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On the Stabilization of a Class of Nonholonomic Systems Using Invariant Manifold Technique

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Abstract

This paper presents an asymptotically stabilizing discontinuous feedback controller for a class of nonholonomic systems. The controller consists of two parts: the first part yields an invariant manifold on which all trajectories of the closed-loop system tend to the origin, and the latter part renders the invariant manifold attractive, while avoiding a discontinuity surface. The controller yields exponential stability so that the convergence can be chosen arbitrarily fast.

1. Introduction

In the past few years, there has been considerable attention paid to the problem of stabilizing nonholonomic systems. It is well-known that nonholonomic systems constitute a remarkable class of controllable nonlinear systems which fail Brockett's necessary condition for the existence of asymptotically stabilizing time-invariant continuous state feedback. As a consequence, research on feedback stabilization of nonholonomic systems has been directed toward the design of time-varying smooth feedback control laws [4], time-varying nonsmooth feedback control laws [3], and time-invariant discontinuous feedback control laws [1],[5].

This paper uses the invariant manifold technique to derive asymptotically stabilizing discontinuous feedback controllers for a class of nonholonomic systems. This technique was successfully employed by Tsiotras et al [5] for the asymptotic stabilization of a symmetric spacecraft.

2. Feedback Stabilization: Kinematics

This section considers the stabilization problem for the class of nonholonomic systems in third-order power form given by

$$\dot{x}_1 = v_1 \ , \ \dot{x}_2 = v_2 \ , \ \dot{x}_3 = x_2 v_1 \ , \tag{1}$$

where $x = (x_1, x_2, x_3) \in \mathbf{X}$ denotes the state and $v = (v_1, v_2) \in \mathbf{V}$ denotes the input, $\mathbf{X}$ and $\mathbf{V}$ are open subsets of R^3 and R^2, respectively, both containing the origin. Note that any three-dimensional nonholonomic kinematic system with nonholonomy degree two can be (locally) converted to the form (1) via a coordinate change followed by a feedback transformation. Examples of such systems arise in the kinematic control formulation of a number of nonholonomic systems including the knife edge, a hopping robot, a synchro-drive mobile robot, a rigid spacecraft with two torque actuators, and a free-floating three-link system (see e.g. [1],[3]).

Consider the problem of constructing a time-invariant state feedback control law $v = v(x)$ which asymptotically stabilizes the system (1) to the origin. Clearly, the control law $v = (-k_1 x_1, -k_1 x_2)$ where k_1 is a positive constant, renders the origin of the (x_1, x_2) subsystem globally exponentially stable. Note that, with this control law, the closed-loop vector field is given by

$$f = -k_1 x_1 \frac{\partial}{\partial x_1} - k_1 x_2 \frac{\partial}{\partial x_2} - k_1 x_1 x_2 \frac{\partial}{\partial x_3} \ . \tag{2}$$

Let s denote a smooth function $s : \mathbf{X} \to R$, with nonzero gradient on X. The set $S = \{x \in \mathbf{X} \mid s(x) = 0\}$ is said to be an invariant manifold of the system defined by the vector field f if $L_f s(x) = 0, \ \ x \in \mathbf{X}$. Here $L_f s$ denotes the Lie derivative of the scalar function (surface coordinate function) s with respect to vector field f.

Consider the surface coordinate function

$$s(x) = x_1 x_2 - 2x_3 \ , \tag{3}$$

which defines an invariant manifold for the system defined by the vector field (2). The time derivative of s along trajectories of (2) can be computed to be $\dot{s}(x) = x_1 v_2 - x_2 v_1$ and as expected the above control law maintains $\dot{s} \equiv 0$ and, once on S, the trajectories remain there. Moreover, since the (x_1, x_2) subsystem has the property that $(x_1(t), x_2(t)) \to 0$ as $t \to \infty$, for any trajectory on S, $x_3(t) \to 0$ as $t \to \infty$, and hence $(x_1(t), x_2(t), x_3(t)) \to 0$ as $t \to \infty$. Note also that S is independent of the control gain k_1. Subsequent development utilizes this manifold for the construction of a stabilizing feedback control law. In order to render S an attractive manifold, the feedback control law must be modified to guarantee that the reaching condition $s\dot{s} < 0$ is satisfied.

Restricting consideration to $x_1^2 + x_2^2 \neq 0$, we can propose the following control law

$$v = \begin{pmatrix} -k_1 x_1 - \frac{x_2 F(s)}{x_1^2 + x_2^2} & -k_1 x_2 + \frac{x_1 F(s)}{x_1^2 + x_2^2} \end{pmatrix} \ , \tag{4}$$

where $s \mapsto F(s)$ is scalar function satisfying $sF(s) < 0$, and hence guaranteeing that the reaching condition $s\dot{s} < 0$ is satisfied. Choosing $F(s) = -k_2 s$, where k_2 is a positive constant, yields the closed-loop system

$$\dot{x}_1 = -k_1 x_1 + \frac{k_2 x_2 s}{x_1^2 + x_2^2} \ , \tag{5}$$

$$\dot{x}_2 = -k_1 x_2 - \frac{k_2 x_1 s}{x_1^2 + x_2^2} \ , \tag{6}$$

$$\dot{s} = -k_2 s \ , \tag{7}$$

where the surface coordinate function s is as given above. Note that the change of variables from (x_1, x_2, x_3) to

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(x_1, x_2, s) is a diffeomorphism.

The following result characterizes the properties of the closed-loop system (5)-(7).

Theorem 1: *Consider the closed-loop system (5)-(7) with $k_2 > 2k_1 > 0$ and let (x_1^0, x_2^0, s^0) denote an initial condition with $(x_1^0)^2 + (x_2^0)^2 \neq 0$. Then the following hold.*

(i) The trajectory $(x_1(t), x_2(t), s(t))$ is bounded for all $t \geq 0$ and converges exponentially to zero with a decay rate of at least k_1.

(ii) The control history $v(t) = (v_1(t), v_2(t))$ is bounded for all $t \geq 0$ and converges exponentially to zero with a decay rate of at least k_1.

The above result demonstrates that for initial conditions satisfying $(x_1^0)^2 + (x_2^0)^2 \neq 0$, the feedback control law (4) is well-defined for all $t \geq 0$. Moreover, it drives the system (1) to the origin, while avoiding the manifold

$$N = \{x \in \mathbf{X} \mid x_1^2 + x_2^2 = 0 \ , \ x_1 x_2 - 2x_3 \neq 0\} \ .$$

Clearly, one can use a finite time feedback control law to move the system away from N. For example,

$$v = \begin{pmatrix} -|x_1 - \epsilon|^\alpha sign(x_1 - \epsilon) & -|x_2 - \epsilon|^\alpha sign(x_2 - \epsilon) \end{pmatrix} \ ,$$

where $\alpha \in [0, 1)$ and $\epsilon \neq 0$ are constants, can be used to move the system away from N in finite time [2].

3. Feedback Stabilization: Dynamic Extension

In this Section, the stabilization problem for the dynamic extension of the system (1) is considered. Dynamic extension results in a nonholonomic system with drift, which significantly complicates the control law design.

Adding an integrator to each input channel of the system (1) yields the dynamic extension given by

$$\dot{x}_1 = v_1 \ , \ \dot{x}_2 = v_2 \ , \ \dot{x}_3 = x_2 v_1 \ , \tag{8}$$

$$\dot{v}_1 = u_1 \ , \ \dot{v}_2 = u_2 \ , \tag{9}$$

where $u = (u_1, u_2) \in \mathbf{U} \subset R^2$ is considered as the control input.

Introduce the variables

$$s_1 = x_1 x_2 - 2x_3 \ \ , \ \ s_2 = (x_1^2 + x_2^2)/2 \ . \tag{10}$$

The first and second time derivatives of $s_1(x)$ and $s_2(x)$ along the trajectories of the system (8)-(9) are given by

$$\dot{s}_1 = x_1 v_2 - x_2 v_1 \ \ , \ \ \ddot{s}_1 = x_1 u_2 - x_2 u_1 \ , \tag{11}$$

$$\dot{s}_2 = x_1 v_1 + x_2 v_2 \ \ , \ \ \ddot{s}_2 = x_1 u_1 + x_2 u_2 + v_1^2 + v_2^2 \ . \tag{12}$$

Let $\lambda_1, \lambda_2, \lambda_3$, and λ_4 denote positive scalars and consider the following second order linear dynamics for s_1 and s_2:

$$\ddot{s}_1 + (\lambda_1 + \lambda_2)\dot{s}_1 + \lambda_1 \lambda_2 s_1 = 0 \ , \tag{13}$$

$$\ddot{s}_2 + (\lambda_3 + \lambda_4)\dot{s}_2 + \lambda_3 \lambda_4 s_2 = 0 \ . \tag{14}$$

Assume that $s_2 \neq 0$. Then, using (11)-(12) in (13)-(14) and solving the resulting equations for u_1, u_2 yield the feedback control law

$$u = \Big(\frac{1}{s_2}[x_2(a_{10}s_1 + a_{11}\dot{s}_1) \ - \ x_1(v_1^2 + v_2^2 + a_{20}s_2 + a_{21}\dot{s}_2)] \ ,$$

$$-\frac{1}{s_2}[x_1(a_{10}s_1 + a_{11}\dot{s}_1) \ + \ x_2(v_1^2 + v_2^2 + a_{20}s_2 + a_{21}\dot{s}_2)] \ , \tag{15}$$

where $a_{10} = \lambda_1 \lambda_2$, $a_{11} = \lambda_1 + \lambda_2$, $a_{20} = \lambda_3 \lambda_4$, $a_{21} = \lambda_3 + \lambda_4$.

The following result can be stated.

Theorem 2: *Consider the system (8)-(9) with the feedback control law (15). Let $\lambda_1 > \lambda_2 > 2\lambda_3 > 2\lambda_4 > 0$ and let $(x_1^0, x_2^0, s_1^0, v_1^0, v_2^0)$ denote an initial condition with $(x_1^0)^2 + (x_2^0)^2 \neq 0$, $v_1^0 = v_2^0 = 0$. Then the following hold.*

(i) The trajectory $(x_1(t), x_2(t), s_1(t), v_1(t), v_2(t))$ is bounded for all $t \geq 0$ and converges exponentially to zero with a decay rate of at least λ_4.

(ii) The control history $u(t) = (u_1(t), u_2(t))$ is bounded for all $t \geq 0$ and converges exponentially to zero with a decay rate of at least λ_4.

The above result demonstrates that for initial conditions satisfying $(x_1^0)^2 + (x_2^0)^2 \neq 0$, $v_1^0 = v_2^0 = 0$, the feedback control law (15) is well-defined for all $t \geq 0$. Moreover, it drives the system (8)-(9) to the origin, while avoiding the manifold

$$N' = \{(x, v) \in \mathbf{X} \times \mathbf{V} \mid x_1^2 + x_2^2 = 0 \ , \ x_1 x_2 - 2x_3 \neq 0\} \ .$$

Clearly, one can use a finite time feedback control law to transfer the system to a state statisfying the conditions of Theorem 2. For example,

$$u_1 = -|x_1 - \epsilon_1|^{a_1} sign(x_1 - \epsilon_1) - |v_1|^{b_1} sign(v_1) \ ,$$

$$u_2 = -|x_2 - \epsilon_2|^{a_2} sign(x_2 - \epsilon_2) - |v_2|^{b_2} sign(v_2) \ ,$$

where $b_i \in (0, 1)$, $a_i > b_i/(2 - b_i)$ and $\epsilon_i \neq 0$, $i = 1, 2$ are constants, can be used to transfer the system to a state statisfying the conditions of Theorem 2 in finite time [2].

4. Conclusions

Discontinuous static feedback control laws have been derived for the stabilization of the third-order power form nonholonomic kinematic systems and their dynamic extension. Future research includes extension of the results obtained in this paper to higher-order power form nonholonomic kinematic systems and their dynamic extensions.

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