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ON THE STABILIZATION OF A FLEXIBLE BEAM WITH A TIP MASS*

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Abstract. We study the stability of a flexible beam that is clamped at one end and free at the other; a mass is also attached to the free end of the beam. To stabilize this system we apply a boundary control force at the free end of the beam. We prove that the closed-loop system is well-posed and is exponentially stable. We then analyze the spectrum of the system for a special case and prove that the spectrum determines the exponential decay rate for the considered case.

Key words. flexible structures, infinite dimensional systems, boundary control, stability, semigroup theory

AMS subject classifications. 93C20, 93D15, 35B35, 35P10

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1. Introduction.

In this paper, we study the stability of a flexible beam that is clamped at one end and is free at the other end; a mass is also attached to the free end. The equations of motion for this system are given by

$$u_{tt} + u_{xxxx} = 0, \qquad 0 < x < 1, \qquad t \geq 0, \tag{1.1}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = 0, \qquad t \geq 0, \tag{1.2}$$

$$-u_{xxx}(1,t) + mu_{tt}(1,t) = w(t), \qquad t \geq 0, \tag{1.3}$$

where $m > 0$ is the tip mass and $w(t)$ is the boundary control force applied at the free end of the beam; a subscript letter denotes the partial derivation with respect to that variable. For simplicity, and without loss of generality, the length of the beam, the mass per unit length, and the flexural rigidity of the beam are chosen to be unity. Our problem is to find a feedback control law for $w(t)$ so that the solutions of the resulting closed-loop system decay uniformly to zero. This can be achieved with a highly unbounded feedback law; see (2.1).

The model given by (1.1)–(1.3) is a variant of the SCOLE model in the sense that one has neglected the moment of inertia at $x = 1$, which has been studied in the past; see, e.g., [1], [9], [14], [15]. It is known that for such types of models the feedback law

$$w(t) = -\alpha u_t(1,t), \qquad \alpha > 0, \qquad t \geq 0, \tag{1.4}$$

is sufficient for strong (i.e., *asymptotic*) stability but not sufficient for uniform (i.e., *exponential*) stability; see [9], where arbitrarily slow decay is proven by using asymptotic estimates of the eigenvalues. In fact, as shown in [14], the control law given by

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(1.4) may be considered as a compact perturbation of the uncontrolled system. It is well known that such compact perturbations are not sufficient to provide uniform stabilization; see [6], [17], [20]. Hence, to obtain uniform stability one has to choose "stronger" feedback terms, such as u_{xxxt} (see [13], [14]), where the lack of uniform stability for the SCOLE model with usual feedback laws (e.g., velocity feedback; see (1.4)) was proven by using the compactness argument, and uniform decay of the energy was obtained by means of higher-order feedback for rather smooth initial data. Also in [15], decay estimates for a flexible cable with a tip mass were given. Let us mention that these papers study the asymptotic or uniform decay for hybrid systems by using energy multipliers; thus the decay is qualitative, and one cannot conclude on the optimality of the decay rate. In [3] a flexible beam with rate control on the bending moment was considered, the uniform decay was proven by using the estimates of the resolvent operator on the imaginary axis, and a careful analysis of the eigenvalues and eigenfunctions was given (similar to the one given in [12] but for a harder problem). In [1] a three-dimensional model for the SCOLE system, including the moment of inertia at $x = 1$, is considered, and then a feedback law similar to (1.4) and another feedback law based on optimal control techniques are studied. As stated above, these results also show the asymptotical or uniform decay of energy for the system considered, but do not give the optimality of the decay rate.

In this paper we investigate the uniform stability of the system given by (1.1)–(1.3). The paper is organized as follows. In the next section we prove the well-posedness and the uniform stability of (1.1)–(1.3) with a proper choice for $w(t)$ for a norm weaker than the one used in [14] by introducing a specific change of variables. Then we study the spectrum of the system for a particular case and prove that for the considered case the spectrum determines the exponential decay rate for almost all $\alpha > 0$. We also show that in case $m = 0$ in (1.1)–(1.4) (i.e., the case of the cantilevered beam with a boundary force control), for almost all $\alpha > 0$, the spectrum determines the exponential growth rate (see Appendix). Finally we give some concluding remarks.

2. Stability results. For the system given by (1.1)–(1.3) we propose the following linear feedback control law for $w(t)$:

$$w(t) = -\alpha u_t(1,t) + \beta u_{xxxt}(1,t), \qquad t \geq 0, \tag{2.1}$$

where α and β are positive constants.

We define the auxiliary function η as

$$\eta(t) = -u_{xxx}(1,t) + \frac{m}{\beta} u_t(1,t), \qquad t \geq 0. \tag{2.2}$$

Upon substituting (2.1) and (2.2) into (1.3), the latter becomes

$$\beta\dot{\eta}(t) + \eta(t) + \left(\alpha - \frac{m}{\beta}\right) u_t(1,t) = 0, \qquad t \geq 0, \tag{2.3}$$

where a dot represents the time derivative. We note that a similar control law has been applied to the stabilization of a cable with a tip mass, see [10].

Let us introduce the following spaces:

$$\mathcal{V} = \{u : [0,1] \to \mathbf{R} | u \in H^2(0,1), \quad u(0) = u_x(0) = 0\}, \tag{2.4}$$

$$\mathcal{H} = \{(u\ v\ \eta)^T | u \in \mathcal{V},\ v \in L^2(0,1),\ \eta \in \mathbf{R}\}, \tag{2.5}$$

where the superscript T stands for the transpose; the spaces $L^2(0,1)$ and $H^k(0,1)$ are defined as

$$L^2(0,1) = \left\{ y : [0,1] \to \mathbf{R} | \int_0^1 y^2 dx < \infty \right\}, \tag{2.6}$$

$$H^k(0,1) = \{y : [0,1] \to \mathbf{R} | y,\ y^{(1)}, \ldots, y^{(k)} \in L^2(0,1)\}. \tag{2.7}$$

In $\mathcal{H}$ we define the following inner-product:

$$< y, \tilde{y} >_{\mathcal{H}} = \int_0^1 (u_{xx}\tilde{u}_{xx} + v\tilde{v})dx + \ K\eta\tilde{\eta}, \tag{2.8}$$

where $y = (u\ v\ \eta)^T \in \mathcal{H}$, $\tilde{y} = (\tilde{u}\ \tilde{v}\ \tilde{\eta})^T \in \mathcal{H}$, $K > 0$ is chosen as

$$K = \frac{\beta^2}{m + \alpha\beta}. \tag{2.9}$$

The reason for this choice will become clear later. Next we define the unbounded operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ as follows:

$$A \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ -u_{xxxx} \\ -\frac{1}{\beta}\eta - \frac{1}{\beta}(\alpha - \frac{m}{\beta})v(1) \end{pmatrix}, \tag{2.10}$$

where the domain $D(A)$ of the operator A is defined as

$$D(A) = \Big\{ (u\ v\ \eta\ )^T | u \in H^4(0,1) \cap \mathcal{V}, v \in \mathcal{V},\ \eta \in \mathbf{R}, \\ u_{xx}(1) = 0, \quad \eta = -u_{xxx}(1) + \frac{m}{\beta}v(1) \Big\}. \tag{2.11}$$

With the previous notation, (1.1)–(1.2) and (2.3) can be written formally as

$$\dot{y} = Ay, \qquad y(0) \in \mathcal{H}, \tag{2.12}$$

where $y = (u\ v\ \eta)^T$, η is defined by (2.2), and $v = u_t$.

THEOREM 2.1. *The operator A, defined by* (2.10) *and* (2.11), *generates a C_0 semigroup of contractions on $\mathcal{H}$.* (*For the terminology on the semigroup theory, the reader is referred to* [11].)

Proof. We apply the Lumer–Phillips theorem; see, e.g., [11, p. 14]. First, for any $y = (u\ v\ \eta)^T \in D(A)$,

$$\begin{aligned} \langle Ay,\ y \rangle_{\mathcal{H}} &= \int_0^1 (u_{xx}v_{xx} - vu_{xxxx})dx - \frac{K}{\beta}\eta \left(\eta + \left(\alpha - \frac{m}{\beta} \right) v(1) \right) \\ &= -\frac{K}{\beta}u_{xxx}^2(1) - \frac{Km\alpha}{\beta^2}v^2(1), \end{aligned} \tag{2.13}$$

where to derive the last equation we integrated by parts twice and used (1.2), (2.2), and (2.9). Note that due to the particular choice of K given by (2.9), the term multiplying $v(1)u_{xxx}(1)$ in (2.13) vanishes. It follows from (2.13) that the operator A is dissipative.

Next we show that the range of the operator $\lambda I - A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is onto for $\lambda > 0$; that is, for any given $z = (f\ g\ h)^T \in \mathcal{H}$, we have to find $y = (u\ v\ \eta)^T \in D(A)$ so that

$$(\lambda I - A)y = z, \tag{2.14}$$

which is equivalent to the following set of equations:

$$\lambda u - v = f, \tag{2.15}$$

$$\lambda v + u_{xxxx} = g, \tag{2.16}$$

$$\left(\lambda + \frac{1}{\beta}\right)\eta + \frac{1}{\beta}\left(\alpha - \frac{m}{\beta}\right)v(1) = h. \tag{2.17}$$

Upon substituting (2.15) into (2.16), the latter becomes

$$\lambda^2 u + u_{xxxx} = \lambda f + g. \tag{2.18}$$

By using (2.15) and (2.2) in (2.17), the latter becomes

$$-\left(\lambda + \frac{1}{\beta}\right)u_{xxx}(1) + \frac{\lambda(\alpha + m\lambda)}{\beta}u(1) = h + \frac{m\lambda + \alpha}{\beta}f(1)\ . \tag{2.19}$$

Therefore to prove that $\lambda I - A$ is onto, we have to prove the existence of a solution for the following set of equations:

$$\lambda^2 u + u_{xxxx} = f^*, \tag{2.20}$$

$$u(0) = u_x(0) = u_{xx}(1) = 0, \tag{2.21}$$

$$-u_{xxx}(1) + cu(1) = h^*, \tag{2.22}$$

where f^*, h^*, and c are given by

$$f^* = \lambda f + g \in L^2(0,1), \qquad h^* = \frac{\beta}{\lambda\beta + 1}h + \frac{m\lambda + \alpha}{\lambda\beta + 1}f(1) \in \mathbf{R}, \tag{2.23}$$
$$c = \frac{\lambda(m\lambda + \alpha)}{\lambda\beta + 1} > 0.$$

The existence, as well as the uniqueness and continuous dependence, of a solution of (2.20)–(2.23) with respect to (f^*, h^*) can be considered as standard. One way to prove it is to use the weak formulation of (2.20)–(2.23), which is

$$\int_0^1 u_{xx}\varphi_{xx}dx + \lambda^2\int_0^1 u\varphi dx + cu(1)\varphi(1) \tag{2.24}$$
$$= \int_0^1 f^*\varphi dx + h^*\varphi(1), \qquad u \in \mathcal{V}, \forall\varphi \in \mathcal{V}.$$

Since $c > 0$, the left-hand side of (2.24) is a coercive bilinear form of φ and u. Then the existence and uniqueness of a $u \in \mathcal{V}$ satisfying (2.24) follow from the well-known

Lax–Milgram theorem; see e.g. [19, p. 26]. By standard regularity $u \in H^4(0,1)$ and by using particular φ, one recovers the boundary conditions in u. Then v given by (2.15) and η given by (2.17) are unique and $(u\ v\ \eta)^T \in D(A)$. This shows that the operator $\lambda I - A$ is onto for $\lambda > 0$, and the proof of the theorem now follows from the Lumer–Phillips theorem. ∎

Remark 1. It follows from Theorum 2.1 that for $(u_0\ v_0\ \eta_0)^T \in D(A)$, the problem (2.12) has a strong solution $(u(t)\ v(t)\ \eta(t))^T \in C^1(\mathbf{R}_+, \mathcal{V} \times L^2(0,1) \times \mathbf{R}) \cap C^0(\mathbf{R}_+, D(A))$. Thus $\eta(t) = -u_{xxx}(1,t) + \frac{m}{\beta} u_t(1,t)$ is differentiable, but $u_{xxx}(1,t)$ and $u_t(1,t)$ are not guaranteed to be separably differentiable. This will be the case if $(u_0\ v_0\ \eta_0)^T \in D(A^2)$.

Next we prove that the semigroup generated by the operator A decays exponentially to zero.

THEOREM 2.2. *Let $T(t)$ be the C_0 semigroup of contractions generated by the operator A on $\mathcal{H}$. Then there exist positive constants M and δ such that the following holds:*

$$\|T(t)\|_{\mathcal{L}(\mathcal{H})} \leq M e^{-\delta t}, \qquad t \geq 0, \tag{2.25}$$

where the norm used is the norm induced by the inner-product given by (2.8).

Proof. We first define the following function:

$$V(t) = tE(t) + \int_0^1 x\ u_t(x,t)\ u_x(x,t) dx, \tag{2.26}$$

where the "energy" $E(t)$ is given by

$$E(t) = \frac{1}{2}\|z(t)\|_{\mathcal{H}}^2 = \frac{1}{2}\int_0^1 (u_t^2(x,t) + u_{xx}^2(x,t))dx + \frac{K}{2}\eta^2(t), \tag{2.27}$$

$z(t) = (u(\cdot,t)\ u_t(\cdot,t)\ \eta(t))^T \in \mathcal{H}$ is the solution of (2.12), and K is given by (2.9). Assume that $z(0) \in D(A)$; then by semigroup property we have $z(t) = T(t)z(0) \in D(A)\ \forall t \geq 0$. Hence, in view of (2.13), we have

$$\dot{E}(t) = < Az(t),\ z(t) >_{\mathcal{H}} = -\frac{K}{\beta} u_{xxx}^2(1,t) - \frac{Km\alpha}{\beta^2} u_t^2(1,t) \ \leq\ 0. \tag{2.28}$$

Next, by using Cauchy–Schwarz and Poincaré's inequalities, it can easily be shown that the following holds for a positive constant C:

$$(t - C)E(t) \leq V(t) \leq (t + C)E(t), \qquad t \geq 0. \tag{2.29}$$

(One can take $C = 1$ or even $C = 1/\sqrt{2}$.) By differentiating (2.26) with respect to time and by using (1.1), we obtain

$$\begin{aligned} \dot{V}(t) = E(t) + t\dot{E}(t) &+ \int_0^1 x\ u_{xt}(x,t)\ u_t(x,t) dx \\ &- \int_0^1 x\ u_x(x,t)\ u_{xxxx}(x,t) dx. \end{aligned} \tag{2.30}$$

Using integration by parts and (1.2), we obtain

$$\int_0^1 x\ u_x(x,t)\ u_{xxxx}(x,t) dx = u_x(1,t)\ u_{xxx}(1,t) + \frac{3}{2}\int_0^1 u_{xx}^2(x,t) dx, \tag{2.31}$$

$$\int_0^1 x\ u_{xt}(x,t)\ u_t(x,t)dx = \frac{1}{2}u_t^2(1,t) - \frac{1}{2}\int_0^1 u_t^2(x,t)\ dx. \tag{2.32}$$

By using (1.2), we obtain

$$u_x^2(1,t) \leq \int_0^1 u_{xx}^2(x,t)dx. \tag{2.33}$$

We also have the following inequalities:

$$u_x(1,t)u_{xxx}(1,t) \leq \delta_1 u_x^2(1,t) + \frac{1}{\delta_1}u_{xxx}^2(1,t), \tag{2.34}$$

$$\eta^2(t) \leq 2u_{xxx}^2(1,t) + 2\frac{m^2}{\beta^2}u_t^2(1,t), \tag{2.35}$$

where $\delta_1 > 0$ is an arbitrary constant. By using (2.28) and (2.31)–(2.35) in (2.30), we obtain

$$\begin{aligned}\dot{V}(t)\ \leq &-(1-\delta_1)\int_0^1 u_{xx}^2(x,t)dx - \left[\frac{K}{\beta}t - K - \frac{1}{\delta_1}\right]u_{xxx}^2(1,t)\\ &- \left[\frac{Km\alpha}{\beta^2}t - \frac{1}{2} - \frac{Km^2}{\beta^2}\right]u_t^2(1,t).\end{aligned} \tag{2.36}$$

By choosing $\delta_1 < 1$, the integral term in (2.36) is negative. Hence there exists a constant $T \geq 0$, which depends only on the constants $K,\ m,\ \alpha,\ \beta,$ and δ_1 such that the following holds:

$$\dot{V}(t) \leq 0, \qquad t \geq T. \tag{2.37}$$

Now, from (2.29) and (2.37) we obtain the following:

$$E(t) \leq \frac{T+C}{t-C}E(0), \qquad t > \max\{C,T\}. \tag{2.38}$$

Note that $E(t) = \frac{1}{2}\|z(t)\|_{\mathcal{H}}^2 = \frac{1}{2}\|T(t)z(0)\|_{\mathcal{H}}^2$; hence from (2.38) it follows that $\|T(t)\|_{\mathcal{L}(\mathcal{H})} < 1$ for $t > 0$ sufficiently large. Hence it follows from the semigroup property that the exponential decay, i.e., (2.25), holds. □

Remark 2. From (2.25) and (2.27) we conclude that both the "energy" associated with the flexible beam (i.e., the integral terms in (2.27)) and η defined by (2.2) decay exponentially to zero. However, we cannot conclude that the same holds separately for the tip mass velocity $u_t(1,t)$ and $u_{xxx}(1,t)$. If we assume that $z(0) \in D(A)$, then we also have for the graph norm

$$\|T(t)z(0)\|_{D(A)} \leq Me^{-\delta t}\|z(0)\|_{D(A)}.$$

In this case, $T(t)z(0)$ decays exponentially to zero in $H^4(0,1) \times H^2(0,1) \times \mathbf{R}$. Since, similar to (2.33), we have

$$u_t^2(1,t) \leq \int_0^1 u_{xt}^2(x,t)dx,$$

we obtain exponential decay of the tip mass velocity $u_t(1,t)$ and $u_{xxx}(1,t)$ uniformly for all smooth initial data $z(0) \in D(A)$ bounded in $D(A)$ for the graph norm.

3. Analysis of the spectrum. In this section we calculate the spectrum of the operator A for a special case and claim that the spectrum determines the optimal exponential decay rate given by (2.25) for the considered case. Our method is to prove that a system of eigenvectors of A forms a Riesz basis in $\mathcal{H}$. To obtain this result we compare the flexible beam with a tip mass to the flexible beam without a tip mass for the spectral properties. Here we have to work in the complexified Hilbert spaces $\mathcal{V}$, $L^2(0,1)$ and $\mathcal{H}$. For convenience we do not change the notation for these spaces.

Let $\lambda \in \mathbf{C}$ be an eigenvalue of A and let $y = (u\ v\ \eta)^T \in D(A)$ be a corresponding eigenvector. To find y we have to solve (2.14), and hence (2.15)–(2.17) for $z = (f\ g\ h)^T = 0$. Using (2.15) in (2.16), the latter, together with the boundary conditions, becomes

$$\lambda^2 u + u_{xxxx} = 0, \tag{3.1}$$

$$u(0) = u_x(0) = u_{xx}(1) = 0. \tag{3.2}$$

Similarly, by using (2.15) and (2.2) in (2.17), the latter becomes (cf. (2.19))

$$-\left(\lambda + \frac{1}{\beta}\right) u_{xxx}(1) + \frac{\lambda(\alpha + m\lambda)}{\beta} u(1) = 0. \tag{3.3}$$

By solving (3.1)–(3.3) one can find u. Then v and η can be found from (2.15) and (2.2), respectively.

The solutions of (3.1), together with the first two boundary conditions in (3.2), can be found as (for $0 \le x \le 1$)

$$u(x) = c_1(\cosh \tau x - \cos \tau x) + c_2(\sinh \tau x - \sin \tau x), \qquad \lambda = i\tau^2, \tag{3.4}$$

where c_1 and c_2 are constants to be determined by the remaining boundary conditions, cosh and sinh are the hyperbolic cosine and sine functions, respectively, and τ is one square root of λ/i. The choice of the sign is not important since by using $-\tau$ instead of τ nothing changes except the signs of the eigenvectors associated with λ.

By using (3.4) in (3.3) and the last boundary condition in (3.2), we obtain

$$\tau^2(\cosh \tau + \cos \tau)c_1 + \tau^2(\sinh \tau + \sin \tau)c_2 = 0, \tag{3.5}$$

$$\begin{aligned}&[-q_1(\lambda)\tau^3(\sinh \tau - \sin \tau) + q_2(\lambda)(\cosh \tau - \cos \tau)]c_1 \\ &\quad +[-q_1(\lambda)\tau^3(\cosh \tau + \cos \tau) + q_2(\lambda)(\sinh \tau - \sin \tau)]c_2 = 0,\end{aligned} \tag{3.6}$$

where

$$q_1(\lambda) = \lambda + \frac{1}{\beta}, \qquad q_2(\lambda) = \frac{\lambda(m\lambda + \alpha)}{\beta}.$$

By writing (3.5)–(3.6) in matrix form and taking the determinant of the coefficient matrix, it can easily be shown that (3.5)–(3.6) admit nontrivial solutions for c_1 and c_2 if and only if λ (hence τ) satisfies the following equation with λ necessarily nonzero:

$$-\tau^3 q_1(\lambda)(1 + \cosh \tau \cos \tau) + q_2(\lambda)(\sinh \tau \cos \tau - \cosh \tau \sin \tau) = 0. \tag{3.7}$$

The solutions of (3.7) give the eigenvalues of A; the corresponding eigenvectors can be found from (3.4)–(3.6), (2.15), and (2.2).

In what follows we analyze the spectrum of A for the case $\alpha = \frac{m}{\beta}$. From (2.3) or (2.10) it is clear that this choice leads to simplifications in the system (1.1)–(1.3) or (2.12), especially for the asymptotic behavior since the system is then uncoupled, except for the initial conditions. Then (3.7) can be written in the following form:

$$\left(\lambda + \frac{1}{\beta}\right)\left[-\tau^3(1+\cosh\tau\cos\tau) + \alpha\lambda(\sinh\tau\cos\tau - \cosh\tau\sin\tau)\right] = 0. \tag{3.8}$$

From (3.8) it follows that $\lambda_* = -\frac{1}{\beta}$ is an eigenvalue of A. To find the remaining eigenvalues of A let us define the function $f(\cdot)$ given by

$$f(\tau) = -\tau^3(1+\cosh\tau\cos\tau) + \alpha\lambda(\sinh\tau\cos\tau - \cosh\tau\sin\tau), \tag{3.9}$$

which is just the remaining factor of (3.8) after the division by the term $(\lambda + \frac{1}{\beta})$. Hence the remaining eigenvalues of A are precisely the (nonzero) roots of this factor:

$$-\tau^3(1+\cosh\tau\cos\tau) + \alpha\lambda(\sinh\tau\cos\tau - \cosh\tau\sin\tau) = 0. \tag{3.10}$$

It is known that (3.10) is just the characteristic equation for the system given by (1.1)–(1.4) with $m = 0$, i.e., the clamped-free (cantilevered) beam with boundary force controller at the free end; see, e.g., [12]. Moreover the eigenvectors of A corresponding to the roots of (3.10) are also related to the eigenvectors of the cantilevered beam in a simple way. For these reasons we will briefly study the spectral properties of the cantilevered beam in the following subsection.

3.1. Spectral analysis of the cantilevered beam. We consider the Euler–Bernoulli beam with boundary force control:

$$u_{tt} + u_{xxxx} = 0, \qquad 0 < x < 1, \quad t \geq 0, \tag{3.11}$$

$$u(0,t) = u_x(0,t) = u_{xx}(1,t) = 0, \quad u_{xxx}(1,t) = \alpha u_t(1,t), \quad t \geq 0, \tag{3.12}$$

where $\alpha > 0$. Note that this system is the same as (1.1)–(1.4) with $m = 0$.

We define the following spaces:

$$V = \{v \in \mathbf{H}^2(0,1);\ \ v(0) = v_x(0) = 0\}, \tag{3.13}$$

$$D(B) = \{(u\ v)^T | u \in \mathbf{H}^4(0,1) \cap V, v \in V, u_{xx}(1) = 0,\ u_{xxx}(1) = \alpha v(1)\}. \tag{3.14}$$

The operator B for the problem (3.11)–(3.12) is

$$B\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -u_{xxxx} \end{pmatrix}, \qquad (u\ v)^T \in D(B). \tag{3.15}$$

The system given by (3.11), (3.12) can be written formally as

$$\dot{z}(t) = Bz(t), \quad z(0) \in V \times \mathbf{L}^2(0,1), \tag{3.16}$$

where $z = (u(\cdot,t)\ u_t(\cdot,t))^T$, and the domain of B is given by (3.14).

We first state the following well-known result.

LEMMA 3.1. *Consider the system given by* (3.16).

i : *B generates an exponentially stable C_0 semigroup of contractions in $V \times \mathbf{L}^2(0,1)$.*

ii : *B has compact resolvent for $\lambda > 0$.*

iii : *The eigenvalues of B are countable and isolated. Moreover each eigenvalue has finite algebraic multiplicity.*

Proof. For **i** and **ii**, see [3]. Then **iii** follows from **ii**; see, e.g., [8, p. 187], [5, p. 2292]. ∎

Writing $z = (u\ v)^T$ and $Bz = \lambda z$, we get the following well-known characteristic equation:

$$f(\tau) = -\tau^3(1 + \cosh\tau\cos\tau) + i\alpha\tau^2(\sinh\tau\cos\tau - \cosh\tau\sin\tau) = 0, \tag{3.17}$$

where $\lambda = i\tau^2$. Note that $\lambda = 0$ is not an eigenvalue of B. Hence the roots of (3.17) are precisely the eigenvalues of B, and by Lemma 3.1, (3.17) has only countably many roots; moreover each root is isolated and has finite algebraic multiplicity. Eigenvectors corresponding to $\lambda = i\tau^2$ can be taken as $(\varphi_1\ \lambda\varphi_1)^T$, where

$$\begin{aligned}\varphi_1(\tau, x) = (&\cosh\tau + \cos\tau)(\sinh\tau x - \sin\tau x)\\ &- (\sinh\tau + \sin\tau)(\cosh\tau x - \cos\tau x).\end{aligned} \tag{3.18}$$

All eigenvalues are geometrically simple. For the algebraic multiplicity we have the following result.

LEMMA 3.2. *Consider the operator B on $V \times \mathbf{L}^2(0,1)$ given by* (3.15)*, where $D(B)$ is given by* (3.14)*. Let λ be an eigenvalue of B and set $\lambda = i\tau^2$. Then the algebraic multiplicity of λ is* 1 *if and only if $f(\tau) = 0$ and $f'(\tau) \neq 0$ (i.e., if and only if τ is a simple root of* (3.17)).

Proof. The algebraic multiplicity of λ is greater than 1 if and only if $\mathrm{Ker}(\mathrm{B} - \lambda\mathrm{I})^2 \setminus \mathrm{Ker}(\mathrm{B} - \lambda\mathrm{I}) \neq \emptyset$, i.e., there exists $(\psi_1\ \psi_2)^T$ which satisfies

$$(B - \lambda I)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\varphi_1\\ \lambda\varphi_1\end{pmatrix}, \tag{3.19}$$

which is equivalent to the following set of equations:

$$\psi_2 - \lambda\psi_1 = \varphi_1, \tag{3.20}$$

$$-\psi_{1xxxx} - \lambda\psi_2 = \lambda\varphi_1, \tag{3.21}$$

$$\psi_1(0) = \psi_{1x}(0) = \psi_{1xx}(1) = 0, \quad \psi_{1xxx}(1) = \alpha\psi_2(1). \tag{3.22}$$

By eliminating ψ_2, we obtain the following set of equations:

$$-\psi_{1xxxx} - \lambda^2\psi_1 = 2\lambda\varphi_1, \tag{3.23}$$

$$\psi_1(0) = \psi_{1x}(0) = \psi_{1xx}(1) = 0, \quad \psi_{1xxx}(1) = \alpha\lambda\psi_1(1) + \alpha\varphi_1(1). \tag{3.24}$$

The general solution of (3.23) satisfying the first three conditions of (3.24) is given for all λ by $\psi_1 = \frac{d\varphi_1}{d\lambda} + z$, where $\frac{d\varphi_1}{d\lambda} = \frac{1}{2i\tau}\frac{d\varphi_1}{d\tau}$ and z satisfies

$$z_{xxxx} + \lambda^2 z = 0, \tag{3.25}$$

$$z(0) = z_x(0) = z_{xx}(1) = 0. \tag{3.26}$$

The last condition of (3.24) becomes

$$z_{xxx}(1) - \alpha\lambda z(1) = \alpha\varphi_1(1) + \alpha\lambda\frac{d\varphi_1}{d\lambda}(1) - \left(\frac{d\varphi_1}{d\lambda}\right)_{xxx}(1) = \mu. \tag{3.27}$$

By computing μ defined in (3.27) from φ_1 and $\frac{d\varphi_1}{d\lambda}$ we get

$$\mu = \frac{d}{d\lambda}[\alpha\lambda\varphi_1(1) - \varphi_{1xxx}(1)] = 2\frac{df(\tau)}{d\lambda} = \frac{1}{i\tau}f'(\tau). \tag{3.28}$$

We conclude that the algebraic multiplicity of λ is larger than 1 if and only if (3.25)–(3.27) admit a solution z. Multiplying (3.25) by φ_1, integrating by parts, and using the boundary conditions on z and φ_1, we obtain

$$z_{xxx}(1)\varphi_1(1) - z(1)\varphi_{1xxx}(1) = [z_{xxx}(1) - \alpha\lambda z(1)]\varphi_1(1) = 0. \tag{3.29}$$

Since φ_1 is an eigenfunction of B, it could easily be shown that $\varphi_1(1) \neq 0$ (otherwise one obtains a contradiction; see, e.g., [4, p. 429]). Hence (3.25)–(3.27) admit a solution if and only if $\mu = 0$, in which case we could choose $z = \varphi_1$. Hence λ is algebraically simple if and only if $f(\tau) = 0$ and $f'(\tau) \neq 0$, i.e., if and only if λ is a simple root of (3.17). ☐

By Lemma 3.1, B has at most countably many and isolated eigenvalues. Let $\lambda_n = i\tau_n^2$, $n \in \mathbf{Z}$, be the roots of (3.17). The corresponding eigenvectors of B can be given as

$$F_{nr} = \begin{pmatrix} \varphi_1(\tau_n, x) \\ \lambda_n\varphi_1(\tau_n, x) \end{pmatrix}, \tag{3.30}$$

where φ_1 is given by (3.18).

THEOREM 3.3. *Consider the operator B on $V \times \mathbf{L}^2(0,1)$ given by* (3.15), *where $D(B)$ is given by* (3.14).

i. *For any $\alpha > 0$, all eigenvalues of B with sufficiently large modulus are algebraically simple.*

ii. *For almost all $\alpha > 0$, the eigenvalues of B are algebraically simple.*

iii. *If all eigenvalues are algebraically simple, then the set of eigenvectors $\{F_{nr}, n \in \mathbf{Z}\}$ is a Riesz basis for $V \times \mathbf{L}^2(0,1)$, provided that the normalization of eigenvectors is suitable.*

Proof. The proof requires detailed and lengthy calculations and is given in the appendix. In this proof we compare the set of eigenfunctions of B for $\alpha = 0$, denoted by $\{G_{nr}, n \in \mathbf{Z}\}$ with $\{F_{nr}, n \in \mathbf{Z}\}$, and show that these two sets are quadratically close. Since the former set is a Riesz basis for $V \times \mathbf{L}^2(0,1)$, we then conclude that the same is true for the latter set. ☐

3.2. Spectral analysis of the operator *A*. We now consider the operator A given by (2.10) for the case $\alpha = m/\beta$. The eigenvalues of A are given by (3.8). From (3.8) it follows that $\lambda_* = -\frac{1}{\beta}$ is an eigenvalue of A. To find the corresponding eigenfunction, we again set $\lambda_* = i\tau_*^2$ and rewrite (3.5) as ($\tau_* \neq 0$):

$$(\cosh\tau_* + \cos\tau_*)c_1 + (\sinh\tau_* + \sin\tau_*)c_2 = 0. \tag{3.31}$$

Note that since τ_* is a solution of (3.8), (3.6) is linearly dependent on (3.5) and hence will not be used to determine c_1 and c_2.

In (3.31) the coefficients c_1 and c_2 cannot be zero simultaneously. This follows easily since τ_* is not a purely imaginary number (note that $\lambda_* = -\frac{1}{\beta} = i\tau_*^2$). So the natural choice for c_1 and c_2 given by (3.5)–(3.6) is:

$$c_1 = -(\sinh \tau_* + \sin \tau_*), \tag{3.32}$$

$$c_2 = (\cosh \tau_* + \cos \tau_*). \tag{3.33}$$

Therefore an eigenfunction F_* corresponding to λ_* is

$$F_* = \begin{pmatrix} u_* \\ v_* \\ \eta_* \end{pmatrix}, \tag{3.34}$$

where

$$\begin{aligned} u_*(x) = \varphi_1(\tau_*, x) = {} & (\cosh \tau_* + \cos \tau_*)(\sinh \tau_* x - \sin \tau_* x) \\ & -(\sinh \tau_* + \sin \tau_*)(\cosh \tau_* x - \cos \tau_* x), \end{aligned} \tag{3.35}$$

$$v_* = \lambda_* u_*(x), \tag{3.36}$$

$$\eta_* = 2f(\tau_*), \tag{3.37}$$

where $f(\cdot)$ and φ_1 are given by (3.17) and (3.18), respectively. The remaining eigenvalues of A are precisely the (nonzero) roots of (3.10). From the preceding section it follows that these eigenvalues are the roots of (3.17), and hence the eigenvalues of B, i.e., the eigenvalues of the cantilevered beam without a tip mass. By Lemma 3.1, (3.17) admits countably many distinct roots $\lambda_n = i\tau_n^2$, $n \in \mathbf{Z}$, $\mathcal{R}e\ \{\lambda_n\} < 0$. We set

$$\begin{aligned} u_n(x) = \varphi_1(\tau_n, x) = {} & (\cosh \tau_n + \cos \tau_n)(\sinh \tau_n x - \sin \tau_n x) \\ & -(\sinh \tau_n + \sin \tau_n)(\cosh \tau_n x - \cos \tau_n x), \end{aligned} \tag{3.38}$$

$$v_n = \lambda_n u_n(x), \tag{3.39}$$

$$\eta_n = 2f(\tau_n) = 0, \tag{3.40}$$

where φ_1 is given in (3.18). As before, since $\mathcal{R}e\{\lambda_n\} < 0$ implies that τ_n is not a purely imaginary number, the constant factors in (3.18) cannot vanish simultaneously. Then

$$F_n = \begin{pmatrix} u_n \\ v_n \\ 0 \end{pmatrix} \tag{3.41}$$

is an eigenvector for A associated with the eigenvalue λ_n. Note that $(u_n\ v_n)^T$ is an eigenvector of B (i.e., of the cantilevered beam) associated with the same eigenvalue.

Assume now that all the eigenvalues λ_n of the cantilevered beam are algebraically simple. By Lemma 3.2, this assumption can be written as

$$f(\tau_n) = 0, \qquad f'(\tau_n) \neq 0, \tag{3.42}$$

where $\lambda = i\tau^2$ and for $f(\tau) = 0$ we have

$$f'(\tau) = -\tau^2 \left(1 + \frac{\tau^2}{i\alpha}\right)(1 + \cosh\tau\cos\tau) - 2i\alpha\tau^2 \sinh\tau \sin\tau. \tag{3.43}$$

Under the assumption given by (3.42), we now compute the algebraic multiplicity of all the eigenvalues $(\lambda_*, \lambda_n, n \in \mathbf{Z})$ of A. We note that the algebraic simplicity of λ_n as an eigenvalue of B does not imply the algebraic simplicity of λ_n as an eigenvalue of A. We have to distinguish two cases: $\eta_* = 2f(\tau_*) \neq 0$, in which case $\lambda_* \neq \lambda_n \ \forall n \in \mathbf{Z}$, or $\eta_* = 0$, in which case $\lambda_* = \lambda_N$ for some $N \in \mathbf{Z}$.

An easy computation shows that $\eta_* = 0$ if and only if

$$\alpha = \frac{\beta_*(2 + \cosh 2\beta_* + \cos 2\beta_*)}{\sinh 2\beta_* - \sin 2\beta_*}, \tag{3.44}$$

where $\beta_* = 1/\sqrt{2\beta}$. Hence the case $\eta_* = 0$ is just an exceptional one in the sense that (α, β) have to belong to the curve defined by (3.44). For instance, if α is sufficiently small, η_* is always nonzero.

Let $\tilde\lambda$ be an eigenvalue of A, and let $(\tilde u\ \tilde v\ \tilde\eta)^T$ be the corresponding eigenvector. Let us study when the algebraic multiplicity of $\tilde\lambda$ is equal to one or not.

Obviously $\mathrm{Ker}(\mathrm{A} - \tilde\lambda \mathrm{I})^2 \setminus \mathrm{Ker}(\mathrm{A} - \tilde\lambda \mathrm{I}) \neq \emptyset$ if and only if there exists $(u\ v\ \eta)^T \in D(A)$ such that

$$A \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} - \tilde\lambda \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} \tilde u \\ \tilde v \\ \tilde\eta \end{pmatrix}, \tag{3.45}$$

which is equivalent to the following:

$$v = \tilde\lambda u + \tilde u, \tag{3.46}$$

$$-u_{xxxx} - \tilde\lambda v = \tilde v = \tilde\lambda \tilde u, \tag{3.47}$$

$$-\left(\tilde\lambda + \frac{1}{\beta}\right)\eta = \tilde\eta, \tag{3.48}$$

where $(u\ v\ \eta)^T \in D(A)$. Equations (3.46)–(3.48) have a solution if and only if the equations

$$-u_{xxxx} - \tilde\lambda^2 u = 2\tilde\lambda \tilde u, \tag{3.49}$$

$$-\left(\tilde\lambda + \frac{1}{\beta}\right)\eta = \tilde\eta, \tag{3.50}$$

$$u(0) = u_x(0) = u_{xx}(1) = 0, \tag{3.51}$$

$$\eta = -u_{xxx}(1) + \alpha\tilde{\lambda}u(1) + \alpha\tilde{u}(1) \tag{3.52}$$

admit a solution.

LEMMA 3.4. *Let $\alpha = m/\beta$ and consider the operator A given by* (2.10). *Let α be such that the eigenvalues of the operator B given by* (3.15) *are algebraically simple* (*note that this is true for almost all $\alpha > 0$ by Theorem* 3.3). *Let (λ_*, F_*) be the eigenvalue-eigenvector pair of A given by $\lambda_* = 1/\beta$ and* (3.34), *respectively, and let (λ_n, F_n), $n \in \mathbf{Z}$ be the remaining eigenvalue-eigenvector pairs of A, where λ_n is a root of* (3.10) *and F_n is given by* (3.41).

i. *If $\eta_* \neq 0$, then all eigenvalues of A are algebraically simple.*

ii. *If $\eta_* = 0$, then the algebraic multiplicity of λ_* is exactly* 2 *and all the eigenvalues $\lambda_n \neq \lambda_*$ are algebraically simple.*

Proof. **i.** Let $\eta_* \neq 0$, which implies $\lambda_* \neq \lambda_n$, $n \in \mathbf{Z}$. Then, for $\tilde{\lambda} = \lambda_*$, (3.50) implies $\eta_* = 0$, which is a contradiction. Thus λ_* is algebraically simple. Choose now $\tilde{\lambda} = \lambda_n$ for $n \in \mathbf{Z}$, and for simplicity, denote by $\lambda = i\tau^2$ the eigenvalue λ_n. Then $\tilde{\eta} = \eta_n = 0$, and since $(\tilde{\lambda} + \frac{1}{\beta}) \neq 0$, we get $\eta = 0$ so that (3.49)–(3.52) reduces to

$$-u_{xxxx} - \lambda^2 u = 2\lambda u_n, \tag{3.53}$$

$$u(0) = u_x(0) = u_{xx}(1) = 0, \tag{3.54}$$

$$u_{xxx}(1) = \alpha\lambda u(1) + \alpha u_n(1). \tag{3.55}$$

Then, proceeding exactly as in Lemma 3.2, we obtain that (3.53)–(3.55) has a solution if and only if $f'(\tau) = 0$ (cf. (3.23), (3.24)). By Lemma 3.2 this implies that λ_n is not algebraically simple as an eigenvalue of B, which is a contradiction. Hence by Lemma 3.2 we see that λ_n is also algebraically simple as an eigenvalue of A.

ii. For the case $\eta_* = 0$, by the argument given above, all the λ_n such that $\lambda_n \neq \lambda_*$ are also algebraically simple.

Let $\lambda_* = \lambda_N$ for some $N \in \mathbf{Z}$, which is denoted by λ for simplicity. Then (3.49)–(3.52) reduces to

$$-u_{xxxx} - \lambda^2 u = 2\lambda u_*, \tag{3.56}$$

$$u(0) = u_x(0) = u_{xx}(1) = 0, \tag{3.57}$$

$$-u_{xxx}(1) + \alpha\lambda u(1) + \alpha u_*(1) = \eta. \tag{3.58}$$

Now proceeding again as in Lemma 3.2, but replacing the right-hand side of (3.27) by $\mu - \eta$, we obtain that (3.56)–(3.58) has a solution if and only if

$$\eta = \frac{f'(\tau)}{i\tau}, \tag{3.59}$$

and hence is nonzero by Lemma 3.2 if λ is an algebraically simple eigenvalue of B. Consequently it is always possible to compute $\eta_{**} \neq 0$ in a unique way such that (3.59) is true for $\eta = \eta_{**}$, and then one has a (nonunique) solution $u = u_{**}$ of (3.56)–(3.58) with $v = v_{**} = \lambda_* u_{**} + u_*$ such that (3.46)–(3.48) is satisfied. Thus in case $\eta_* = 0$, λ_* has algebraic multiplicity at least two, $F_* = (u_* \ v_* \ 0)^T$ is an eigenvector of A, and $F_{**} = (u_{**}\ v_{**}\ \eta_{**})^T$ with $\eta_{**} \neq 0$ is a generalized eigenvector (but not an eigenvector) of A. In fact, λ_* has algebraic multiplicity exactly two, since for $w = (u\ v\ \eta)^T \in D(A)$, $w \in \mathrm{Ker}(\mathrm{A}-\lambda_*\mathrm{I})^3 \setminus \mathrm{Ker}(\mathrm{A}-\lambda_*\mathrm{I})^2$ implies $(A-\lambda_* I)w \in \mathrm{Ker}(\mathrm{A}-\lambda_*\mathrm{I})^2 \setminus \mathrm{Ker}(\mathrm{A}-\lambda_*\mathrm{I})$; thus $(A-\lambda_* I)w = (u_{**}\ v_{**}\ \eta_{**})^T$. But this implies that $-(\lambda_* + 1/\beta)\eta = \eta_{**}$ (cf.(3.48)), hence $\eta_{**} = 0$, which is impossible. □

We have now the material to write down the Riesz basis property. Recall that $(u_n\ v_n)^T$ are not the functions given exactly by (3.38)–(3.39) but have been suitably normalized to posses the adequate Riesz basis property for the cantilevered beam, (see Theorem 3.3).

THEOREM 3.5. *Let* $\alpha = m/\beta$, $\lambda_* = -1/\beta$, λ_n, $n \in \mathbf{Z}$, *be the roots of* (3.10). *Assume* (3.42) *and* $\eta_* = 2f(\tau_*) \neq 0$. *Then* $\{F_*,\ F_n,\ n \in \mathbf{Z}\}$ *is a Riesz basis for* $\mathcal{H}$. *Moreover the estimate* (2.25) *is valid with* $\delta > 0$ *such that*

$$-\delta \ = \ \max\{-1/\beta,\ \mathcal{R}e\{\lambda_n\},\ n \in \mathbf{Z}\}, \tag{3.60}$$

which is the optimal rate of decay.

Proof. Let $z = (u\ v\ \eta)^T \in \mathcal{H}$ be given. Since $\eta_* \neq 0$, we can write

$$z = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ 0 \end{pmatrix} + c_* F_*, \tag{3.61}$$

where

$$c_* = \eta/\eta_*, \quad \tilde{u} = u - c_* u_* \in \mathcal{V}, \quad \tilde{v} = v - c_* v_* \in \mathbf{L}^2(0,1). \tag{3.62}$$

Since $(u_n\ v_n)^T$, $n \in \mathbf{Z}$, is a Riesz basis for $\mathcal{V} \times \mathbf{L}^2(0,1)$, we can write

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \sum_{n \in \mathbf{Z}} c_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \tag{3.63}$$

where $c_n \in l^2(\mathbf{Z})$, and there exist positive constants C_1, C_2 such that

$$C_1 \sum_{n \in \mathbf{Z}} |c_n|^2 \leq \left\| \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\|^2_{\mathcal{V} \times L^2} \leq C_2 \sum_{n \in \mathbf{Z}} |c_n|^2. \tag{3.64}$$

Note that

$$\left\| \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\|^2_{\mathcal{V} \times L^2} = \int_0^1 (\tilde{u}_{xx}^2 + \tilde{v}^2) dx \ , \ \|z\|^2_{\mathcal{H}} = \int_0^1 (u_{xx}^2 + v^2) dx + K\eta^2, \tag{3.65}$$

where K is given by (2.9). By using (3.62) we obtain

$$u_{xx}^2 = \tilde{u}_{xx}^2 + 2c_* \tilde{u}_{xx} u_{*xx} + c_*^2 u_{*xx}^2, \tag{3.66}$$

$$v^2 = \tilde{v}^2 + 2c_* \tilde{v} v_* + c_*^2 v_*^2. \tag{3.67}$$

It follows from Young's inequality that

$$\begin{aligned} \left| \int_0^1 2c_* \tilde{u}_{xx} u_{*xx} dx \right| &\leq \sigma \int_0^1 \tilde{u}_{xx}^2 dx + \frac{1}{\sigma} \int_0^1 \mid c_* \mid^2 u_{*xx}^2 dx \\ &\leq \sigma \int_0^1 \tilde{u}_{xx}^2 dx + \frac{M_1}{\sigma} \eta^2, \end{aligned} \tag{3.68}$$

where $\sigma > 0$ is an arbitrary constant and (using (3.62))

$$M_1 = \frac{\int_0^1 u_{*xx}^2}{\eta_*^2}.$$

Similarly we obtain

$$\left| \int_0^1 2c_* \tilde{v} v_* dx \right| \le \sigma \int_0^1 \tilde{v}^2 dx + \frac{M_2}{\sigma} \eta^2, \tag{3.69}$$

where

$$M_2 = \frac{\int_0^1 v_*^2}{\eta_*^2}.$$

By using (3.66)–(3.69) in (3.65), we obtain

$$\|z\|_{\mathcal{H}}^2 \le (1+\sigma) \left\| \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\|_{\mathcal{V} \times L^2}^2 + \left(K + M_1 + M_2 + \frac{M_1}{\sigma} + \frac{M_2}{\sigma} \right) \eta^2, \tag{3.70}$$

$$\|z\|_{\mathcal{H}}^2 \ge (1-\sigma) \left\| \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \right\|_{\mathcal{V} \times L^2}^2 + \left(K + M_1 + M_2 - \frac{M_1}{\sigma} - \frac{M_2}{\sigma} \right) \eta^2 . \tag{3.71}$$

From (3.61)–(3.63) it follows that

$$z = \sum_{n \in \mathbf{Z}} c_n F_n + c_* F_*. \tag{3.72}$$

Next we choose $\sigma > 0$ such that

$$\frac{M_1 + M_2}{K + M_1 + M_2} < \sigma < 1,$$

which implies that all coefficients in (3.71) are positive. Since η is proportional to c_* (see (3.62)), it follows from (3.64), (3.70)–(3.71) that there exist positive constants C_3 and C_4 such that the following holds:

$$C_3 \Big(\sum_{n \in \mathbf{Z}} |c_n|^2 + |c_*|^2 \Big) \le \|z\|_{\mathcal{H}}^2 \le C_4 \Big(\sum_{n \in \mathbf{Z}} |c_n|^2 + |c_*|^2 \Big). \tag{3.73}$$

It follows from (3.72)–(3.73) that the system $\{F_*, F_n, n \in \mathbf{Z}\}$ is a Riesz basis in $\mathcal{H}$.

Since $F_*,\ F_n,\ n \in \mathbf{Z}$ are all eigenvectors of A, we then have

$$T(t)z = T(t) \left[\sum_{n \in \mathbf{Z}} c_n F_n + c_* F_* \right] = \sum_{n \in \mathbf{Z}} e^{\lambda_n t} c_n F_n + e^{\lambda_* t} c_* F_*. \tag{3.74}$$

That (3.60) determines the optimal decay rate for the semigroup is now an immediate and general consequence of the Riesz basis property in $\mathcal{H}$. ∎

THEOREM 3.6. *Let $\alpha = m/\beta$, $\lambda_* = -1/\beta$, λ_n, $n \in \mathbf{Z}$, be the roots of* (3.10). *Assume* (3.42) *and $\eta_* = 2f(\tau_*) = 0$. Then $\{F_{**},\ F_n,\ n \in \mathbf{Z}\}$ is a Riesz basis for $\mathcal{H}$. Moreover, for any $\epsilon > 0$, the estimate* (2.25) *is valid for $\delta - \epsilon$, where $-\delta$ is given by* (3.60). *Hence $-\delta$ is again the optimal decay rate.*

Proof. We recall that here $F_* = F_N$ for some $N \in \mathbf{Z}$, F_n being suitably normalized eigenvectors of A. The Riesz basis property can be proven as in Theorem 3.5 by just replacing F_* by F_{**} and using the fact that $\eta_{**} \neq 0$, so that with $c_{**} = \eta/\eta_{**}$ we have

$$z = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ 0 \end{pmatrix} + c_{**}F_{**}, \tag{3.75}$$

and for $(\tilde{u}\ \tilde{v})^T$ we use the Riesz basis property of F_n, $n \in \mathbf{Z}$. Then we get

$$z = \sum_{n\in\mathbf{Z}} c_n F_n + c_{**}F_{**}, \tag{3.76}$$

where F_n, $n \in \mathbf{Z}$, are the eigenvectors of A, with $F_N = F_*$, $\lambda_N = \lambda_*$, but $F_{**} \in \mathrm{Ker}(\mathrm{A} - \lambda_\mathrm{N}\mathrm{I})^2 \setminus \mathrm{Ker}(\mathrm{A} - \lambda_\mathrm{N}\mathrm{I})$. Since F_{**} satisfies $(A - \lambda_* I)F_{**} = F_*$, we get

$$\frac{d}{dt}\Big(e^{\lambda_* t}(tF_* + F_{**})\Big) = e^{\lambda_* t}A(tF_* + F_{**}). \tag{3.77}$$

From (3.76)–(3.77) we get

$$T(t)z = \sum_{n\neq N,\ n\in\mathbf{Z}} e^{\lambda_n t}c_n F_n + e^{\lambda_* t}\Big((c_N + tc_{**})F_* + c_{**}F_{**}\Big). \tag{3.78}$$

Now the fact that the estimate (2.25) holds for $\delta - \epsilon$, for any $\epsilon > 0$, is an immediate and general consequence of the Riesz basis property. Due to the fact that $-\delta$ given by (3.60) may be achieved by λ_*, $\epsilon > 0$ comes from the possible compensation of $e^{2Re\lambda_* t}t^2$ by $e^{(2Re\lambda_* + \epsilon)t}$. If $-\delta > \lambda_* = -1/\beta$, then ϵ is unnecessary. ∎

4. Conclusion. In this paper we studied the stability of a flexible beam with a tip mass. The flexible beam is assumed to be clamped at one end and is free at the other, where a mass is also attached. This model is a variant of the SCOLE model and has been studied before; see, e.g., [1], [9], [13]. To stabilize this hybrid system we apply a boundary control force at the free end of the beam. It is well known that for this model the standard velocity feedback for the control force (e.g., (1.4)), which is widely used in boundary control systems, yields only *asymptotic*, but not *exponential*, stability; see e.g., [9], [13]. In this paper we proposed a (new) control law (see (2.1)), which contains the term $u_{xxxt}(1,t)$ in addition to the standard feedback term $u_t(1,t)$. We then proved that the system is well-posed and that the energy associated with the system decays exponentially to zero if the initial data are in $\mathcal{H}$. We also showed that if the initial data are sufficiently smooth (i.e., in $D(A)$), then the tip mass velocity also decays exponentially to zero. Then we analyzed the spectrum of the system for the special case $m = \alpha\beta$ and proved that the spectrum determines the exponential decay rate for the considered case for almost all $\alpha > 0$.

Appendix A. On the Riesz basis property of eigenvectors of the cantilevered beam with boundary force control. Here our aim is to prove Theorem 3.3. We will consider the set of eigenvectors of the operator B given by (3.15) for the cases $\alpha = 0$ (i.e., uncontrolled cantilevered beam) and $\alpha > 0$ (i.e., controlled cantilevered beam) and show that these two sets are quadratically close. Since the former set of eigenvectors is known to be a Riesz basis in $V \times \mathbf{L}^2(0,1)$, we conclude that the latter set is also a Riesz basis in the same space.

Before we prove the Riesz basis property, first we will show that the number of eigenvalues of the uncontrolled and controlled cantilevered beam are the same, counting multiplicities, in sufficiently large disks. This result will enable us to enumerate the eigenvalues of both systems in a similar way. We recall that the eigenvalues of B for $\alpha \geq 0$ are precisely the roots of (3.10). Since $\lambda = 0$ is not an eigenvalue, equivalently the eigenvalues are the roots of the following function (for $\lambda = i\tau^2$)

$$h(\tau) = \frac{f(\tau)}{\tau^2} = \tau(1 + \cosh\tau\cos\tau) - i\alpha(\sinh\tau\cos\tau - \cosh\tau\sin\tau); \tag{A.1}$$

hence for the uncontrolled case (i.e., $\alpha = 0$), the eigenvalues are the roots of the following function

$$g(\tau) = \tau(1 + \cosh\tau\cos\tau). \tag{A.2}$$

Note that $\tau = 0$ is a simple root of both (A.1) and (A.2) but not an eigenvalue of B for $\alpha \geq 0$. Hence it follows that if $h(\cdot)$ and $g(\cdot)$ have the same number of roots in a large disk, then the same is true for the eigenvalues of the operator B for $\alpha = 0$ and $\alpha > 0$.

LEMMA A.1. *There exists a sequence $R_k \in \mathbf{R}$ such that $R_k \to \infty$ as $k \to \infty$ and the number of roots of* (A.1) *and* (A.2) *are the same, counting multiplicities, in $B(0, R_k)$ where $B(0, R)$ is defined as*

$$B(0, R) = \{ \ \tau \in \mathbf{C} \ \ | \ \ | \tau | \leq R \ \ \}. \tag{A.3}$$

Proof. Let $R > 0$ be given and $\gamma = \{ \ \tau \in \mathbf{C} \ \ | \ \ | \tau | = R \ \ \}$, i.e., a circle of radius R. Since both $h(\cdot)$ and $g(\cdot)$ are analytic in $B(0, R)$, by Rouché's theorem they have the same number of roots, counting multiplicities, if $| \ h(\tau) - g(\tau) \ | < | \ g(\tau) \ |$ for $\tau \in \gamma$. We will show that this is true for some sufficiently large R. For convenience let us define

$$s(\tau) = i\alpha(\sinh\tau\cos\tau - \cosh\tau\sin\tau); \tag{A.4}$$

hence equivalently we need to show the following:

$$\left| \frac{s(\tau)}{g(\tau)} \right| \ \ < \ \ 1, \qquad \tau \in \gamma. \tag{A.5}$$

Since both $g(\cdot)$ and $s(\cdot)$ are odd functions it is sufficient to consider the upper half plane, and since $\cosh i\tau = \cos\tau$, $\cos i\tau = \cosh\tau$, $\sinh i\tau = i\sin\tau$, $\sin i\tau = i\sinh\tau$, it is sufficient to consider only the first quadrant, i.e., $\tau = Re^{i\theta}$ for $0 \leq \theta \leq \pi/2$.

Let $\tau = Re^{i\theta}$. After straightforward calculations it could be shown that the following holds:

$$| \ s(\tau) \ | \leq \frac{\alpha}{2}(e^{RD} + e^{-RD} + e^{RS} + e^{-RS}), \tag{A.6}$$

$$4\cosh\tau\cos\tau = e^{RD}e^{iRS} + e^{RS}e^{-iRD} + e^{-RS}e^{iRD} + e^{-RD}e^{-iRS}, \tag{A.7}$$

where $D = \cos\theta - \sin\theta$, $S = \cos\theta + \sin\theta$.

For $0 \leq \theta \leq \pi/2$ we have $S \geq 1$ and $S \geq \mid D \mid$; hence $\mid s(\tau) \mid \leq 2\alpha e^{RS}$. For $0 < \theta < \pi/4$ we have $D > 0$; hence for sufficiently large R the following holds:

$$\text{(A.8)} \qquad \frac{\mid s(\tau) \mid}{\mid \cosh\tau\cos\tau \mid} \leq \frac{2\alpha}{\mid e^{-RS}\cosh\tau\cos\tau \mid} \leq M$$

for some $M > 0$. For $\pi/4 < \theta < \pi/2$ we have $D < 0$, and from (A.6) and (A.7) it easily follows that an estimate similar to (A.8) holds. Hence for $0 < \theta < \pi/2$ and $\theta \neq \pi/4$ we have $\lim_{R\to\infty} \mid \frac{s(\tau)}{g(\tau)} \mid = 0$. For $\theta = 0$ or $\theta = \pi/2$ we have $D = 1$ or $D = -1$, respectively; $S = 1$ and $1 + \cosh\tau\cos\tau = 1 + \cosh R\cos R$ in both cases. Hence if we choose $R = 2n\pi$, we have $\lim_{n\to\infty} \mid \frac{s(\tau)}{g(\tau)} \mid = 0$. We note that this holds if $R \to \infty$ in such a way that $\mid \cos R \mid \geq \delta$ for any $\delta > 0$. For $\theta = \pi/4$ we have $D = 0$, $S > 1$, and $4\cosh\tau\cos\tau = 2\cos RS + 2\cosh RS$; hence $\lim_{R\to\infty} \mid \frac{s(\tau)}{g(\tau)} \mid = 0$. Therefore, for $\tau = Re^{i\theta}$, $R = 2n\pi$, and $0 \leq \theta \leq 2\pi$ we have $\lim_{n\to\infty} \mid \frac{s(\tau)}{g(\tau)} \mid = 0$. Hence there exists a sequence $R_k = 2k\pi$, $k \in \mathbf{N}$, and $k \to \infty$ such that $\lim_{k\to\infty} \mid \frac{s(\tau)}{g(\tau)} \mid < 1$ for $\mid \tau \mid = R_k$. Therefore, by Rouchée's theorem, the number of roots of $g(\cdot)$ and $h(\cdot)$, or equivalently the eigenvalues of the operator B for the cases $\alpha = 0$ and $\alpha > 0$, respectively, are the same in $B(0, R_k)$, counting multiplicities. □

The lemma given above lets us enumerate the eigenvalues of uncontrolled and controlled cantilevered beam in a similar way, at least if they are algebraically simple (see Remark 3 for an extension). In what follows we will give asymptotic formulas for these eigenvalues and then compare the corresponding eigenvectors.

Consider the system and the corresponding eigenvalue problem given by (3.11)–(3.18). From (3.17) it follows that the eigenvalues occur in complex conjugate pairs. Since there are countably many eigenvalues and each eigenvalue is isolated (see Lemma 3.1), the eigenvalues which have positive imaginary part can be numerated by considering the imaginary parts with increasing order. By using asymptotic analysis it can be shown that asymptotically the solutions of (3.17) can be given as ($\lambda = i\tau^2$):

$$\text{(A.9)} \qquad \lambda_k = -2\alpha + \mathcal{O}(1/k^2) + i((m\pi)^2 + \alpha\mathcal{O}(1/k)),$$

for sufficiently large $k \in \mathbf{N}$, where $m = k + 1/2$; see [12, p. 76]. We note that this estimate can also be obtained by using the wave propagation method (see [2]) for similar estimates. Here the symbol $\mathcal{O}(f(k))$ denotes any function such that $\lim_{k\to\infty} \mathcal{O}(f(k))/f(k)$ exists and is finite.

By using $\lambda_k = i\tau_k^2$, the corresponding τ_k can easily be found as

$$\text{(A.10)} \qquad \tau_k = \pm\left[(m\pi + \mathcal{O}(1/k^2)) + i\left(\frac{\alpha}{m\pi} + \mathcal{O}(1/k^3)\right)\right]$$

for sufficiently large k. In what follows we will consider (A.10) with + sign; the same conclusions hold with - sign as well (see below). By using (A.10), with + sign, we obtain the following estimates:

$$\text{(A.11)} \qquad e^{\tau_k x} = e^{m\pi x}\left((1 + \mathcal{O}(1/k^2)f_1(x)) + i\left(\frac{\alpha x}{m\pi} + \mathcal{O}(1/k^3)f_2(x)\right)\right),$$

$$\text{(A.12)} \qquad e^{-\tau_k x} = e^{-m\pi x}\left((1 + \mathcal{O}(1/k^2)f_3(x)) - i\left(\frac{\alpha x}{m\pi} + \mathcal{O}(1/k^3)f_4(x)\right)\right),$$

$$\begin{aligned} e^{i\tau_k x} &= e^{-\frac{\alpha x}{m\pi}}((\cos m\pi x + \cos m\pi x \quad \mathcal{O}(1/k^3) f_5(x) \\ &\quad - \sin m\pi x \quad \mathcal{O}(1/k^2) f_6(x)) + i(\sin m\pi x + \sin m\pi x \quad \mathcal{O}(1/k^3) f_7(x) \\ &\quad + \cos m\pi x \quad \mathcal{O}(1/k^2) f_8(x))), \end{aligned} \tag{A.13}$$

$$\begin{aligned} e^{-i\tau_k x} &= e^{\frac{\alpha x}{m\pi}}((\cos m\pi x + \cos m\pi x \quad \mathcal{O}(1/k^3) f_9(x) \\ &\quad - \sin m\pi x \quad \mathcal{O}(1/k^2) f_{10}(x)) - i(\sin m\pi x + \sin m\pi x \quad \mathcal{O}(1/k^3) f_{11}(x) \\ &\quad + \cos m\pi x \quad \mathcal{O}(1/k^2) f_{12}(x))), \end{aligned} \tag{A.14}$$

where the functions $f_i(\cdot)$, $i = 1, \ldots, 12$, are smooth and bounded functions with bounded derivatives. By using (A.11)–(A.14), we obtain the following estimates:

$$\begin{aligned} &(\cosh \tau_k + \cos \tau_k)(\sinh \tau_k x - \sin \tau_k x) = \frac{e^{\tau_k} e^{\tau_k x}}{4} + \Bigg(- \frac{e^{m\pi} e^{-m\pi x}}{4} \\ &+ e^{m\pi} \mathcal{O}(1/k^2) o_1(x) + e^{m\pi x} o_2(x) - \frac{e^{m\pi}}{2} \sin m\pi x + o_3(x) \Bigg) \\ &+ i(e^{m\pi} \mathcal{O}(1/k) o_4(x) + e^{m\pi x} \mathcal{O}(1/k) o_5(x) + o_6(x)), \end{aligned} \tag{A.15}$$

$$\begin{aligned} &(\sinh \tau_k + \sin \tau_k)(\cosh \tau_k x - \cos \tau_k x) = \frac{e^{\tau_k} e^{\tau_k x}}{4} + \Bigg(\frac{e^{m\pi} e^{-m\pi x}}{4} \\ &+ e^{m\pi} \mathcal{O}(1/k^2) o_7(x) + e^{m\pi x} o_8(x) - \frac{e^{m\pi}}{2} \cos m\pi x + o_9(x) \Bigg) \\ &+ i(e^{m\pi} \mathcal{O}(1/k) o_{10}(x) + e^{m\pi x} \mathcal{O}(1/k) o_{11}(x) + o_{12}(x)), \end{aligned} \tag{A.16}$$

where the functions $o_i(\cdot)$, $i = 1, \ldots, 12$, are smooth and bounded functions (as a function of k), and their derivatives are either bounded or satisfy the following:

$$o_i^{(n)}(x) = (k\pi)^n \hat{o}_i(x), \qquad i = 1, \ldots, 12, \qquad n \in \mathbf{N}, \tag{A.17}$$

where the functions $\hat{o}_i(\cdot)$ are also smooth and bounded functions. By using (A.15) and (A.16) in (3.18) we obtain

$$\begin{aligned} \varphi_1(\tau_k, x) = \Bigg[&- \frac{e^{m\pi} e^{-m\pi x}}{2} + e^{m\pi} \mathcal{O}(1/k^2) o_{13}(x) + e^{m\pi x} o_{14}(x) \\ &+ \frac{e^{m\pi}}{2} \cos m\pi x - \frac{e^{m\pi}}{2} \sin m\pi x + o_{15}(x) \Bigg] \\ &+ i[e^{m\pi} \mathcal{O}(1/k) o_{16}(x) + e^{m\pi x} \mathcal{O}(1/k) o_{17}(x) + o_{18}(x)], \end{aligned} \tag{A.18}$$

where the functions $o_i(\cdot)$ are of the same form as given in (A.15)–(A.16).

Let $\lambda \in \mathbf{C}$ be an eigenvalue of B (see (3.15)), and let $E \in H = V \times L^2(0,1)$ be the corresponding (unnormalized) eigenvector given by

$$E = \begin{pmatrix} \varphi_1(\tau, x) \\ i\tau^2 \varphi_1(\tau, x) \end{pmatrix}; \tag{A.19}$$

see (3.18). The norm of E can be found as

$$\|E\|_H^2 = (|\ \lambda\ |^2 - \lambda^2) \int_0^1 \varphi_1 \bar{\varphi}_1 dx - \alpha\lambda |\ \varphi_1(1)\ |^2, \tag{A.20}$$

where a bar denotes the complex conjugate. Let λ_k and E_k be an eigenvalue, (unnormalized) eigenvector pair. By using (A.18) it easily follows that

$$\int_0^1 (\mathrm{Im}\{\varphi_1\})^2 dx = \mathcal{O}(e^{2k\pi}/(k\pi)^2) \tag{A.21}$$

for k sufficiently large. By using the simple integrals

$$\int_0^1 \cos^2 m\pi x dx = \int_0^1 \sin^2 m\pi x dx = 1/2, \qquad \int_0^1 \sin m\pi x \cos m\pi x dx = \frac{1}{2m\pi}$$

(note that $m = k + 1/2$), it follows from (A.18) that

$$\int_0^1 (Re\{\varphi_1\})^2 dx = C_1 e^{2k\pi} + \mathcal{O}(e^{2k\pi}/(k\pi)) \tag{A.22}$$

for k sufficiently large, where $C_1 > 0$ is a constant. By using (A.9), (A.21), and (A.22) in (A.20) it follows that

$$\|E_k\|_H^2 = C_2 (k\pi)^4 e^{2k\pi} + \mathcal{O}(e^{2k\pi}(k\pi)^3) \tag{A.23}$$

for k sufficiently large, where $C_2 > 0$ is a constant. Hence we define the (approximately) normalized eigenvectors as

$$F_{kr} = \frac{1}{(k\pi)^2 e^{k\pi}} \begin{pmatrix} \varphi_1(\tau_k, x) \\ i\tau_k^2 \varphi_1(\tau_k, x) \end{pmatrix}, \tag{A.24}$$

where τ_k and φ_1 are given by (3.17) and (3.18), respectively.

Now consider the system (3.11)–(3.12) with $\alpha = 0$, i.e., uncontrolled system. By using μ instead of τ, the characteristic equation (3.17) becomes

$$1 + \cosh \mu \cos \mu = 0, \qquad \lambda = i\mu^2, \tag{A.25}$$

whose roots are asymptotically given by

$$\mu_k = m\pi + \mathcal{O}(e^{-m\pi}), \qquad m = k + 1/2 \tag{A.26}$$

for k sufficiently large. It follows that the corresponding function $\varphi_1(\mu_k, x)$ is real. By following the analysis given above, similar to (A.18), we obtain

$$\begin{aligned} \varphi_1(\mu_k, x) &= -\frac{e^{m\pi} e^{-m\pi x}}{2} + e^{m\pi}\mathcal{O}(e^{-m\pi}) o_{19}(x) + e^{m\pi x} o_{20}(x) \\ &\quad + \frac{e^{m\pi}}{2} \cos m\pi x - \frac{e^{m\pi}}{2} \sin m\pi x + o_{21}(x), \end{aligned} \tag{A.27}$$

where the functions $o_i(\cdot)$ are as given in (A.15)–(A.16). Hence, by following the analysis given above, we define the (approximately) normalized eigenvector corresponding to μ_k as

$$G_{kr} = \frac{1}{(k\pi)^2 e^{k\pi}} \begin{pmatrix} \varphi_1(\mu_k, x) \\ i\mu_k^2 \varphi_1(\mu_k, x) \end{pmatrix}. \tag{A.28}$$

THEOREM A.2. *Consider the (approximately) normalized eigenvectors F_k and G_k given by* (A.24) *and* (A.28), *respectively. Then the estimate*

$$\|F_{kr} - G_{kr}\|_H = \mathcal{O}(1/k) \tag{A.29}$$

holds for sufficiently large k.

Proof. From (A.18) and (A.27) it follows that

$$\begin{aligned}\varphi_1(\tau_k, x) - \varphi_1(\mu_k, x) = e^{m\pi}\mathcal{O}(1/k^2)o_{22}(x) + e^{m\pi x}o_{23}(x) + o_{24}(x) \\ + i[e^{m\pi}\mathcal{O}(1/k)o_{25}(x) + e^{m\pi x}\mathcal{O}(1/k)o_{26}(x) + o_{27}(x)],\end{aligned} \tag{A.30}$$

where the functions $o_i(\cdot)$ are as given in (A.15). Also note that

$$\begin{aligned}i\tau_k^2\varphi_1(\tau_k, x) - i\mu_k^2\varphi_1(\mu_k, x) = i\tau_k^2[\varphi_1(\tau_k, x) - \varphi_1(\mu_k, x)] \\ + i(\tau_k^2 - \mu_k^2)\varphi_1(\mu_k, x).\end{aligned} \tag{A.31}$$

From (A.17), (A.30), and (A.31) it follows that

$$\int_0^1 \mid \varphi_{1xx}(\tau_k, x) - \varphi_{1xx}(\mu_k, x) \mid^2 dx = \mathcal{O}(e^{2k\pi}(k\pi)^2), \tag{A.32}$$

$$\int_0^1 \mid \tau_k^2\varphi_1(\tau_k, x) - \mu_k^2\varphi_1(\mu_k, x) \mid^2 dx = \mathcal{O}(e^{2k\pi}(k\pi)^2) \tag{A.33}$$

for k sufficiently large. Hence (A.29) easily follows from (A.32) and (A.33). ∎

Now we consider the algebraic simplicity of the eigenvalues of B for the case $\alpha > 0$ and prove the statement **i** of Theorem 3.3.

LEMMA A.3. *Consider the system given by* (3.11)–(3.12) *for* $\alpha > 0$. *All eigenvalues of B with sufficiently large modulus are algebraically simple.*

Proof. Since the operator B has compact resolvent (see Lemma 3.1), it follows that the spectrum of B consists entirely of isolated points, at most countable, and each eigenvalue has a finite algebraic multiplicity.

Let τ be a root of (3.17), and let $\lambda = i\tau^2$ be the corresponding eigenvalue. From Lemma 3.2 it follows that λ has algebraic multiplicity greater than 1 if and only if $f'(\tau) = 0$; see (3.43).

First note that by using (A.10), (A.13), (A.14), it follows that

$$\cos\tau_k = -(-1)^k\mathcal{O}(1/k^2) - i\left((-1)^k\frac{\alpha}{m\pi} + (-1)^k\mathcal{O}(1/k^3)\right), \tag{A.34}$$

$$\sin\tau_k = (-1)^k + (-1)^k\mathcal{O}(1/k^2) - i((-1)^k\mathcal{O}(1/k^3)). \tag{A.35}$$

By using (A.11), (A.12), (A.34), (A.35) in (3.43) we obtain

$$-i\frac{f'(\tau_k)}{\tau_k^2} = (e^{m\pi}o_1(k) + o_2(k)) + i(-m\pi e^{m\pi}/2 + e^{m\pi}o_3(k) + o_4(k)), \tag{A.36}$$

where $o_i(k)$, $i = 1, \ldots, 4$ are bounded functions of k. Hence it follows that, for sufficiently large k, we have $f'(\tau_k) \neq 0$, which implies that all eigenvalues with sufficiently large modulus are algebraically simple. ∎

Next we prove that, for almost all $\alpha > 0$, the eigenvalues of B are algebraically simple. Moreover the set of $\alpha > 0$, for which there exists at least one eigenvalue which is not algebraically simple, does not contain a limit point, i.e., any such $\alpha > 0$ is necessarily isolated.

Let $F(\tau)$ be defined as

$$F(\tau) = G(\tau) + i\alpha S(\tau), \tag{A.37}$$

where

$$G(\tau) = -\tau(1 + \cosh\tau\cos\tau), \qquad S(\tau) = \sinh\tau\cos\tau - \cosh\tau\sin\tau. \tag{A.38}$$

We know that for a given $\alpha > 0$, $\lambda = i\tau^2$ is an algebraically simple eigenvalue of B if and only if $F(\tau) = 0,\ F'(\tau) \neq 0$, (see Lemma 3.2). Note that we have

$$F'(\tau) = G'(\tau) + i\alpha S'(\tau). \tag{A.39}$$

Also note that if for some $\alpha > 0$ and $\tau \in \mathbf{C}$ we have $F(\tau) = F'(\tau) = 0$, then by eliminating α in (A.37) and (A.39) we obtain $R(\tau) = 0$, where $R(\tau)$ is given by

$$R(\tau) = G'(\tau)S(\tau) - G(\tau)S'(\tau). \tag{A.40}$$

Note that $G(\tau) = 0$ and $S(\tau) = 0$ cannot be satisfied simultaneously. To see that, assume that for some $\tau \in \mathbf{C}$ we have $G(\tau) = S(\tau) = 0$. Then, since $\tau = 0$ is not an eigenvalue, from (A.38) we obtain $\cos\tau = -1/\cosh\tau$, $\sin\tau = -\sinh\tau/\cosh^2\tau$. Then, by using $\sin^2\tau + \cos^2\tau = 1$, we obtain $\cosh\tau = \pm 1$, and then (A.38) implies $\cos\tau = \mp 1$. It can now easily be shown that such a $\tau \in \mathbf{C}$ does not exist. Hence if $F(\tau) = 0$, then both $G(\tau) \neq 0$ and $S(\tau) \neq 0$ must be true.

LEMMA A.4. *Let, for $a > 0$, the sets $\mathcal{C}_a$ and $\mathcal{C}_\infty$ be defined as*

$$\mathcal{C}_a = \{\alpha \in \mathbf{R},\ 0 < \alpha < a | \exists\tau \in \mathbf{C}, F(\tau) = F'(\tau) = 0\}, \tag{A.41}$$

$$\mathcal{C}_\infty = \{\alpha \in \mathbf{R},\ \alpha > 0 | \exists\tau \in \mathbf{C}, F(\tau) = F'(\tau) = 0\}. \tag{A.42}$$

Then

i. *The set $\mathcal{C}_\infty$, if not empty, is at most countable.*

ii. *The set $\mathcal{C}_a$, if not empty, contains finitely many points.*

Proof. **i.** For some $\alpha > 0$ and $\tau \in \mathbf{C}$ we have $F(\tau) = F'(\tau) = 0$. Then we assume $R(\tau) = 0$, where $R(\tau)$ is given by (A.40). Since $R(\tau)$ is a nonconstant analytic function, it follows that its zero set (i.e., the roots of $R(\tau) = 0$) is at most countable; see, e.g., [16, p. 209, Thm. 10.18]. This also shows that the eigenvalues $\lambda = i\tau^2$ which are not algebraically simple also satisfy $R(\tau) = 0$, and hence are independent of α. From (A.37) we obtain

$$\alpha = i\frac{G(\tau)}{S(\tau)}. \tag{A.43}$$

Since there are countably many $\tau \in \mathbf{C}$ for which the eigenvalues $\lambda = i\tau^2$ are not algebraically simple, and since for these τ (A.43) is satisfied, it follows that there are at most countable many values for $\alpha > 0$ such that there exists at least one eigenvalue with algebraic multiplicity greater that one. Hence the set $\mathcal{C}_\infty$ is countable.

ii. Let $a > 0$ be given and let $0 < \alpha < a$. From Lemma A.3 we know that all eigenvalues with sufficiently large modulus are algebraically simple. Hence there exists a $M > 0$ such that, for all $0 < \alpha < a$ and for all eigenvalues $\lambda = i\tau^2$ which are not algebraically simple, we have $\tau \in B(0, M)$, defined by (A.3). Moreover (A.36) implies that $f'(\tau_k) \neq 0$ for sufficiently large k, uniformly with respect to α, for $0 < \alpha \leq a$.

This fact implies that the constant M is independent of α, for $0 < \alpha \leq a$. However such τ must also satisfy $R(\tau) = 0$, where $R(\tau)$ is given by (A.40). Since $B(0, M)$ is a compact set, the number of roots of $R(\tau) = 0$ in $B(0, M)$ must be finite, for otherwise there will be a limit point of zeros of $R(\tau)$ in $B(0, M)$, which is a contradiction; see, e.g., [16, p. 209, Thm. 10.18]. Since in $B(0, M)$ there are at most finitely many candidates of τ for eigenvalues which are not algebraically simple, it follows from (A.43) that the set $\mathcal{C}_a$ also contains finitely many points. ∎

The next corollary now proves assertion **ii** of Theorem 3.3.

Corollary A.5. **i.** *For almost all $\alpha > 0$ the eigenvalues of the operator B given by* (3.15) *are algebraically simple.*

ii. *If for some $\alpha_0 > 0$ and $\tau_0 \in \mathbf{C}$, $\lambda_0 = i\tau_0^2$ is an eigenvalue which is not algebraically simple, then there exists an open set $U \subset \mathbf{R}$ such that $\alpha_0 \in U$, and for $\alpha \in U$, $\alpha \neq \alpha_0$, the eigenvalues of B are algebraically simple.*

Proof. **i.** This follows easily from Lemma A.4.

ii. Note that the right-hand side of (A.43) is an analytic function around any possible $\tau \in \mathbf{C}$ such that the eigenvalue $\lambda = i\tau^2$ is not algebraically simple. Then the result follows from, e.g., [16, p. 216, Thm. 10.32], and from the fact that all eigenvalues with sufficiently large modulus are algebraically simple. ∎

To prove that the generalized eigenfunctions of B form a Riesz basis in $\mathcal{H}$, we need the following simple fact.

Lemma A.6. *Let B be a densely defined closed linear operator in a Hilbert space $\mathcal{H}$. Assume that the spectrum of B consists entirely of, at most countable, isolated points, each of which has a finite algebraic multiplicity. Moreover assume that the eigenvalues are distinct. Then the generalized eigenfunctions are ω-linearly independent* (*for the definition of ω-independence, see, e.g.,*[7, *p.* 316], *or* [18, *p.* 50]).

Proof. Proof of this fact is essentially the same as given in [7, p. 329] for bounded operators. For closed (unbounded) operators with compact resolvent (discrete in the notation of [5]), we may proceed by using [8, p. 178] or [5, pp. 2292–2293] as follows. Let λ_n and ν_n denote the eigenvalues and their algebraic multiplicity of B, respectively. Let ψ_{ij}, $i = 1, 2, \ldots, n, \ldots$, $j = 1, \ldots, \nu_i$, denote the set of generalized eigenfunctions. Since the spectrum of A does not contain an accumulation point, for each λ_i we can find a constant $r_i > 0$ such that the circle $C_i = \{\lambda \in \mathbf{C} | \mid \lambda - \lambda_i \mid = r_i\}$ does not encircle any eigenvalue other than λ_i. It is well known that the operator

$$P_i = \frac{1}{2\pi i}\int_{C_i} (\lambda I - A)^{-1} d\lambda, \tag{A.44}$$

is well defined and is the projection operator onto the generalized eigenspace corresponding to λ_i; see, e.g., [8, p. 178], [5, pp. 2292–2293]. Now consider the following equation:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\nu_i} c_{ij}\psi_{ij} = 0. \tag{A.45}$$

By using the projection operator P_i given by (A.44), we obtain

$$P_i\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\nu_i} c_{ij}\psi_{ij}\right) = \sum_{j=1}^{\nu_i} c_{ij}\psi_{ij} = 0. \tag{A.46}$$

Since $\nu_i < \infty$ and the generalized eigenfunctions are linearly independent, it follows from (A.46) that $c_{ij} = 0$, $j = 1, \dots, \nu_i$. Since this is true for each $i \in \mathbf{N}$, it follows that the generalized eigenfunctions are ω-linearly independent. ∎

THEOREM A.7. *Let $\alpha > 0$ be given and assume the eigenvalues of the operator B are all algebraically simple (note that this condition holds for almost all $\alpha > 0$; see* Corollary A.5). *Then the set of eigenvectors of B forms a Riesz basis for $\mathcal{H}$.*

Proof. Let F_{kr} and G_{kr} be given by (A.24) and (A.28), respectively. Note that F_{kr} and G_{kr} are the (appropriately) normalized eigenvectors of the operator B, corresponding to given $\alpha > 0$ and $\alpha = 0$, respectively. We note that by Lemma A.1, it is possible to enumerate these eigenvectors similarly, and because of algebraic simplicity we consider only the eigenvectors and not the generalized eigenvectors. This point is important in Theorum 3.5 and Theorum 3.6 in proving the spectrum-determined growth property, which is our main aim.

From Theorum A.2 it follows that for some N we have

$$\sum_{|k|>N} \|F_{kr} - G_{kr}\|^2_{\mathcal{H}} < \infty; \tag{A.47}$$

see (A.29). Since $N < \infty$, it follows that

$$\sum_{k \in \mathbf{Z}} \|F_{kr} - G_{kr}\|^2_{\mathcal{H}} < \infty. \tag{A.48}$$

Hence the set of vectors $\{F_{kr}\}$ is quadratically close to the set of vectors $\{G_{kr}\}$. It is well known that the latter set of vectors forms a Riesz basis for $\mathcal{H}$, since for $\alpha = 0$ the operator B becomes a skew adjoint operator. Also by Lemma A.6, the former set of vectors is ω-linearly independent. This implies that the set of vectors $\{F_{kr}\}$ also forms a Riesz basis in $\mathcal{H}$; see, e.g., [18, p. 347, Thm. 11.3]. ∎

Remark 3. The requirement that the eigenvalues of B for $\alpha > 0$ be algebraically simple is not essential and could be relaxed. Let $\alpha > 0$, and let $\lambda \in \mathbf{C}$ be a root of (A.1), i.e., an eigenvalue of B. It is not known a priori whether the multiplicity of λ as a root of (A.1) and the algebraic multiplicity of λ as an eigenvalue of B are the same. Let us assume that these two multiplicities are the same, and let the set of vectors $\{F_{kr}\}$ include all eigenvectors and the generalized eigenvectors of B. Then by using Lemma A.1, Lemma A.3, Theorem A.2, and Theorem A.7, we conclude that the sets $\{F_{kr}\}$ and $\{G_{kr}\}$ are quadratically close; i.e., (A.48) holds. Hence the set $\{F_{kr}\}$ also forms a Riesz basis in $\mathcal{H}$, and the spectrum-determined growth property stated in Theorum 3.5 and Theorum 3.6 holds. The assumption on the equality of the multiplicities stated above seems to be true; however, the proof of this statement could be rather tedious. If we assume algebraic simplicity, which is generic (i.e., holds for almost all $\alpha > 0$), then these two multiplicities are the same; see Lemma 3.2. This is the basic reason for the assumption on algebraic simplicity.

COROLLARY A.8. *There exists an $a > 0$ such that, for all $0 < \alpha < a$, the set of eigenfunctions of B forms a Riesz basis for $\mathcal{H}$.*

Proof. This fact was proven in [4]. Here we may obtain this result as a corollary by using Lemma A.4, part ii, and Theorem A.7. ∎

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