parameters.) Also, the input must retain this property for all time. If these conditions, intuitively reasonable for adaptive identification, are fulfilled, then the lower bound in (3.2) holds, while the upper bound reflects boundedness of $u_p(\cdot)$.

A common procedure to ensure fulfillment of those requirements is to take $u_p(\cdot)$ to be a finite sum of sinusoids or periodic signals. In this way, $u_p(\cdot)$ is periodic, or almost periodic, and if there are sufficient different frequencies within $u_p(\cdot)$, the persistently exciting condition holds for $V(\cdot)$.

A-III Origin of (3.4)

An alternative approach to the above (useful because, as it turns out, integrators are saved) is developed in, e.g., [4] and [9]. The model, this time, partly in Laplace transform notation and neglecting the transform of exponentially decaying quantities, is

$$V_1(s)=\frac{1}{s^{n-1}+\sum_{i=1}^{n-1}\beta_i s^{i-1}}\begin{bmatrix}1\\ s\\ \cdot\\ \cdot\\ s^{n-1}\end{bmatrix}Y_p(s)$$

$$V_2(s)=\frac{1}{s^{n-1}+\sum_{i=1}^{n-1}\beta_i s^{i-1}}\begin{bmatrix}1\\ s\\ \cdot\\ \cdot\\ s^{n-1}\end{bmatrix}U_p(s) \tag{A5}$$

$$w_m(t)=l_1'(t)v_1(t)+l_2(t)v_2(t)$$

$$Y_m(s)=B'(sI-A)^{-1}BW_m(s), \qquad A+A'=-I.$$

One can show that $Y_m(s)=Y_p(s)$ if and only if $l_1(t)=\tilde{k}_1$, $l_2(t)=\tilde{k}_2$ for two constant n-vectors $\tilde{k}_1,\tilde{k}_2$ determined by and determining the plant transfer function. The task therefore is to ensure that $l_i(t)\rightarrow\tilde{k}_i$ as $t\rightarrow\infty$. One still adjusts $l_1(t)$, $l_2(t)$ using the error $y_m(t)-y_p(t)$:

$$\begin{bmatrix}\dot{l}_1\\ \dot{l}_2\end{bmatrix}=-\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix}\left[y_m(t)-y_p(t)\right] \tag{A6}$$

although the error $y_m(\cdot)-y_p(\cdot)$ is not formed in the same way as before. By taking

$$x=\begin{bmatrix}l_1(t)-\tilde{k}_1\\ l_2(t)-\tilde{k}_2\\ x_2\end{bmatrix}$$

where $\dot{x}_2=Ax_2+B[w_m(t)-\tilde{k}_1'v_1(t)-\tilde{k}_2'v_2(t)]$, (3.4) follows, other than for an additive, exponentially decaying term.

A-IV The Observability Condition (3.5)

The remarks concerning (3.2) apply of course, but there is additional intuition regarding the need for the integral in (3.5). In forming the error $y_m(t)-y_p(t)$ which is used for adjusting the $l_i(\cdot)$ in (A6), the $v_i(t)$ are integrated [see last equation in (A5)] in the second scheme, while no integration occurs in the first scheme [see last equation in (A1)]. The persistently exciting condition is required of the integrated $v_i(\cdot)$.

Proposition 1 states that if a persistently exciting condition is absent, it cannot be regained by integration, while Proposition 2 states that if it is present, and if $V(\cdot)\in\mathcal{V}$, then it is retained by integration. Requiring $V(\cdot)\in\mathcal{V}$ is equivalent to not allowing $V(\cdot)$, as time evolves, to contain less and less low frequency content. Since the effect of integration is to cut down high frequency content, taking $V(\cdot)\in\mathcal{V}$ therefore ensures that the integral of $V(\cdot)$ does not die away as $t\rightarrow\infty$.

A-V Origin of (3.9)

The thinking is much as for the origin of (3.4), save that instead of having $Y_m(s)=B'(sI-A)^{-1}BW_m(s)$ where the constraint $A+A'=-I$ forces $B'(sI-A)^{-1}B$ to be positive real one allows $Y_m(s)=Z(s)W_m(s)$ where $Z(s)$ is positive real (in a strict sense described in Theorem 4). Equation (3.9) is thus a generalization of (3.4).

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On the Stability of Solutions to Minimal and Nonminimal Design Problems

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Abstract—**A partial resolution to the question of stability of solutions to the minimal design problem is given in terms of transfer matrix factorizations employing the new notions of common system poles and common systems zeros as well as the fixed poles of all solutions and the fixed poles of minimal solutions. The results are employed to more directly and easily resolve questions involving the attainment of stable solutions to the model matching problem and stable minimal-order state observers.**

I. Introduction

The primary purpose of this paper is to investigate various questions involving minimal-order dynamic compensation. In particular, in Section II we present some preliminary mathematical notions involving minimal bases of rational vector spaces. In Section III we formulate the minimal design problem and illustrate how it can be rather easily and directly resolved via prime polynomial matrix reductions to either row or column proper form.

The question of obtaining stable solutions to the minimal design problem is then considered in Section IV. Here we define the new and intuitively appealing notions of the common poles and the common zeros of dynamical systems, as well as the fixed poles of all solutions and

Manuscript received October 6, 1975; revised February 20, 1976 and August 5, 1976. Paper recommended by J. B. Pearson, Chairman of the IEEE S-CS Linear Systems Committee. This work was supported in part by the National Science Foundation under Grant ENG 73-03846A01 and in part by the Air Force Office of Scientific Research under Grant AFOSR 71-2078C.

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the fixed poles of minimal solutions. These definitions are then employed to completely resolve the question of obtaining stable solutions to the model matching problem, and to partially resolve the question of obtaining stable solutions to the minimal design problem.

In Section V, the question of obtaining minimal-order dynamic observers of any linear function of the state of a given system is shown to be analogous to the minimal design problem. The allied question of observer stability is then considered in view of earlier results, and some concluding remarks are presented in Section VI.

II. Mathematical Preliminaries

Let $\mathfrak{F}(s)$ denote the *field of rational functions* and $\mathfrak{F}[s]$ the *ring of polynomials* in s (the Laplace operator) all with real coefficients. Any set of m (column or row) vectors $v_j(s)$ with elements in $\mathfrak{F}(s)$ or $\mathfrak{F}[s]$ will be called *linearly independent* if and only if $\Sigma_1^m\ \alpha_j(s)v_j(s)=0$ implies that $\alpha_j(s)=0$ for all $j\in\mathbf{m}$ (where $\mathbf{m}$ denotes the set $1,2,\cdots,m$). It should be noted that the $\alpha_j(s)$ can belong to $\mathfrak{F}(s)$ or $\mathfrak{F}[s]$ regardless of the $v_j(s)$. The *rank* of any $p\times m$ matrix $M(s)$ with elements $m_{ij}(s)$ in $\mathfrak{F}(s)$ or $\mathfrak{F}[s]$ will be defined as the number of linearly independent (column or row) vectors which comprise $M(s)$.

We now recall certain of the notation and results given in [1] and [2]. In particular, let $M(s)$ represent any $p\times m$ matrix with elements in $\mathfrak{F}[s]$, i.e., a *polynomial matrix*. Let $\Gamma_c[M(s)]$ ($\Gamma_r[M(s)]$) represent the $(p\times m)$ real matrix consisting of the coefficients of the highest degree polynomial or polynomials in each column (row) of $M(s)$. $M(s)$ will be called *column (row)* proper [1] if and only if the rank, denoted by ρ, of $\Gamma_c[M(s)]$ ($\Gamma_r[M(s)]$) equals $\min\{p,m\}$. It has been shown [1] that any full rank polynomial matrix can be reduced to column (row) proper form via elementary column (row) operations, i.e., if $\rho\{M(s)\}=\min\{p,m\}$, there exists a *unimodular matrix* max $n\times 1$, $U_R(s)$,[1] $U_R(s)$ ($U_L(s)$), such that

$$M(s)U_R(s)=M_R(s) \qquad (U_L(s)M(s)=M_L(s)) \tag{2.1}$$

with $M_R(s)$ ($M_L(s)$) column (row) proper. An algorithm for performing this reduction is given in [1], and a computer program is also available for performing this reduction as well as other useful polynomial and rational matrix operations [3].

If $\rho\{M(s)\}=p\le m$ (the columns of) $M(s)$ will be called *relatively left prime* (rlp) [1], [2] if and only if any *greatest common left divisor* (gcld) [1], [2] $G_L(s)$ of (the columns of) $M(s)$ is unimodular. Similarly, if $\rho\{M(s)\}=m\le p$, (the rows of) $M(s)$ will be called *relatively right prime* (rrp) if and only if any *greatest common right divisor* (gcrd) of (the rows of) $M(s)$ is unimodular. It is well known [1], [2] that any gcld (gcrd) of a rank $p(m)$ polynomial matrix $M(s)$ can be obtained via elementary column (row) operations, i.e., if $\rho\{M(s)\}=p(m)$, there exists a $U_R(s)$ ($U_L(s)$) such that

$$M(s)U_R(s)=\left[\,G_L(s)\ \vdots\ 0\,\right] \qquad \left(U_L(s)M(s)=\begin{bmatrix} G_R(s) \\ \hline 0 \end{bmatrix}\right) \tag{2.2}$$

where $G_L(s)$ ($G_R(s)$) is a gcld (gcrd) of $M(s)$. Algorithms for performing this reduction can be found in [1] and [2], and one is included in the computer program noted earlier [3].

Let $M(s)$ be any $p\times m$ rational matrix of rank $p(\le m)$. If the rows of $M(s)$ are thought of as representing a *basis* of a *vector space* V *over* $\mathfrak{F}(s)$, the *dual space* $V^\perp$ *over* $\mathfrak{F}(s)$ consists of all m-dimensional rational column vectors $k(s)$ such that $M(s)k(s)=0$. Since V has dimension p, $V^\perp$ will have dimension $m-p$, and a basis $K(s)$ of $V^\perp$ will consist of any set of $(m-p)$ linearly independent column vectors in ker $M(s)$, the kernel of $M(s)$, i.e., (the columns of) any $m\times(m-p)$ rational matrix $K(s)$ will be called a basis of $V^\perp$ if $\rho\{K(s)\}=m-p$ and

$$M(s)K(s)=0. \tag{2.3}$$

A *prime basis* of the vector space $V(V^\perp)$ over $\mathfrak{F}(s)$ is now defined as (the rows (columns) of) any $p\times m$ ($m\times(m-p)$) polynomial matrix $\bar{M}(s)$ ($\bar{K}(s)$) whose columns (rows) are rlp (rrp). A *minimal basis*[2] of $V(V^\perp)$ is defined as any row proper (column proper) prime basis. Finally, if the rows of any minimal basis $\bar{\bar{M}}(s)$ of V are permuted so that $\partial_{ri}[\bar{\bar{M}}(s)]$, the degree of each ith row of $\bar{\bar{M}}(s)$ is less than or equal to the degree of subsequent rows, then (the rows of) $\bar{\bar{M}}(s)$ will be called a *degree ordered* minimal basis of V as well. The notion of a degree ordered minimal basis of $V^\perp$ is defined in an analogous fashion, i.e., $\bar{\bar{K}}(s)$ is a degree ordered minimal basis of $V^\perp$ if and only if the rows of $\bar{\bar{K}}(s)$ are rrp, $\bar{\bar{K}}(s)$ is column proper, and $\partial_{cj}[\bar{\bar{K}}(s)]\le\partial_{c,j+1}[\bar{\bar{K}}(s)]$ for $j\in\mathbf{m-p-1}$. For convenience, we have and will continue to refer to a matrix, such as $\bar{\bar{M}}(s)$ or $\bar{\bar{K}}(s)$, as a basis of a vector space although, strictly speaking, we realize that the rows or columns of the matrix actually comprise the basis. Appropriate algorithms for the construction of minimal bases of various rational vector spaces are given in [4].

III. The Minimal Design Problem

The primary purpose of this relatively short section is simply to introduce the minimal design problem and to present a solution to it based on the earlier work of a number of investigators. The results presented in this section will then serve to motivate much of what follows.

The *minimal design problem* (MDP) can be stated as follows. Given a $p\times m$ rational transfer matrix $T_1(s)$ of rank $p(<m)$[3] and a $p\times q$ rational transfer matrix $T_2(s)$ find a $(m\times q)$ proper rational transfer matrix $T(s)$ of minimal dynamic order[4] (if such a transfer matrix exists) such that

$$T_1(s)T(s)=T_2(s). \tag{3.1}$$

It might be noted that if the minimality of $T(s)$ is irrelevant, then (3.1) represents the well-known "exact model matching problem," which has been the subject of numerous investigations. To resolve the MDP, in light of the notation employed in Section II, we require one additional definition. In particular, suppose that $K(s)$ is a $q\times r$ polynomial matrix with $q>r$. It is clear that $K(s)$ can be partitioned as $\begin{bmatrix} K_r(s) \\ \hline K_{q-r}(s)\end{bmatrix}$, where $K_r(s)$ denotes the first r rows of $K(s)$ and $K_{q-r}(s)$ denotes the final $q-r$ rows. $\Gamma_c[K(s)]$ will now be written as $\begin{bmatrix} K_{r\gamma} \\ \hline K_{q-r,\gamma}\end{bmatrix}$, noting that $K_{r\gamma}$ (or $K_{q-r,\gamma}$) does not necessarily equal $\Gamma_c[K_r(s)]$ (or $\Gamma_c[K_{q-r}(s)]$). With this notation in mind, we can now resolve the MDP.

Theorem 3.2: Let $K(s)=\begin{bmatrix} K_m(s) \\ \hline K_q(s)\end{bmatrix}$ be any $(m+q)\times(m+q-p)$ degree ordered, minimal basis for ker $\left[\,T_1(s)\ \vdots\ -T_2(s)\,\right]$. The MDP has a solution $T(s)$ if and only if

$$\rho[K_{q\gamma}]=q. \tag{3.3}$$

Furthermore, if (3.3) holds, the minimal dynamic order of an appropriate $T(s)$ is equal to the sum of the column degrees of the first (ordered from left to right) q columns of $K(s)$ for which the corresponding (numbered) columns of $K_{q\gamma}$ are linearly independent. These q columns of $K(s)$, $\begin{bmatrix} R(s) \\ \hline P(s)\end{bmatrix}$, represent a proper, minimal-order solution $T(s)=R(s)P(s)^{-1}$ to (3.1).

Proof: Since Theorem 3.2 is not original, except for the particular way it is stated using the (new) notion of a degree ordered, minimal basis, a formal proof will not be given here and the interested reader is referred to [4]. It should be noted that Wang and Davison [6], [7] were

[1]A unimodular matrix is a nonsingular polynomial matrix whose inverse is a polynomial matrix [1], [2].

[2]This definition corresponds to that given by Forney [4].

[3]If $p\ge m$, the MDP either has no solution or a unique solution (which can easily be found).

[4]If $R(s)P^{-1}(s)$ is a rrp factorization of $T(s)$, $\partial[|P(s)|]$ is the *dynamic order* of (a minimal state-space realization of) $T(s)$ [1].

first to investigate the MDP. Forney [4] later employed the notion of minimal bases of rational vector space in order to resolve the MDP. More recently, Sain [8] has presented a more direct procedure for obtaining minimal bases which facilitates certain of the computational steps outlined by Forney, while Morse [9] has established the equivalence between the MDP and the problem of finding an (A,B)-invariant subspace of least dimension, which contains a given subspace.

Although we will not formally establish Theorem 3.2 here, we will illustrate its employment by example. In particular, if

$$T_1(s)=\begin{bmatrix} \dfrac{s}{s^2+3s+2} & 0 & \dfrac{s^2+2s+2}{s^2+3s+2} \\ \dfrac{2s+1}{s+1} & \dfrac{s-1}{s+2} & 0 \end{bmatrix} \quad \text{and} \quad T_2(s)=I_2,$$

i.e., if we wish to find a proper "right inverse" of $T_1(s)$, then we can first determine a minimal basis of ker $\left[\, T_1(s) \;\vdots\; -T_2(s) \,\right]$. By employing appropriate algorithms in either [4] or [8], in conjunction with our computer program [3], we readily determine that

$$K(s)=\left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & s+2 & 0 \\ 1 & 0 & s^2+3s+2 \\ \hline 1 & 0 & s^2+2s+2 \\ 1 & s-1 & 0 \end{array}\right]=\left[\begin{array}{c} K_m(s) \\ \hline K_q(s) \end{array}\right]$$

is a degree ordered minimal basis of ker $\left[\, T_1(s) \;\vdots\; -T_2(s) \,\right]$. We now note that

$$\Gamma_c[K(s)]=\left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]=\left[\begin{array}{c} K_{m\gamma} \\ \hline K_{q\gamma} \end{array}\right], \quad \rho[K_{q\gamma}]=q=2$$

which establishes the existence of a proper right inverse. Since columns 1 and 2 of $K(s)$ are the first two $(=q)$ for which the corresponding columns of $K_{q\gamma}$ are linearly independent, the minimal dynamic order of a proper right inverse is $1=\partial_{c1}[K(s)]+\partial_{c2}[K(s)]$, and

$$T(s)=R(s)P(s)^{-1}=\begin{bmatrix} 1 & 0 \\ -1 & S+2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & S-1 \end{bmatrix}^{-1}$$

$$=\begin{bmatrix} 1 & 0 \\ \dfrac{-2S-1}{S-1} & \dfrac{S+2}{S-1} \\ 1 & 0 \end{bmatrix}$$

represents a minimal order right inverse.

It should be noted that the above inverse has an unstable pole at $s=1$ and, as we will show in the next section, all first-order inverses of $T_1(s)$ will have a single unstable pole at $s=1$, an observation which now serves to motivate the next section.

IV. The Stability Question

The purpose of this section will be to investigate the ability and/or inability to achieve stable solutions to the MDP. It should be noted that Morse [10] has partially resolved this question by presenting a necessary and sufficient condition for achieving stable solutions to the "model following problem," i.e., in view of the formulation employed here Morse's model following problem can be shown to be analogous to the question of obtaining proper solutions to (3.1) when $T_1(s)$ and $T_2(s)$ are *strictly* proper transfer matrices. By employing transfer matrix factorizations, rather that the state-space approach employed in [10], we will obtain more general and hopefully more illuminating answers to the MDP stability question. As in [10], our initial result (Theorem 4.2) only partially resolves the MDP stability question, since minimality of the dynamic order of stable solutions cannot always be assured. Unlike [10], however, a sufficient condition (Theorem 4.13) for the instability of the MDP is also presented. It might be noted that the question of obtaining both a stable and a minimal dynamic order solution to (3.1) is similar to the difficult question of stabilizing a linear system via constant gain output feedback.

Before we consider the stability question, some preliminary observations and definitions are required. In particular, if $T_1(s)$ and $T_2(s)$ are factored as the rlp products $P_{1Q}(s)^{-1}Q_1(s)$ and $P_{2Q}^{-1}(s)Q_2(s)$, respectively, the zeros of the determinant $\Delta_p(s)$ of any gcrd $G_{RP}(s)$ of $P_{1Q}(s)$ and $P_{2Q}(s)$ will be called the *common poles* of $T_1(s)$ and $T_2(s)$. It now follows that

$$P_{2Q}(s)P_{1Q}^{-1}(s)=\hat{P}_{2Q}(s)G_{RP}(s)G_{RP}^{-1}(s)\hat{P}_{1Q}^{-1}(s)=\tilde{P}_1^{-1}(s)\tilde{P}_2(s) \tag{4.1}$$

for some rlp pair $\{\tilde{P}_1(s),\tilde{P}_2(s)\}$. The zeros of the determinant $\Delta_Q(s)$ of any gcld, $G_{LQ}(s)$, of $\tilde{P}_2(s)Q_1(s)$ and $\tilde{P}_1(s)Q_2(s)$ will be called the *common zeros of* $T_1(s)$ *and* $T_2(s)$. Finally, if we let $\Delta_T(s)$ represent the determinant of any gcld, $G_L(s)$, of (the columns of) $G_{LQ}^{-1}(s)\tilde{P}_2(s)Q_1(s)$, the zeros of $\Delta_T(s)=|G_L(s)|$ will be defined as *the fixed poles of* $T(s)$. The motivation for this definition will become apparent once we state and constructively establish the following theorem.

Theorem 4.2: If $T(s)=R(s)P^{-1}(s)$, with $R(s)$ and $P(s)$ rrp represents any solution to (3.1), then the zeros of $|P(s)|$ equal those of $\Delta_T(s)\cdot\Delta_D(s)$. Furthermore, if (3.3) holds, a proper solution can be found which arbitrarily assigns the zeros of $\Delta_D(s)$.

Proof: For notational convenience, let

$$M(s)=\left[\, M_m(s) \;\vdots\; -M_q(s) \,\right]=G_{LQ}^{-1}(s)\left[\, \tilde{P}_2(s)Q_1(s) \;\vdots\; -\tilde{P}_1(s)Q_2(s) \,\right].$$

Suppose $T(s)=R(s)P^{-1}(s)$ is a solution to (3.1) with $R(s)$ and $P(s)$ rrp. Clearly,

$$\left[\, T_1(s) \;\vdots\; -T_2(s) \,\right]\left[\begin{array}{c} R(s) \\ \hline P(s) \end{array}\right]$$

$$=\left[\, P_{1Q}^{-1}(s)Q_1(s) \;\vdots\; -P_{2Q}^{-1}(s)Q_2(s) \,\right]\left[\begin{array}{c} R(s) \\ \hline P(s) \end{array}\right]=0 \tag{4.3}$$

then

$$\left[\, P_{2Q}(s)P_{1Q}^{-1}(s)Q_1(s) \;\vdots\; -Q_2(s) \,\right]\left[\begin{array}{c} R(s) \\ \hline P(s) \end{array}\right]=0$$

and

$$G_{LQ}^{-1}(s)\left[\, \tilde{P}_2(s)Q_1(s) \;\vdots\; -\tilde{P}_1(s)Q_2(s) \,\right]\left[\begin{array}{c} R(s) \\ \hline P(s) \end{array}\right]$$

$$=\left[\, M_m(s) \;\vdots\; -M_q(s) \,\right]\left[\begin{array}{c} R(s) \\ \hline P(s) \end{array}\right]=0. \tag{4.4}$$

Consequently, $\left[\, M_m(s) \;\vdots\; -M_q(s) \,\right] \in \ker \left[\begin{array}{c} R(s) \\ \hline P(s) \end{array}\right]$.[5] From [8] we note that there must exist a rlp factorization $T(s)=\bar{P}^{-1}(s)Q(s)$ with $|\bar{P}(s)| =|P(s)|$ such that the rows of $\left[\, \bar{P}(s) \;\vdots\; -Q(s) \,\right]$ are a basis for $\ker \left[\begin{array}{c} R(s) \\ \hline P(s) \end{array}\right]$. This implies the existence of a unimodular matrix $U(s)$ such that $U(s)\left[\, \bar{P}(s) \;\vdots\; Q(s) \,\right]=\left[\begin{array}{c|c} M_m(s) & M_q(s) \\ \hline \beta(s) & \gamma(s) \end{array}\right]$ or that $U(s)\bar{P}(s)$

[5] In this case, ker $X(s)$ denotes those row vectors $v(s)$ for which $v(s)X(s)=0$.

$=\begin{bmatrix} M_m(s) \\ \hline \beta(s) \end{bmatrix}$. Since $G_L(s)$ is a gcld of $M_m(s)$ there must also exist a polynomial matrix $\bar{M}_m(s)$ such that $M_m(s)=G_L(s)\bar{M}_m(s)$. We then obtain

$$U(s)\bar{P}(s)=\begin{bmatrix} G_L(s) & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \bar{M}_m(s) \\ \beta(s) \end{bmatrix}$$

and

$$|\bar{P}(s)|=|U^{-1}(s)|\cdot|G_L(s)|\cdot\left|\begin{matrix} \bar{M}_m(s) \\ \beta(s) \end{matrix}\right|=|P(s)|$$

or

$$|P(s)|=\Delta_T(s)\Delta_D(s). \tag{4.5}$$

For some polynomial $\Delta_D(s)=|U^{-1}(s)|\cdot\left|\begin{matrix} \bar{M}_m(s) \\ \beta(s) \end{matrix}\right|$ which establishes the first part of Theorem 4.2.

To show that the zeros of $\Delta_D(s)$, can be arbitrarily assigned when (3.3) holds, we first assume, without loss of generality, that $M(s)$ is row proper, since $G_{LQ}^{-1}(s)$ can always be chosen to insure this [1]. If (3.3) holds, a proper solution $T(s)=R(s)P(s)^{-1}$ to (3.1) can be found with $P(s)$ column proper and $\partial_{cj}[R(s)]\leqslant\partial_{cj}[P(s)]$ for $j\in\boldsymbol{q}$, i.e., $\partial_c[R(s)]\leqslant\partial_c[P(s)]$. For any such solution, $\begin{bmatrix} R(s) \\ P(s) \end{bmatrix}\in\ker M(s)$ which in turn implies that $M_m(s)$ is row proper since

$$\partial_{ri}[M_m(s)]\geqslant\partial_{ri}[M_q(s)], \qquad \text{for } i\in\boldsymbol{p}. \tag{4.6}$$

To explicitly verify (4.6), we note that if $\partial_{rk}[M_m(s)]<\partial_{rk}[M_q(s)]$ for some $k\in\boldsymbol{p}$, then $\partial_{rk}[M_m(s)R(s)]<\partial_{rk}[M_q(s)P(s)]$ since $P(s)$ is column proper and $\partial_c[P(s)]\geqslant\partial_c[R(s)]$. Therefore, the kth row of $M_m(s)R(s)-M_q(s)P(s)\neq 0$, contrary to the fact that $\begin{bmatrix} R(s) \\ P(s) \end{bmatrix}\in\ker M(s)$.

We next assume, for convenience,[6] that the first p columns of $\Gamma_r[M_m(s)]$ are linearly independent (since $M_m(s)$ is row proper $\Gamma_r[M_m(s)]$ will have rank p) and partition $M_m(s)$ as $[M_{mp}(s) \mid M_{m,m-p}(s)]$, noting that $M_{mp}^{-1}(s)M_{m,m-p}(s)$ will be proper transfer matrix. Let $\tilde{N}(s)\tilde{M}(s)^{-1}$ represent any rrp factorization of $M_{mp}^{-1}(s)M_{m,m-p}(s)$ with $\tilde{M}(s)$ column proper and $\nu\triangleq\max_j\partial_{rj}[M_{mp}(s)]$. The *eliminant matrix* [1] of $\tilde{N}(s)$ and $\tilde{M}(s)$ can now be employed to obtain a pair $\{\alpha_{m-p}(s),\alpha_p(s)\}$ of polynomial matrices of dimensions $(m-p)\times(m-p)$ and $(m-p)\times p$, respectively, such that $\alpha_{m-p}(s)$ is row proper with $\partial_{ri}[\alpha_{m-p}(s)]=\nu-1$ for $i\in\boldsymbol{m-p}$, $\alpha_{m-p}^{-1}(s)\alpha_p(s)$ a proper rlp transfer matrix factorization, and

$$\alpha_{m-p}(s)\tilde{M}(s)+\alpha_p(s)\tilde{N}(s)=D(s) \tag{4.7}$$

a column proper polynomial matrix with arbitrary determinant $\Delta_D(s)$ of degree $d=\partial[|\tilde{M}(s)|]+(m-p)(\nu-1)$, i.e., $\partial_{cj}[D(s)]=\partial_{cj}[\tilde{M}(s)]+\nu-1$ for $j\in\boldsymbol{m-p}$.

We now let $U_R(s)=\begin{bmatrix} \delta(s) & \begin{matrix} -\gamma_p(s) \\ \gamma_{m-p}(s) \end{matrix} \end{bmatrix}$ be any unimodular matrix such that

$$M_m(s)U_R(s)=[M_{mp}(s) \mid M_{m,m-p}(s)]U_R(s)=[G_L(s) \mid 0] \tag{4.8}$$

where $G_L(s)$ is a gcld of $M_m(s)$. Since $U_R(s)$ is unimodular, (4.8) implies that the final m columns of $U_R(s)$, $\begin{bmatrix} -\gamma_p(s) \\ \gamma_{m-p}(s) \end{bmatrix}$, form a basis for ker $M_m(s)$ and thus represent a dual, rrp factorization of $M_{mp}^{-1}(s)M_{m,m-p}(s)$, i.e.,

$$M_{mp}^{-1}(s)M_{m,m-p}(s)=\gamma_p(s)\gamma_{m-p}^{-1}(s)=\tilde{N}(s)\tilde{M}(s)^{-1}. \tag{4.9}$$

Since $\begin{bmatrix} -\gamma_p(s) \\ \gamma_{m-p}(s) \end{bmatrix}$ and $\begin{bmatrix} -\tilde{N}(s) \\ \tilde{M}(s) \end{bmatrix}$ are column equivalent [5],

$$\begin{bmatrix} M_{mp}(s) & M_{m,m-p}(s) \\ -\alpha_p(s) & \alpha_{m-p}(s) \end{bmatrix}\begin{bmatrix} \delta(s) & \begin{matrix} -\tilde{N}(s) \\ \tilde{M}(s) \end{matrix} \end{bmatrix}=\begin{bmatrix} G_L(s) & 0 \\ X(s) & D(s) \end{bmatrix} \tag{4.10}$$

with the determinant of $\begin{bmatrix} M_{mp}(s) & M_{m,m-p}(s) \\ -\alpha_p(s) & \alpha_{m-p}(s) \end{bmatrix}$ equal to $\beta\Delta_T(s)\Delta_D(s)$, i.e.,

$$\left|\begin{bmatrix} M_m(s) \\ \alpha(s) \end{bmatrix}\right|=\beta|G_L(s)|\times|D(s)|=\beta\Delta_T(s)\Delta_D(s) \tag{4.11}$$

for some nonzero real scalar β.

We finally note that

$$\ker\begin{bmatrix} M_m(s) & -M_q(s) \\ \alpha(s) & 0 \end{bmatrix}\triangleq K_\alpha(s)=\begin{bmatrix} R_\alpha(s) \\ P_\alpha(s) \end{bmatrix}\in\ker[M_m(s) \mid -M_q(s)]$$

$$=\ker[T_1(s) \mid -T_2(s)]$$

and, therefore, that

$$T(s)=R_\alpha(s)P_\alpha^{-1}(s)=\begin{bmatrix} M_m(s) \\ \alpha(s) \end{bmatrix}^{-1}\begin{bmatrix} M_q(s) \\ 0 \end{bmatrix} \tag{4.12}$$

is a proper (since $\alpha_{m-p}(s)$ is row proper and $\partial_r[\alpha_{m-p}(s)]\geqslant\partial_r[\alpha_p(s)]$[1]) solution to the MDP with poles equal to the zero of $\Delta_T(s)$ and $\Delta_D(s)$. Theorem 4.2 is therefore established.

While the various steps taken to constructively establish Theorem 4.2 may appear somewhat formidable, it should be noted that the proof is based on the key observation that when a proper $T(s)$ does exist, one can append to

$$M(s)=[M_m(s) \mid -M_q(s)]=G_{LQ}^{-1}(s)[\tilde{P}_2(s)Q_1(s) \mid -\tilde{P}_1(s)Q_2(s)]$$

$(m-p)$ additional rows, $[\alpha(s) \mid 0]$, such that $\left|\begin{bmatrix} M_m(s) \\ \alpha(s) \end{bmatrix}\right|=\beta\Delta_T(s)\Delta_D(s)$, with $\begin{bmatrix} M_m(s) \\ \alpha(s) \end{bmatrix}^{-1}\begin{bmatrix} M_q(s) \\ 0 \end{bmatrix}=T(s)$, a proper solution to (3.1).

It is of interest to note that neither the common zeros nor the common poles of $T_1(s)$ and $T_2(s)$ affect $T(s)$ since they can be "cancelled" on both sides of (3.1). We further note that *the fixed poles of T(s)* [those poles which characterize all solutions of (3.1)] *do correspond to all of the zeros of $T_1(s)$ which are not common to $T_2(s)$, as well as the zeros of $|\tilde{P}_2(s)|$, which represent the poles of $T_2(s)$ which are not common to $T_1(s)$.* In order to achieve stable solutions to the MDP, it is therefore necesssary that the uncommon [to $T_1(s)$] poles of $T_2(s)$ be chosen stable and that $T_2(s)$ have in common with $T_1(s)$ any and all unstable zeros of $T_1(s)$. This observation, which is rather obvious in the scalar case, therefore has an analogous interpretation in the more general multivariable case.

It is finally of interest to note that a sufficient condition for the

[6]If $M_{mp}(s)$ is not row proper, a column permutation matrix W can be found such that the first p columns of $M_m(s)W$ are row proper. An $\alpha(s)$ can then be found so that (4.11) holds with $M_m(s)$ replaced by $M_m(s)W$, which implies that $T(s)=\begin{bmatrix} M_m(s) \\ \alpha(s)W^{-1} \end{bmatrix}^{-1}\begin{bmatrix} M_q(s) \\ 0 \end{bmatrix}$ is an appropriate solution to (3.1).

W. A. Wolovich, P. J. Antsaklis and H. Elliott, "On the Stability of Solutions to Minimal and Nonminimal Design Problems," I E EE T r a ns. A utomatic C ontro l , Vol. AC-22, No.1, pp. 88-94, Feb. 1977.

instability of all solutions to the MDP can now be presented in view of our earlier definitions and results. In particular, let $\bar{d}$ denote the degree of the final column of $\begin{bmatrix} R(s) \\ \hline P(s) \end{bmatrix} \in K(s) = \begin{bmatrix} K_m(s) \\ \hline K_q(s) \end{bmatrix}$, as defined in Theorem 3.2, i.e., $T(s) = R(s)P(s)^{-1}$ represents a proper minimal order solution to (3.1) with $P(s)$ column proper and degree ordered with $\bar{d} = \max \partial_{cj}[P(s)]$. Let $G_{Lq}(s)$ represent any gcld of those jth columns $K_{qj}(s)$ of $K_q(s)$ of degree no greater than $\bar{d}$ for which $K_{q\gamma j} \neq 0$, i.e., all of the columns of $K_q(s)$ which can be used to construct a $P(s)$ such that $\partial[|P(s)|]$ is minimal and $R(s)P(s)^{-1}$ is a proper solution of (3.1). Since *all* of the zeros of $|G_{Lq}(s)| \triangleq \Delta_{Tq}(s)$ will represent some of the poles of *any* minimal order solution of (3.1), they will be called *the fixed poles of the MDP*.[7] In view of this observation, we clearly have Theorem 4.13.

Theorem 4.13: All (minimal) solutions of the MDP are unstable when $\Delta_{Tq}(s)$ is not a Hurwitz polynomial.

It might finally be noted that a Hurwitz $\Delta_{Tq}(s)$ does not necessarily insure a stable solution to the MDP.

To illustrate the results of this section, let us recall the example employed in the previous section, i.e.,

$$T_1(s) = \begin{bmatrix} \dfrac{s}{s^2+3s+2} & 0 & \dfrac{s^2+2s+2}{s^2+3s+2} \\ \dfrac{2s+1}{s+2} & \dfrac{s-1}{s+2} & 0 \end{bmatrix}, \quad T_2(s) = I_2$$

and

$$K_q(s) = \begin{bmatrix} 1 & 0 & s^2+2s+2 \\ 1 & s-1 & 0 \end{bmatrix}.$$

In this example, $\bar{d} = 1$ and $K_{qj}(s) = \begin{bmatrix} 1 & 0 \\ 1 & s-1 \end{bmatrix} = G_{Lq}(s)$. In light of Theorem 4.13, $\Delta T_q(s) = s-1$, with $s=1$ the fixed pole of this MDP, and any minimal (first-)-order solution to the MDP will be unstable. We recall that this observation was first made, but not formally established, at the end of the previous section.

In light of the Theorem 4.2, we can now resolve the question of whether or not *any* stable right inverse exists, regardless of dynamical order. In particular, since

$$P_{1Q}^{-1}(s)Q_1(s) = \begin{bmatrix} s^2+3s+2 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} s & 0 & s^2+2s+2 \\ 2s+1 & s-1 & 0 \end{bmatrix}$$

and

$$P_{2Q}^{-1}(s)Q_2(s) = I_2^{-1}I_2$$

represent rlp factorizations of $T_1(s)$ and $T_2(s)$, respectively, it is clear that $T_1(s)$ and $T_2(s)$ have no poles or zeros in common (actually $T_2(s) = I_2$ has no poles or zeros). Therefore any $G_{LQ}(s)$ will be unimodular and $\tilde{P}_1(s)$ can be equated to $P_{1Q}(s)$. Since $G_{LQ}^{-1}(s)\tilde{P}_2(s)$ is unimodular, the zeros of $\Delta_T(s)$ will correspond to the zeros of the determinant of any gcld $G_L(s)$ of $Q_1(s)$ which represent the zeros of $T_1(s)$ [11]. However, it should be noted that since the columns of $Q_1(s)$ are rlp, any $G_L(s)$ of $Q_1(s)$ will be unimodular, i.e., $T_1(s)$ has no zeros and, consequently, there are no fixed poles of $T(s)$. In view of Theorem 4.2 therefore, it should be possible to find a right inverse of $T_1(s)$ with arbitrarily assignable poles. To show that this is indeed the case, we first determine that

$$M(s) = \left[M_m(s) \,\vdots\, -M_q(s) \right] = \left[Q_1(s) \,\vdots\, -P_{1Q}(s) \right]$$
$$= \left[\begin{array}{ccc|cc} s & 0 & s^2+2s+2 & -s^2-3s-2 & 0 \\ 2s+1 & s-1 & 0 & 0 & -s-2 \end{array} \right].$$

[7]The fixed poles of the MDP will, of course, include any and all fixed poles of $T(s)$.

It therefore follows that $\nu = 2$ and, consequently, that an $\alpha(s)$ of the form $[\alpha_{11}s + \alpha_{10}, \alpha_{21}s + \alpha_{20}, \alpha_{31}s + \alpha_{30}]$ with $2\alpha_{21} \neq \alpha_{11}$ will insure a nonsingular, row proper $\begin{bmatrix} M_m(s) \\ \hline \alpha(s) \end{bmatrix}$ with a fourth degree, arbitrarily assignable determinant $\Delta_D(s)$. In particular, if $\Delta_D(s) = (s+2-j)(s+2+j)(s+3)(s+4)$, then $\alpha(s) = [s+18, s+12, s-7]$ will insure the desired determinant and

$$T(s) = \begin{bmatrix} M_m(s) \\ \hline \alpha(s) \end{bmatrix}^{-1} \begin{bmatrix} M_q(s) \\ \hline 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} (s^2-8s+7)(s^2+3s+2) & (s^3+14s^2+26s+24)(s+2) \\ (-2s^2+13s+7)(s^2+3s+2) & -(s^3+19s^2+45s+36)(s+2) \\ (s^2+8s+30)(s^2+3s+2) & -(s^2+12s)(s+2) \end{bmatrix}}{s^4+11s^3+40s^2+83s+60}$$

will represent a stable right inverse of $T_1(s)$ with (arbitrarily assignable) poles at $s = -2 \pm j$, -3, and -4.

V. Minimal-Order Observers

The results which have been presented thus far will next be employed to investigate the question of obtaining observers of minimal order whose output exponentially approaches some desired linear function of the state of the given system. This section is motivated by some earlier work of Wang and Davison [6] and some results which have only recently appeared [12], [13]. Before presenting the main result of this section some preliminary observations are required. In particular, as noted earlier, if $T(s)$ represents the $p \times m$ proper transfer matrix of a given dynamical system, it can be factored as the product $R(s)P(s)^{-1}$ with $R(s)$ and $P(s)$ rrp, $P(s)$ column proper, and $\partial_{ci}[R(s)] \leq \partial_{ci}[P(s)] \triangleq d_i$ for $i \in \mathbf{m}$. We now note in light of [1] that any such factorization of $T(s)$ implies a corresponding *minimal* (both controllable and observable) *differential operator realization* of $T(s)$ of the form

$$P(D)z(t) = u(t); y(t) = R(D)z(t) \tag{5.1}$$

with $D = d/dt$, $z(t)$ the *partial state*, $u(t)$ the *input*, and $y(t)$ the *output* of the system. As noted in [1], an appropriate (entire) state, $x(t)$ of (5.1) is given by $S(D)z(t)$, with

$$S(D) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ D & 0 & \cdots & 0 \\ & & & \cdot \\ D^{d_1-1} & 0 & & \cdot \\ & & & \cdot \\ 0 & 1 & \cdots & 0 \\ 0 & D & \cdots & 0 \\ 0 & D^{d_2-1} & & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & 1 \\ & & & D \\ 0 & 0 & \cdots & D^{d_m-1} \end{bmatrix},$$

i.e.,

$$x(t) = S(D)z(t) \tag{5.2}$$

is an appropriate state of a minimal state-space realization of $T(s)$. It therefore follows that any q-dimensional linear function of the state of the given system $Fx(t)$ can be expressed in differential operator form as $F(D)z(t)$, where

$$Fx(t) = FS(D)z(t) \triangleq F(D)z(t) \tag{5.3}$$

W. A. Wolovich, P. J. Antsaklis and H. Elliott, "On the Stability of Solutions to Minimal and Nonminimal Design Problems," I E EE T r a ns. A utomatic C ontro l , Vol. AC-22, No.1, pp. 88-94, Feb. 1977.

with $F(D)$ a $q\times m$ polynomial matrix which satisfies the condition, $\partial_c[F(D)]<\partial_c[P(D)]$.

Let us now consider the dynamical system defined by the differential operator representation

$$Q(D)w(t)=K(D)u(t)+H(D)y(t) \tag{5.4}$$

which is "driven by" $u(t)$ and $y(t)$ of the given system. This system or equivalently, $\{Q(D),K(D),H(D)\}$, will be called an *observer* of $F(D)z(t)$ of the differential operator system (5.1) if and only if

1) $|Q(D)|$ is a Hurwitz polynomial
2) $Q(s)^{-1}\left[H(s)\mid K(s)\right]$ is a proper transfer matrix[8]

and

3) $K(D)P(D)+H(D)R(D)=Q(D)F(D)$. (5.5)

As shown in [1], these three conditions are both necessary and sufficient to insure that $w(t)$ will exponentially approach $F(D)z(t)$ with increasing time regardless of any initial condition differences.[9] It is also shown in [1] that an observer of $F(D)z(t)$ of (5.1) can always be found with arbitrarily assignable poles (the zeros of $|Q(s)|$), although the minimality of the dynamical order of such an observer cannot be assured. With these preliminaries in mind, we now state and establish the main result of this section.

Theorem 5.6: The differential operator system (5.4) is an observer of $F(D)z(t)$ of (5.1) if and only if $\left[H(s)\mid Q(s)\mid K(s)\right]$ $\in\ker\begin{bmatrix}R(s)\\ \hline -F(s)\\ \hline P(s)\end{bmatrix}$ with $Q(s)^{-1}\left[H(s)\mid K(s)\right]$ a proper, stable transfer matrix. Furthermore, if $\left[\tilde H(s)\mid\tilde Q(s)\mid\tilde K(s)\right]$ represents a degree ordered minimal basis of $\ker\begin{bmatrix}R(s)\\ \hline -F(s)\\ \hline P(s)\end{bmatrix}$ with $\Gamma_r\left[\tilde H(s)\mid\tilde Q(s)\mid\tilde K(s)\right]\triangleq\left[\tilde H_\gamma\mid\tilde Q_\gamma\mid\tilde K_\gamma\right]$, then the dynamical order of any observer of $F(D)z(t)$ can be no less than the sum of the degrees of the first q rows (ordered from top to bottom) of $\left[\tilde H(s)\mid\tilde Q(s)\mid\tilde K(s)\right]$ for which the corresponding rows of $\tilde Q_\gamma$ are linearly independent. These q rows, $\left[\tilde H_q(s)\mid\tilde Q_q(s)\mid\tilde K_q(s)\right]$ of $\left[\tilde H(s)\mid\tilde Q(s)\mid\tilde K(s)\right]$ will represent an observer, $\{\tilde Q_q(D),\tilde K_q(D),\tilde H_q(D)\}$ of $F(D)z(t)$ provided $|\tilde Q_q(s)|$ is a Hurwitz polynomial.

Proof: The first statement of this theorem is a direct consequence of the definition of an observer of $F(D)z(t)$ as well as the observation that (5.5) holds if and only if $\left[H(s)\mid Q(s)\mid K(s)\right]\in\ker\begin{bmatrix}R(s)\\ \hline -F(s)\\ \hline P(s)\end{bmatrix}$. To establish the dynamic order bound associated with the observer, we now note that a minimal, degree ordered basis for $\ker\begin{bmatrix}R(s)\\ \hline -F(s)\\ \hline P(s)\end{bmatrix}$ is given by any dual rlp row proper and degree ordered factorization $\left[\tilde H(s)\mid\tilde Q(s)\right]^{-1}\tilde K(s)$ of the proper transfer matrix $\begin{bmatrix}-R(s)\\ \hline F(s)\end{bmatrix}$ $P(s)^{-1}$, i.e.,

$$\left[\tilde H(s)\mid\tilde Q(s)\right]^{-1}\tilde K(s)=\begin{bmatrix}-R(s)\\ \hline F(s)\end{bmatrix}P(s)^{-1} \tag{5.7}$$

with $\partial_r\left[\tilde H(s)\mid\tilde Q(s)\right]\geqslant\partial_r[\tilde K(s)]$, and $\left[\tilde H(s)\mid\tilde Q(s)\right]$ both row proper and degree ordered. It now follows that $\left[\tilde H_\gamma\mid\tilde Q_\gamma\right]$ will be nonsingular and, therefore, that $\rho(\tilde Q_\gamma]=q$. In light of Theorem 3.2 therefore, the minimal order condition is established. Finally, if $\left[\tilde H_q(s)\mid\tilde Q_q(s)\mid\tilde K_q(s)\right]$ represents the first q rows of $\left[\tilde H(s)\mid\tilde Q(s)\mid\tilde K(s)\right]$ for which the corresponding rows of $\tilde Q_\gamma$ are linearly independent, then $\tilde Q_q(s)$ is row proper and $\partial_r[\tilde Q_q(s)]\geqslant\partial_r\left[\tilde H_q(s)\mid\tilde K_q(s)\right]$, which implies [1] that $\tilde Q_q^{-1}(s)\left[\tilde H_q(s)\mid\tilde K_q(s)\right]$ is a proper transfer matrix. It thus follows that if $|\tilde Q_q(s)|$ is a Hurwitz polynomial, then $\tilde Q_q(D)w(t)=\tilde K_q(D)u(t)+\tilde H_q(D)y(t)$ is an observer of $F(D)z(t)$ of the given differential operator system (5.1).

It should be noted at this point that a $Q(s)$ of minimal row degree can always be found which satisfies all of the observer conditions except 1). In other words, as in the case of the MDP, there is no guarantee that a minimal-order solution to (5.5) will also be stable, an observation analogous to that made in [12]. It might be noted that an alternative procedure is outlined in [12] for determining the minimal order of an observer, and that ρ_M of that paper corresponds to the degree of $|\tilde Q_q(s)|$ as defined here in Theorem 5.6.

We further remark that a notion of "fixed poles of any minimal-order observer of $F(D)z(t)$" can be defined in a manner analogous to that used to define the fixed poles of the MDP in Section IV. In particular, let $\tilde d$ denote the degree of the qth row of $\tilde Q_q(s)$ and $0_m=\partial[|\tilde Q_q(s)|]$. The zeros of the determinant of any gcrd, $G_R(s)$, of those ith rows $Q_i(s)$ of $\tilde Q(s)$ of degree no greater than $\tilde d$ for which $\tilde Q_{\gamma i}\neq 0$, i.e., those ith rows of $\tilde Q(s)$ which can be used to construct an "observer" of $F(D)z(t)$ of lowest possible order 0_m will be called *the fixed poles of any* 0_m*th-order "observer" of F(D)z(t).* The reader is cautioned that there is no guarantee that an observer of order 0_m actually exists, even when $|G_R(s)|$ is a Hurwitz polynomial; hence, the quotation marks around the term observer. It is clear, however, that if $|G_R(s)|$ is not a Hurwitz polynomial, then an observer of $F(D)z(t)$ of order 0_m does not exist. We finally note that an observer of $F(D)z(t)$ can always be constructed with completely arbitrary poles [1]. However, the minimality of such observers has generally been established only in the rather restrictive single-input and/or single-output cases.

It should be noted that the results given in this section can also be employed if one begins with a controllable and observable state-space representation of the form: $\dot x(t)=Ax(t)+Bu(t)$; $y(t)=Cx(t)$, by first factoring the system transfer matrix $T(s)=C(sI-A)^{-1}B$ as the relatively right prime product $R(s)P^{-1}(s)$ with $P(s)$ column proper. We will now illustrate this point by employing an example used in [12].

In particular, suppose we are given a state-space system with

$$A=\begin{bmatrix}1&0&0&-1&1\\-2&0&0&2&-1\\0&1&-1&0&-1\\1&0&0&0&1\\0&0&1&1&0\end{bmatrix},\quad B=\begin{bmatrix}1&-1\\-1&2\\-1&0\\1&1\\1&1\end{bmatrix}$$

$$C=\begin{bmatrix}0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

and we wish to construct a minimal-order observer of $Fx(t)$, with $F=\frac{1}{4}\begin{bmatrix}-6&-1&-1&-2&-2\\2&-3&1&-2&-2\end{bmatrix}$. By employing our computer program [3], we readily determine that an appropriate factorization $R(s)P^{-1}(s)$ of $C(sI-A)^{-1}B$ is given by the pair

$$R(s)=\begin{bmatrix}s^2&-s-2\\s^2+s+1&-s\end{bmatrix},\quad P(s)=\begin{bmatrix}-1&-s^2\\s^3&1\end{bmatrix}$$

and that the corresponding $F(s)$ [1] is

$$F(s)=\begin{bmatrix}-1&2s+2\\-3s^2-3s-1&0\end{bmatrix}.$$

[8] The Laplace operator s and the differential operator D can be interchanged freely due to the assumption of zero initial conditions.

[9] In this paper we might note that if $P(D)\,z(t)$ is substituted for $u(t)$ and $R(D)\,z(t)$ for $y(t)$ in (5.4), then in view of (5.5), $Q(D)\,[w(t)-F(D)\,z(t)]=0$, which implies the noted observation.

W. A. Wolovich, P. J. Antsaklis and H. Elliott, "On the Stability of Solutions to Minimal and Nonminimal Design Problems," I E EE T r a ns. A utomatic C ontro l , Vol. AC-22, No.1, pp. 88-94, Feb. 1977.

We next determine [3] that

$$\left[\begin{array}{cc|cc|cc} 4s+7.8 & 2.7s-7.2 & 4s-8.3 & -1.9s+0.8 & -14.7 & -1 \\ 4s & 2.7s+0.6 & -3.8s-0.5 & -1.9s+0.8 & 0.9 & -1 \\ -5s & -3s & 4s+1 & 2s-1 & 0 & 2 \\ -5.2s & -1.8s^2+4.8s-2 & -5.5s^2+5.7s+0.5 & 0.6s^2-0.2s+1.5 & 12.85 & 1 \end{array}\right] = \left[\, \tilde{H}(s) \mid \tilde{Q}(s) \mid \tilde{K}(s) \,\right]$$

represents a degree ordered, minimal basis of ker $\begin{bmatrix} R(s) \\ \hdashline -F(s) \\ \hdashline P(s) \end{bmatrix}$, with

$$\Gamma_r\left[\, \tilde{H}(s) \mid \tilde{Q}(s) \mid \tilde{K}(s) \,\right] = \left[\, \tilde{H}_\gamma \mid \tilde{Q}_\gamma \mid \tilde{K}_\gamma \,\right]$$

$$= \left[\begin{array}{cc|cc|cc} 4 & 2.7 & 4 & -1.9 & 0 & 0 \\ 4 & 2.7 & -3.8 & -1.9 & 0 & 0 \\ -5 & -3 & 4 & 2 & 0 & 0 \\ 0 & -1.8 & -5.5 & 0.6 & 0 & 0 \end{array}\right].$$

Since $q=2$ in this example, and the first two rows of $\tilde{Q}_\gamma$ are linearly independent, $\tilde{d}=1$ while $0_m=2$. We next observe that the $i=1$, 2, and 3 rows of $\tilde{Q}(s)$ are of degree no greater than $\tilde{d}=1$ with $\tilde{Q}_{\gamma i} \neq 0$ and, consequently, can be used to construct an observer of $Fx(t)$ of minimal order 2 (provided one exists). Since $G_R(s)=I_2$ is a gcrd of the first three rows of $\tilde{Q}(s)$ [3], it follows that there are no fixed poles associated with any minimal- (second-) order observer of $Fx(t)$. In fact, for this relatively simple example, it can be shown that both poles of a minimal-order observer can be arbitrarily assigned.

In particular, if we premultiply $\left[\, \tilde{H}(s) \mid \tilde{Q}(s) \mid \tilde{K}(s) \,\right]$ by the (2×4) scalar matrix $\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \end{bmatrix}$, with a and b arbitrary, the resulting (2×6) matrix, $\left[\, H(s) \mid Q(s) \mid K(s) \,\right] \in$ ker $\begin{bmatrix} R(s) \\ \hdashline -F(s) \\ \hdashline P(s) \end{bmatrix}$, will be such that $|Q(s)| = (-14.8+15.6b)\ s^2 + (21.1-7.9a-21.9b)s + (-6.2+0.3a+7.5b)$; i.e., by appropriately choosing a and b, we can obtain any pair of observer poles.[10] To illustrate, if $a=-0.2$ and $b=0.985$, then $|Q(s)| = 0.54s^2+1.08s+1.08$ and

$$\{Q(D), K(D), H(D)\} = \left\{ \begin{bmatrix} 5D+7.8, & 3.3D-7.2 \\ -0.9D & -0.25D+0.6 \end{bmatrix}, \begin{bmatrix} 3.2D-8.5, & -2.3D+1 \\ 0.14D+0.5, & 0.1D-0.2 \end{bmatrix}, \begin{bmatrix} -14.7 & -1.4 \\ 0.9 & 1 \end{bmatrix} \right\}$$

would represent a minimal-(second-) order observer of $Fx(t)$ in differential operator form. An equivalent state space observer could now be easily obtained using the results given in [1].

We finally observe that if the original dynamical equations of a system are given in the differential operator form (5.1), then the results of this section can be employed to design a minimal-order observer without employing any form of state space representation.

VI. Concluding Remarks

The question of obtaining stable solutions to the model matching question was resolved through the employment of prime transfer matrix factorizations and the new and intuitively natural notions of the common zeros and poles of dynamical systems as well as the fixed poles of various minimal and nonminimal solutions. More specifically, it was shown that the fixed poles of any solution to the model matching problem must correspond to certain zeros of the given system which are not common to the model system as well as certain poles of the model system which are not common to the given system. This observation extends, to the multivariable case, a rather obvious result which characterizes the scalar case. The notion of the fixed poles of any minimal-order solution was also defined for the first time and employed to obtain a sufficient condition for the instability of all solutions to the MDP.

Finally, it was shown that the question of designing minimal-order state function observers is analogous to the minimal design problem, and a bound on the minimal order of an observer was given in terms of differential operator system representations. The question of stability of minimal-order "observers" was also investigated, and a new notion of the "fixed poles of a minimal-order observer" was introduced and discussed.

[10] This observation is also made in [12] using an alternative procedure.

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Closed-Loop Structural Stability for Linear-Quadratic Optimal Systems

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Abstract—**This paper contains an explicit parametrization of a subclass of linear constant gain feedback maps that will not destabilize an originally open-loop stable system. These results can then be used to obtain several new structural stability results for multiinput linear-quadratic feedback optimal designs.**

I. Introduction and Motivation

This paper presents preliminary results which, in our opinion, represent a first necessary step in the systematic computer aided design of reliable control systems for multivariable control systems. A specific motivating example arises in the context of future high performance aircraft. It is widely recognized that advances in active control aircraft and control configured vehicles will require the automatic control of several actuators so as to be able to fly future aircraft characterized by reduced stability margins and additional flexure modes.

As a starting point for our motivation we must postulate that the design of future stability augmentation systems will have to be a multivariable design problem. As such, traditional single input-single output system design tools based on classical control theory cannot be effectively used, especially in a computer aided design context. Since modern

Manuscript received January 28, 1976; revised August 31, 1976. Paper recommended by J. B. Pearson, Chairman of the IEEE S-CS Linear Systems Committee. This work was supported in part by the NASA Ames Research Center under Grant NGL-22-009-124 and by the Air Force Office of Scientific Research under Grant AF-AFOSR-72-2273.

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