On the Stability of Positive Linear Switched Systems Under Arbitrary Switching Laws

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***Abstract*—We consider n-dimensional positive linear switched systems. A necessary condition for stability under arbitrary switching is that every matrix in the convex hull of the matrices defining the subsystems is Hurwitz. Several researchers conjectured that for positive linear switched systems this condition is also sufficient. Recently, Gurvits, Shorten, and Mason showed that this conjecture is true for the case $n = 2$, but is not true in general. Their results imply that there exists some minimal integer n_p such that the conjecture is true for all $n < n_p$, but is not true for $n = n_p$. We show that $n_p = 3$.**

***Index Terms*—Switched systems, stability under arbitrary switching law, positive linear systems, Metzler matrix.**

I. Introduction

Consider the linear switched system:

$$\dot{\boldsymbol{x}}(t) = A_{\sigma(t)}\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{1}$$

where $\boldsymbol{x}(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$, $A_0, A_1 \in \mathbb{R}^{n\times n}$, and $\sigma(\cdot) : \mathbb{R}_+ \to \{0, 1\}$ is a piecewise constant function of time, referred to as the *switching law*. Roughly speaking, this models a system that may switch between the two linear subsystems: $\dot{\boldsymbol{x}} = A_0\boldsymbol{x}$ and $\dot{\boldsymbol{x}} = A_1\boldsymbol{x}$.

Recall that a function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\beta : [0,\infty)\times[0,\infty) \to [0,\infty)$ is said to be of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(s, t)$ decreases to 0 as $t \to \infty$ for each fixed $s \ge 0$. We say that (1) is *globally uniformly asymptotically stable* (GUAS) if there exists a class $\mathcal{KL}$ function β such that for any initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and *any* switching law σ the corresponding solution of (1) satisfies $|\boldsymbol{x}(t)| \le \beta(|\boldsymbol{x}_0|, t)$ for all $t \ge 0$. This implies in particular that $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{0}$.

Denote $A(k) := kA_0 + (1-k)A_1$. By the classic *Lie-Trotter product formula* [1, Chapter 2]:

$$\lim_{n\to\infty}\Big(\exp(kA_0/n)\exp((1-k)A_1/n)\Big)^n = \exp(A(k))$$

for any $k \in [0, 1]$. It follows from this that if $A(k)$ is not Hurwitz for some $k \in [0, 1]$, then (1) is not GUAS. Thus, from hereon we assume the following.

Assumption 1 *Every matrix in*

$$\mathrm{co}\{A_0, A_1\} := \{A(k) : k \in [0, 1]\}$$

is Hurwitz.

Assumption 1 is a necessary (but not sufficient) condition for GUAS of (1).

Recently, the problem of establishing conditions that guarantee GUAS of (1) has attracted considerable interest [2], [3], [4], [5], [6], [7]. A natural idea, first suggested by Pyatnitskiy and his colleagues [8], is to try and characterize the "most unstable" switching law. If the corresponding trajectory is asymptotically stable, then so are all the other trajectories. Thus, the problem can be reduced to analyzing the behavior of this single trajectory. The "most unstable" switching law can be characterized using variational principles (see the survey paper [9]).

Recall that a linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, with $A \in \mathbb{R}^{n\times n}$, is called *positive* if $\mathbb{R}^n_+ := \{\boldsymbol{x} \in \mathbb{R}^n | x_i \ge 0, i = 1, \ldots, n\}$ is an invariant set of the dynamics, that is, if $\boldsymbol{x}(0) \in \mathbb{R}^n_+$ implies that $\boldsymbol{x}(t) \in \mathbb{R}^n_+$ for all $t \ge 0$. A necessary and sufficient condition for this is that A is a *Metzler matrix*, that is, all the non-diagonal elements of A are non-negative. Positive linear systems play an important role in system and control theory because in many physical systems the state-variables represent quantities that can never attain negative values (see e.g. [10]).

If both A_0 and A_1 are Metzler and $\boldsymbol{x}_0 \in \mathbb{R}^n_+$, then we refer to (1) as a *positive linear switched system* (PLS). PLSs were used for modeling communication systems [11] and formation flying [12] (see also [7]).

Mason and Shorten [13], and independently David Angeli, posed the following conjecture.

Conjecture 1 *A PLS that satisfies Assumption 1 is GUAS.*

Recently, Gurvits, Shorten, and Mason [14] proved that this conjecture is true for the case $n = 2$ (even when the number of subsystems is arbitrary). This result was also proved using the variational approach in [15].

Gurvits, Shorten, and Mason [14] also showed that Conjecture 1 is in general false. As noted in [14], this naturally raises the following question. What is the *minimal* integer n_p for which there exists a PLS that satisfies Assumption 1 but is not GUAS?

In this paper, we solve this problem by presenting a specific three-dimensional system for which Conjecture 1 is false. Since $n_p > 2$, this proves that $n_p = 3$.

The remainder of this note is organized as follows. The next section provides a brief review of the results in [14]. Section III presents a specific three-dimensional counterexample to Conjecture 1. The final section concludes.

II. The Gurvits et al. Counterexample

In this section, we explain the construction in [14]. The first step is an argument that allows transforming a linear switched system with an invariant cone into a PLS.

Recall that a set $\Omega \in \mathbb{R}^n$ is called a *closed convex cone* if for any $\boldsymbol{y}^1, \boldsymbol{y}^2 \in \Omega$ and any $c_1, c_2 \ge 0$, $c_1\boldsymbol{y}^1 + c_2\boldsymbol{y}^2 \in \Omega$. The cone is said to be *proper* if its interior is non-empty, and $\Omega \cap (-\Omega) = \{\boldsymbol{0}\}$.

Suppose that there exists a proper polyhedral convex cone

$$\Omega = \Big\{\sum_{i=1}^{k} c_i\boldsymbol{z}^i : c_i \ge 0\Big\},$$

with $\boldsymbol{z}^i \in \mathbb{R}^n$, that is an invariant set of (1). In other words, if $\boldsymbol{x}_0 \in \Omega$ then

$$\exp(A_i t)\boldsymbol{x}_0 \in \Omega, \qquad t \ge 0,\ i = 0, 1.$$

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It then follows [16] that there exists $\tau > 0$ such that for any $\boldsymbol{x}_0 \in \Omega$:

$$(I + \tau A_i)\boldsymbol{x}_0 \in \Omega, \qquad i = 0, 1.$$

This implies in particular that

$$(I + \tau A_i)\boldsymbol{z}^j \in \Omega, \qquad i = 0, 1.$$

By the definition of Ω, there exist $c^i_{pj} \geq 0$ such that

$$(I + \tau A_i)\boldsymbol{z}^j = \sum_{p=1}^{k} c^i_{pj} \boldsymbol{z}^p,$$

i.e.

$$A_i \boldsymbol{z}^j = \frac{1}{\tau}(\sum_{p=1}^{k} c^i_{pj} \boldsymbol{z}^p - \boldsymbol{z}^j).$$

In other words,

$$A_i \boldsymbol{z}^j = \sum_{p=1}^{k} a^i_{pj} \boldsymbol{z}^p,$$

with $a^i_{pj} \geq 0$ if $p \neq j$. Thus, the matrices $B_0 = \{a^0_{pj}\}^k_{p,j=1}$ and $B_1 := \{a^1_{pj}\}^k_{p,j=1}$ are Metzler. It can be shown that every matrix in $\mathrm{co}\{A_0, A_1\}$ is Hurwitz if and only if every matrix in $\mathrm{co}\{B_0, B_1\}$ is Hurwitz. Furthermore, the n-dimensional switched system (1) is GUAS if and only if the k-dimensional PLS $\dot{\boldsymbol{x}}(t) = B_{\sigma(t)}\boldsymbol{x}(t)$ is GUAS.

Now fix two matrices A_0 and A_1 such that every matrix in $\mathrm{co}\{A_0, A_1\}$ is Hurwitz, yet the switched system (1) is not GUAS. Define

$$Z(t) := \boldsymbol{x}(t)\boldsymbol{x}'(t). \tag{2}$$

Then

$$\dot{Z} = A_\sigma Z + Z A'_\sigma, \tag{3}$$

with $Z(0) = \boldsymbol{x}(0)\boldsymbol{x}'(0)$. Since the original system is not GUAS, it follows from (2) that so is the switched system (3) (see also [17] for some related considerations). The system (3) evolves on the cone of symmetric non-negative definite matrices. It is possible to approximate this cone, to arbitrary accuracy, using a proper polyhedral cone. It follows from the arguments above that this yields a PLS that satisfies Assumption 1 yet is *not* GUAS. This elegant construction proves that Conjecture 1 is, in general, false.

Note that the dimension of the resulting PLS is k, where k is the number of vertices defining the approximating polyhedral cone. Intuitively, this suggests that constructing an *explicit* counterexample using this scheme is non trivial and, furthermore, that the resulting PLS will have a very large dimension.

III. A Three-Dimensional Counterexample

From here on we consider the switched system (1) with $n = 3$,

$$A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 0 & -10 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -10 & 0 & 10 \\ 0 & -10 & 0 \\ 0 & 10 & -1 \end{pmatrix}.$$

The eigenvalues of A_0 (A_1) are $\{-1, -1, -10\}$ ($\{-1, -10, -10\}$), so both matrices are Hurwitz. Also both A_0 and A_1 are Metzler, so clearly every matrix in $\mathrm{co}\{A_0, A_1\}$ is Metzler. We claim that this system provides a counterexample to Conjecture 1.

We now show that every matrix in $\mathrm{co}\{A_0, A_1\}$ is Hurwitz. This can be done directly by using the Routh-Hurwitz criterion. We use a simpler approach, suggested to us by Leonid Gurvits and also by one of the anonymous reviewers, and based on an idea from [18, Chapter 2, Section 5, Exercise 5].

Fix some arbitrary $k \in [0, 1]$. We begin by showing that $A(k)$ is non-singular. If this is not true, then clearly $k \in (0, 1)$ and there exists $\boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ such that $A(k)\boldsymbol{v} = \boldsymbol{0}$. Using the definition of $A(k)$ yields

$$A_0^{-1} A_1 \boldsymbol{v} = \frac{k}{k-1} \boldsymbol{v},$$

i.e. $A_0^{-1} A_1$ has a real and negative eigenvalue. However, the characteristic polynomial of $A_0^{-1} A_1$ is

$$p(s) = s^3 - 20.1 s^2 + 2s - 10,$$

and clearly it has no real negative roots. Thus we conclude that $A(k)$ is non-singular for all $k \in [0, 1]$.

Since $A(k) = kA_0 + (1-k)A_0$ is a Metzler matrix, $\exp(A(k))$ has nonnegative entries. By the Perron-Frobenius theorem (see e.g. [18]), the eigenvalue of $\exp(A(k))$ with the maximal absolute value is real and nonnegative. Consequently, if $\lambda(k)$ is the eigenvalue of $A(k)$ with the largest real part, then $\lambda(k)$ is real. As both A_0 and A_1 are Hurwitz, $\lambda(0) < 0$ and $\lambda(1) < 0$. The function $l \mapsto \lambda(l)$ is continuous, and we already know that $A(k)$ is non-singular for all $k \in (0, 1)$. We conclude that $\lambda(l) < 0$ for all $l \in [0, 1]$, so every matrix in $\mathrm{co}\{A_0, A_1\}$ is Hurwitz. Thus our PLS satisfies Assumption 1.

Finally, a calculation reveals that the matrix $\exp(A_0)\exp(A_1)$ has one real eigenvalue $\eta \approx 1.66879$. The corresponding eigenvector $\boldsymbol{\zeta}$ satisfies $\boldsymbol{\zeta} \in \mathbb{R}^3_+$ (this is again a consequence of the Perron-Frobenius theorem). Thus, for any integer j:

$$\big(\exp(A_0)\exp(A_1)\big)^j \boldsymbol{\zeta} = \eta^j \boldsymbol{\zeta}.$$

Since $\eta > 1$, this implies that the switched system admits a trajectory $\boldsymbol{x}(t)$, with $\boldsymbol{x}(0) = \boldsymbol{\zeta}$, corresponding to a *periodic* switching law, such that $\boldsymbol{x}(t)$ is unbounded. Hence, the switched system is clearly not GUAS.

Fig. 1 depicts the trajectory of the switched system emanating from $\boldsymbol{\zeta}$, following $\dot{\boldsymbol{x}} = A_1\boldsymbol{x}$ for $t = 1$ seconds (dashed line), and then $\dot{\boldsymbol{x}} = A_0\boldsymbol{x}$ for $t = 1$ seconds (solid line). The final point is then $\exp(A_0)\exp(A_1)\boldsymbol{\zeta} = \eta\boldsymbol{\zeta}$.

Fig. 2 depicts the trajectory of the linear system $\dot{\boldsymbol{x}} = A_1\boldsymbol{x}$ emanating from $\boldsymbol{x}(0) = \boldsymbol{\zeta}$. The trajectory is attracted to the line $c(1, 0, 0.9)$, $c \in \mathbb{R}$, which is an eigenvector of A_1, and then converges, of course, to the origin.

Fig. 3 depicts the trajectory of the linear system $\dot{\boldsymbol{x}} = A_0\boldsymbol{x}$ emanating from $\boldsymbol{\mu} := \exp(A_1)\boldsymbol{\zeta}$. The trajectory is attracted to the line $c(0, 1, 0)$, $c \in \mathbb{R}$, which is an eigenvector of A_0, and then converges, of course, to the origin. During the transient, the trajectory moves away from the origin. Indeed,

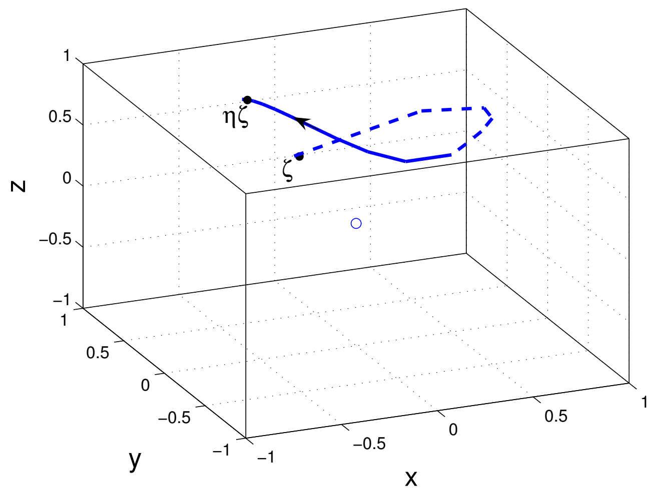

Fig. 1. A trajectory of the switched system with $\boldsymbol{x}(0) = \boldsymbol{\zeta}$.

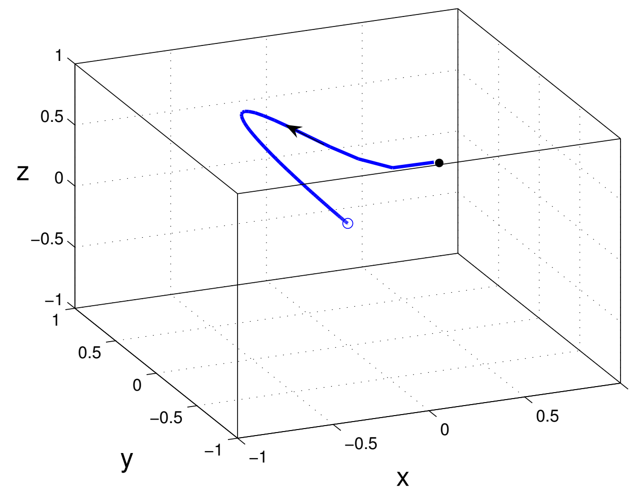

Fig. 3. Trajectory of $\dot{\boldsymbol{x}} = A_0\boldsymbol{x}$ with $\boldsymbol{x}(0) = \exp(A_1)\boldsymbol{\zeta}$.

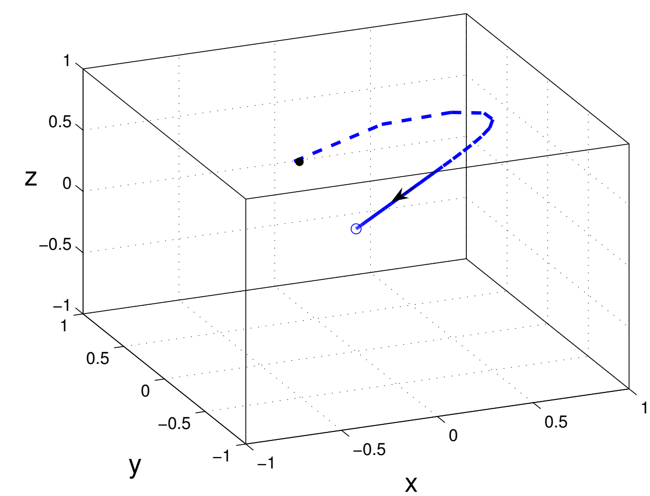

Fig. 2. Trajectory of $\dot{\boldsymbol{x}} = A_1\boldsymbol{x}$ with $\boldsymbol{x}(0) = \boldsymbol{\zeta}$.

a calculation shows that

$$\frac{|\exp(A_0)\boldsymbol{\mu}|}{|\boldsymbol{\mu}|} \approx 2.747.$$

Summarizing, by properly combining the transient behaviors of the two linear subsystems, it is possible to obtain a diverging trajectory of the switched system. This is of course the standard mechanism yielding instability in a switched system that satisfies Assumption 1. The fact that the system is a PLS does not preclude the possibility of this mechanism here. The interesting question is then why Conjecture 1 does hold for the case $n = 2$. As noted above, two answers to this question can be found in [14], [15].

IV. Conclusions

We considered an n-dimensional PLS satisfying the following condition: every matrix in $\mathrm{co}\{A_0, A_1\}$ is Hurwitz. This condition is necessary for GUAS, and several researchers conjectured that for PLSs it is also sufficient for GUAS.

Recently, Gurvits, Shorten, and Mason showed that this conjecture holds for $n = 2$, but is not true in general. In other words, there exists some finite integer n_p such that the conjecture is true for all $n < n_p$, but is not true for $n = n_p$.

In this paper, we presented a three-dimensional counterexample to this conjecture. Since $n_p > 2$, this proves that $n_p = 3$. This result provides another demonstration of the fact that linear switched systems, even in relatively low dimensions, may exhibit surprising and non trivial behavior. It also suggests that new behaviors may emerge when we move from the planar case $n = 2$ to the three-dimensional case $n = 3$. This is perhaps not surprising, as linear switched systems behave, in many respects, as nonlinear systems.

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