x by $x_n + x$ and $x_n - x$ in (2.1), we can, with an application of (2.9), state the existence of a limit function

$$k_2(x) := \lim_{n\to\infty} \frac{g(x_n + x) + g(x_n - x)}{g(x_n)},$$

where the function $k_2 : G \to \mathbb{C}$ satisfies the equation

$$h(x + y) - h(x - y) = k_2(x)h(y) \quad \forall x, y \in G. \tag{2.10}$$

From the definition of k_2, we get the equality $k_2(0) = 2$, which jointly with (2.10) implies that h has to be odd.

(i) Run along the same lines applied after (2.6), which states nothing else but (S).

(ii) if g satisfies (A), the defined limit function k_2 is simply $2g$, so (2.10) implies $h(x + y) - h(x - y) = 2g(x)h(y)$.

Secondly, as above, replacing x by $x_n + y$ and $x_n - y$, and replacing y by x in (2.1), respectively. Then, since g satisfies (A), we obtain, with an application of (2.9), the equation $h(x+y)+h(x-y) = 2h(x)g(y)$. □

By replacing h by f, g by f, h by g in Theorem 1 and Theorem 2, we obtain the following corollaries.

Corollary 1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y) - f(x - y) - 2g(x)f(y)| \le \varepsilon \quad \forall\, x, y \in G. \tag{2.11}$$

Then either f is bounded or g satisfies (A).

Proof. Replacing h by f in (2.1) of Theorem 1. An obvious slight change in the steps of the proof of Theorem 1, gives that k_1 of (2.5) states nothing else but $2g$. Hence, from (2.6), g satisfies the equation (A). □

Corollary 2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality* (2.11)

If g fails to be bounded, then

(i) g satisfies (A),

(ii) f and g are solutions of (T_{gf}) and (A_{fg}).

Proof. Replacing h by f in (2.1) of Theorem 2.

(i) It is enough from Corollary 1 to show that the boundedness of f implies the boundedness of g. Namely, If f is bounded, choose $y_0 \in G$ such that $f(y_0) \neq 0$, and then by (2.11) we obtain

$$\begin{aligned}|g(x)| - \Big|\frac{f(x+y_0)-f(x-y_0)}{2f(y_0)}\Big| \\ &\leq \Big|\frac{f(x+y_0)-f(x-y_0)}{2f(y_0)} - g(x)\Big| \\ &\leq \frac{\varepsilon}{2|f(y_0)|},\end{aligned}$$

from which follows that g is also bounded on G. Since f is nonzero, the unboundedness of g implies the unboundedness of f. Hence, from Corollary 1, g satisfies (A).

(ii) Since we known that g satisfies (A) by (i), Following steps applied in Theorem 2, then the equation (T_{gf}) holds from (2.10).

For the case (A_{gf}), replacing x by $x_n + y$ and $x_n - y$, and y by x in (2.11), respectively. other step runs same as above. □

Corollary 3. *Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)h(y)| \leq \varepsilon \quad \forall\, x, y \in G. \tag{2.12}$$

If f fails to be bounded, then

(i) h satisfies (S) under 2-divisibility,

(ii) If, additionally, f satisfies (A), f and h are solutions of $h(x+y) - h(x-y) = 2f(x)h(y)$ and $h(x+y) + h(x-y) = 2h(x)f(y)$.

Corollary 4. *Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)h(y)| \leq \varepsilon \quad \forall\, x, y \in G.$$

If h fails to be bounded, then

(i) f satisfies (S) under 2-divisibility and one of the cases $f(0) = 0$, $f(x) = f(-x)$,

(ii) if, additionally, h satisfies (A) or (T), f and h are solutions of $f(x+y) + f(x-y) = 2f(x)h(y)$,

(iii) h satisfies (S) under 2-divisibility,

(iv) if, additionally, f satisfies (A), f and h are solutions of $h(x+y) - h(x-y) = 2f(x)h(y)$ and $h(x+y) + h(x-y) = 2h(x)f(y)$.

Proof. It is trivial from Theorem 2 except for (iii) and (iv).

As a boundedness proof of (ii) in Corollary 1, we can see that the unboundedness of h implies the unboundedness of f, so (v) and (vi) follows from (i) and (ii) immediately . □

From Theorem 2, we obtain easily the following result :

Corollary 5. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2g(x)g(y)| \le \varepsilon \quad \forall\, x, y \in G.$$

Then either g is bounded or g satisfies (S) under 2-divisibility.

Proof. It is trivial from (i) of Theorem 2. □

Corollary 6. ([12]) *Suppose that a non-zero function $f : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)f(y)| \le \varepsilon \qquad \forall\, x, y \in G.$$

Then f is bounded.

Proof. Assume that f is not bounded. Then, by applying $g = h = f$ in Corollary 1 and Corollary 2, f satisfies simultaneously (A) and (T). This forces that f is a zero function. But we know that there exists the cosine function which satisfies (A) as a non-zero. Hence we arrive the result by a contradiction. □

3. Extension to the Banach algebra

The obtained results for the functional equations (T_{gh}) in section 2 can be extended to the Banach algebra. For simplify, we only will represent one of them, and the applications to the other theorems and corollaries will be omitted.

Given mappings $f, g, h : G \to \mathbb{C}$, for above equations, we will denote a difference for each equation by an operator $DT_{gh} : G \times G \to \mathbb{C}$ as

$$DT_{gh}(x, y) := f(x+y) - f(x-y) - 2g(x)h(y).$$

Theorem 3. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h : G \to E$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - 2g(x)h(y)\| \le \varepsilon \quad \forall\, x, y \in G. \tag{3.1}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

if the superposition $x^* \circ g$ fails to be bounded, then
(i) h satisfies (S) under 2-divisibility,
(ii) if, additionally, $x^* \circ g$ satisfies (A), g and h are solutions of the equation $h(x+y) - h(x-y) = 2g(x)h(y)$.

Proof. (i) Assume that (3.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E^*$. As is well known, we have $\|x^*\| = 1$ whence, for every $x, y \in G$, we have

$$\begin{aligned}\varepsilon &\geq \|f(x+y) - f(x-y) - 2g(x)h(y)\| \\ &= \sup_{\|y^*\|=1} \left|y^*\big(f(x+y) - f(x-y) - 2g(x)h(y)\big)\right| \\ &\geq \left|x^*\big(f(x+y)\big) - x^*\big(f(x-y)\big) - 2x^*\big(g(x)\big)x^*\big(h(y)\big)\right|,\end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$ and $x^* \circ h$ yield a solution of inequality (2.1).

Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 2 shows that the function $x^* \circ h$ solves the equation (S). In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the difference $DS_h(x,y) := h(x)h(y) - h\big(\frac{x+y}{2}\big)^2 + h\big(\frac{x-y}{2}\big)^2$ falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$DS_h(x,y) \in \bigcap \{\ker x^* : x^* \in E^*\}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$h(x)h(y) - h\Big(\frac{x+y}{2}\Big)^2 + h\Big(\frac{x-y}{2}\Big)^2 = 0 \qquad \text{for all} \quad x, y \in G,$$

as claimed.

(ii) Under the assumption that the superposition $x^* \circ g$ satisfies (A), we know from Theorem 2 that the superpositions $x^* \circ h$ and $x^* \circ g$ are solutions of the equation

$$x^*\big(h(x+y)\big) - x^*\big(h(x-y)\big) = 2x^*\big(g(x)\big)x^*\big(h(y)\big).$$

Namely,

$$\begin{aligned}&h(x+y) - h(x-y) - 2g(x)h(y) \\ &\qquad \in \bigcap \{\ker x^* : x^* \in E^*\}.\end{aligned}$$

The other argument is similar. □

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