—NOTES—

ON THE STABILITY OF SOLUTIONS OF A SECOND-ORDER DIFFERENTIAL EQUATION*

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Abstract. This paper deals with the Hill differential equation

$$d^2y/dx^2 + \frac{r}{1 - 2a\cos x + a^2}\, y = 0.$$

Although this equation looks more difficult than Mathieu's, it can be dealt with somewhat more simply than the latter. Stability criteria are obtained in terms of r and a (at least in principle).

1. Introduction. The stability of the second-order, nonsingular, parametric equation

$$d^2y/dx^2 + \frac{r}{1 - 2a\cos x + a^2}\, y = 0, \qquad a^2 < 1 \tag{1}$$

can be investigated quite simply. As this is a Hill equation, solutions can be put in the form

$$y = \exp(\mu x)P(x) \tag{2}$$

where $P(x)$ is periodic with period 2π and r is real or pure imaginary [1]. Now if r is real the solution will evidently be unstable, and if imaginary, stable. Determination of which it is seems at first glance to be rather difficult, as an infinite determinant must somehow be evaluated. For the general Hill equation this cannot be done algebraically (by giving a recurrence formula, say). We shall perform this evaluation. The result will, in fact, be given by a recurrence relation. A particular case will then be discussed in some detail ($r = 1$). The general situation is stability for some range of parameters, as is the case for Mathieu's equation.

First we Fourier-expand the function multiplying y and change to $\xi = (1/2)x$, obtaining the standard form of Hill's equation

$$\frac{d^2y}{d\xi^2} + y\left[r(1 - a^2)^{-1} + 2r(1 - a^2)\sum_{n=1}^{\infty} a^n \cos 2n\xi\right]y = 0. \tag{3}$$

2. **Stability.** To apply the stability analysis outlined in McLachlan's book [1] it is convenient to use the notation

$$\Theta_0 = 4r(1 - a^2)^{-1}, \ \Theta_{2s} = 4a^{|s|}r(1 - a^2)^{-1}. \tag{4}$$

This notation follows naturally from expanding $P(x)$ in (2) in exponential functions. In the present note $\Theta_0 > 0$, though extension to negative Θ_0 is straightforward.

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Following McLachlan, we see that solutions of (3) are stable if an infinite determinant $\Delta(0)$ is positive and smaller than $\sin^{-2}(\pi/2\Theta_0^{1/2})$. This determinant can be denoted by $|\mathcal{A}_{mn}|$, where the integer m goes from $-\infty$ to ∞ from the bottom of the page to the top, and n from left to right, and

$$\begin{aligned} \mathcal{A}_{mn} &= 1, & m + n &= 0 \\ &= \Theta_{2m+2n}(\Theta_0 - 4m^2)^{-1}, & m + n &\neq 0. \end{aligned} \tag{5}$$

As $(r > 0)$ $\Theta_{\pm 2r} = a^{r-1}\Theta_2$, we can simplify the determinant by subtracting column n multiplied by a from column $n + 1$ if n is positive, and column n times a from column $n - 1$ if it is negative. This leads to an expression for $|\mathcal{A}|$ which is a product of two semi-infinite determinants, the upper right-hand and lower left-hand sectors now containing nothing but zeros. After some further manipulation we see that the determinant is the square of

$$D = \lim_{N\to\infty} D^{(N)} = \lim_{N\to\infty} \left(\prod_{s=1}^{N} \frac{as^2\sqrt{1-c^2}}{r - s^2\sqrt{1-c^2}} \right) d^{(N)}, \qquad c = \frac{2a}{1+a^2}$$

$$d^{(1)} = 0, \qquad d^{(2)} = -1, \qquad d^{(k)} = \frac{2(1-k^2)}{k^2c} d^{(k-1)} - d^{(k-2)}, \qquad N > k > 2 \tag{6}$$

$$d^{(N)} = \left(\frac{2}{N^2c} - \frac{1}{a} \right) d^{(N-1)} - d^{(N-2)}.$$

It is easy to see that $D(r, a \to 1) = 0$.

When the product formula for $\sin x$ is used we see that

$$D = \pi(1 - c^2)^{-1/4} \sin^{-1}[\pi(1 - c^2)^{-1/4}] \lim_{N\to\infty} a^N d^{(N)}. \tag{7}$$

and the stability criterion simplifies to

$$d' = \pi(1 - c^2)^{-1/4} \lim_{N\to\infty} a^N d^{(N)} \leq 1. \tag{8}$$

This has been checked to be true for $r = 1$, all a. For this value of r the crucial expression (8) goes from 0 at $a = 0$ to a maximum near $a = 0.9$, and back to 0 at $a = 1$. The function $y(x)$ is bounded everywhere for all $a \leq 1$ and is quasi-oscillating with two periods (2π and $\pi^2 \operatorname{arc} \sin^{-1} d'$) [1]. The function $d'(C)$ has been plotted.

When evaluating D it is more to the point to use the Nth product in (6) than the elementary function it leads to. This is so because the $O(N^{-2})$ terms in $d^{(N)}/d^{(N-1)}$ cancel those coming from the corresponding ratio in the product and the term-to-term error is then $O(N^{-4})$, as can be shown by an approximate evaluation of $d^{(N)}/d^{(N-1)}$ from the recurrence relation. One can see the difference even at small N. For $C = 0.9$, for example, $(d^{(7)} - ad^{(6)})/d^{(7)}$ is about 3%, whereas $(D^{(7)} - D^{(6)})/D^{(7)}$ is less than 1%, and $(d^{(14)} - ad^{(13)})/d^{(14)}$ is about 0.8% whereas $(D^{(14)} - D^{(13)})/D^{(14)}$ is only 0.03%. A term-to-term error of $O(N^{-4})$ means an overall error of $O(N^{-3})$ and so $N = 7$ is quite sufficient.

For general values of r the term-to-term error would be $O(N^{-2})$, leading to an overall error of $O(N^{-1})$ in using any particular $D^{(N)}$.

Summary. A method for determining stability and finding periods of oscillation or growth rates for solutions to a class of second-order differential equations has been

given. One specific example has been dealt with in some detail. This example has appeared in a physical context [3].

References

[1] N. W. McLachlan, *Theory and application of Mathieu functions*, Clarendon, 1947
[2] I. S. Gradshteyn and I. M. Ryshik, *Table of integrals, series and products*, Academic Press, 1965
[3] E. Infeld and G. Rowlands, *On the stability of non-linear cold plasma waves II*, J. Plasma Phys. **10**, 233 (1973)