ON THE STABILIZATION OF LINEAR SYSTEMS

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1. In a recent paper V. N. Romanenko [3] has given a necessary and sufficient condition that the system

$$\frac{dx}{dt} = Ax + bu, \qquad \frac{du}{dt} = px + qu \tag{1.1}$$

be "stabilizable." Here A is an n by n matrix, x and b are n by 1 column matrices (or vectors), p is a 1 by n row matrix and q and u are scalars. We shall assume that the elements of all these may be complex numbers. The vector x can be interpreted physically as the output of a linear system characterized by the matrix A. The vector b corresponds to some feedback or control mechanism with u the controlling signal and p and q adjustable parameters in the controlling circuit. Romanenko calls the system (A, b) *stabilizable* if for any nonempty set S of $n+1$ or less complex numbers there exist p and q such that

$$G = \begin{pmatrix} A & b \\ p & q \end{pmatrix} \tag{1.2}$$

has S as its set of characteristic values (spectrum). In particular then, if (A, b) is stabilizable there exist p and q such that all characteristic values of G have negative real parts and every solution of (1.1) is such that $x(t) \to 0$ and $u(t) \to 0$ as $t \to +\infty$.

In his paper Romanenko claims to be generalizing a known condition, which he attributes to Yu. M. Berezanskiĭ, namely, that if the characteristic values of A are all distinct, then (A, b) is stabilizable if and only if

$$b,\ Ab,\ \cdots,\ A^{n-1}b \text{ are linearly independent.} \tag{1.3}$$

Romanenko's condition appears to be considerably more complicated than (1.3) while it is our intention here to show that in fact (1.3) is necessary and sufficient for (A, b) to be stabilizable irrespective of any condition on the characteristic values of A. Actually this is a corollary to our more general Theorem 1 given below.

As pointed out to the author by Dr. J. P. LaSalle the condition (1.3) and related ones have considerable significance in a seemingly

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different aspect of control theory, namely, the controllability of linear dynamical systems [2]. Undoubtedly there is a deeper connection between stabilizability and controllability. At any rate a certain canonical form developed in the study of the latter concept provides a simplification of our original proof and a generalization of the Berezanskiĭ-Romanenko result.

2. Rather than (1.2) we consider matrices of the form

$$G = \begin{pmatrix} A & B \\ P & Q \end{pmatrix} \tag{2.1}$$

where A is n by n, Q is m by m and B and P are correspondingly sized submatrices. We assume that the elements of all these may be complex. By S_r we denote a nonempty set of r or less complex numbers μ_k; that is, $S_r = \{\mu_k \mid k = 1, 2, \cdots, r\}$, where the μ_k need not all be distinct. The rank h of the matrix $H = (B, AB, \cdots, A^{n-1}B)$ is the significant feature of our results, the first of which is

THEOREM 1. *The condition $h \geqq r - m$ is necessary and sufficient that for each S_r there exist P and Q such that the spectrum of G contains S_r.*

PROOF OF NECESSITY. Take any set S_r such that the μ_k in S_r are distinct and different from any characteristic value of A, and let P and Q be such that the spectrum of G contains this S_r. Then there exist vectors

$$\begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}, \qquad k = 1, 2, \cdots, r,$$

where ξ_k and η_k are n by 1 and m by 1 column matrices, respectively, such that

$$\begin{pmatrix} A & B \\ P & Q \end{pmatrix} \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix} = \mu_k \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}$$

or

$$\begin{aligned} A\xi_k + B\eta_k &= \mu_k \xi_k, \\ P\xi_k + Q\eta_k &= \mu_k \eta_k, \end{aligned} \qquad k = 1, 2, \cdots, r. \tag{2.2}$$

From the first of (2.2) we may write

$$\xi_k = (\mu_k I - A)^{-1} B\eta_k \tag{2.3}$$

since $A - \mu_k I$ is nonsingular. Using the characteristic equation of A, one may write

$$(\mu_k I - A)^{-1} = \sum_{j=1}^{n} c_j(\mu_k) A^{j-1} \tag{2.4}$$

for suitable scalar functions $c_j(\mu)$. Substituting (2.4) into (2.3), we have

$$\xi_k = \sum_{j=1}^{n} A^{j-1} B c_j(\mu_k)\eta_k = H\zeta_k, \qquad k = 1, 2, \cdots, r, \tag{2.5}$$

where the transpose of the nm by 1 column matrix ζ_k is defined by

$$\zeta_k' = (c_1(\mu_k)\eta_k', c_2(\mu_k)\eta_k', \cdots, c_n(\mu_k)\eta_k').$$

Now by choice of S_r the vectors

$$\begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}, \qquad k = 1, 2, \cdots, r,$$

are linearly independent and it is readily seen that the rank of the matrix $(\xi_1, \xi_2, \cdots, \xi_r)$ must therefore be at least $r-m$. From (2.5) it is then clear that the rank of H must likewise be at least $r-m$.

Proof of sufficiency. The proof is accomplished by means of a similarity transformation on G to convert A and B to convenient canonical forms. We first dispose of the case $B=0$, however. In this event $h=0$ and the condition $h \geqq r-m$ becomes $m \geqq r$. It is clear then that we may choose P arbitrarily and Q such that the spectrum of Q and hence also that of G contains S_r. Henceforth, then, we assume $B \neq 0$.

Consider now a matrix J of the form

$$J = \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \tag{2.6}$$

where I is an n by n identity and R is a nonsingular m by m matrix. Then

$$JGJ^{-1} = \begin{pmatrix} A & BR^{-1} \\ RP & PQR^{-1} \end{pmatrix} \tag{2.7}$$

and it is evident that we may achieve any reordering of the columns of B with no essential change in the statement of the proposition to be proved. Thus if the columns of B are denoted by b_i, $i = 1, 2, \cdots, m$, we may assume without loss of generality that the set of h columns of H

$$b_1, Ab_1, \cdots, A^{h_1-1}b_1, \cdots, b_s, Ab_s, \cdots, A^{h_s-1}b_s \tag{2.8}$$

is linearly independent. Here $1 \leqq s \leqq m$, $h_i \geqq 1$, $i=1, 2, \cdots, s$, and $\sum_{i=1}^{s} h_i = h$. We may further assume (see Chapter VII of [1]) that the sequence (2.8) is such that for $i \geqq 2$ the linear subspace V_i spanned by $A^{j-1}b_i$, $j=1, 2, \cdots, h_i$, is invariant modulo $V_1 + V_2 + \cdots + V_{i-1}$ under premultiplication by A. That is,

$$A^{h_i}b_i = \sum_{j=1}^{h_i} \alpha_{ij} A^{j-1} b_i + \sum_{k=1}^{i-1} \sum_{j=1}^{h_k} \beta_{ijk} A^{j-1} b_k, \tag{2.9}$$

for some set of scalars α_{ij}, β_{ijk} and where the double summation in (2.9) does not appear if $i=1$.

We now introduce a similarity transformation on G by means of a matrix K of the form

$$K = \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}, \tag{2.10}$$

where I is an m by m identity and T is a nonsingular n by n matrix. Then

$$KGK^{-1} = \begin{pmatrix} TAT^{-1} & TB \\ PT^{-1} & Q \end{pmatrix}. \tag{2.11}$$

The matrix T is defined by choosing certain combinations of the vectors in (2.8) as a new basis system for the space of n by 1 column matrices. Thus we introduce

$$e_{ik} = A^{h_i-k}b_i - \sum_{j=k+1}^{h_i} \alpha_{ij} A^{j-k-1} b_i, \qquad \begin{matrix} k = 1, 2, \cdots, h_i, \\ i = 1, 2, \cdots, s, \end{matrix} \tag{2.12}$$

and where the summation does not appear in case $k=h_i$. That is

$$e_{ih_i} = b_i, \qquad i = 1, 2, \cdots, s. \tag{2.13}$$

Note from (2.12) and (2.13) we have for $k \geqq 2$

$$Ae_{ik} = e_{i,k-1} + \alpha_{ik} e_{ih_i}, \qquad i = 1, 2, \cdots, s, \tag{2.14}$$

and, using (2.9) in addition, we have, since the $A^{j-1}b_i$'s may be expressed as linear combinations of the e_{ij}'s,

$$Ae_{i1} = \alpha_{i1} e_{ih_i} + \sum_{k=1}^{i-1} \sum_{j=1}^{h_k} \gamma_{ijk}\, e_{kj}, \qquad i = 1, 2, \cdots, s, \tag{2.15}$$

for some set of scalars γ_{ijk} and where, again, the double summation is absent in the case $i=1$.

From the linear independence of the set (2.8) and the remark just preceding (2.15) it is evident that the set

$$e_{11}, \cdots, e_{1h_1}, e_{21}, \cdots, e_{2h_2}, \cdots, e_{s1}, \cdots, e_{sh_s} \tag{2.16}$$

is likewise linearly independent. If $h = \sum_{i=1}^{s} h_i < n$ we may adjoin to the set (2.16) $n-h$ additional vectors to get a set which spans the space of n by 1 matrices. We now view the matrix A as defining the linear transformation $x \to Ax$ and we let T be defined so that for any n by 1 matrix x the matrix Tx is the column of components of x relative to the set (2.16) (augmented if required) as basis. With respect to this basis the matrix of the linear transformation $x \to Ax$ is the matrix TAT^{-1} and, moreover, $Tb_i = u_{\sigma_i}$, $\sigma_i = h_1 + \cdots + h_i$ and u_σ is an n by 1 column all of whose components are zero except the σth which is 1. From (2.14) and (2.15) we may thus infer that TAT^{-1} has the following block form:

$$TAT^{-1} = \begin{pmatrix} C_1 & E_{12} \cdots E_{1s} & F_1 \\ 0 & C_2 \cdots E_{2s} & F_2 \\ \cdot & \cdot \quad\quad \cdot & \cdot \\ \cdot & \cdot \quad\quad \cdot & \cdot \\ \cdot & \cdot \quad\quad \cdot & \cdot \\ 0 & 0 \quad \cdots C_s & F_s \\ 0 & 0 \quad \cdots 0 & L \end{pmatrix}, \tag{2.17}$$

where C_i has the canonical form

$$C_i = \begin{pmatrix} 0 & 1 & \cdots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \cdots 1 \\ \alpha_{i1} & \alpha_{i2} & \cdots \alpha_{ih_i} \end{pmatrix}. \tag{2.18}$$

Moreover,

$$TB = \begin{pmatrix} v_1 & 0 \cdots 0 & w_1 \\ 0 & v_2 \cdots 0 & w_2 \\ \cdot & \cdot \quad\quad \cdot & \cdot \\ \cdot & \cdot \quad\quad \cdot & \cdot \\ \cdot & \cdot \quad\quad \cdot & \cdot \\ 0 & 0 \cdots v_s & w_s \\ 0 & 0 \cdots 0 & 0 \end{pmatrix}, \tag{2.19}$$

where v_i is an h_i by 1 column all of whose components are zero except the last which is 1. The submatrices w_i have $m-s$ columns.

We now choose P and Q in convenient form. Let

$$Q = \operatorname{diag}(q_1, q_2, \cdots, q_m)$$

and let P be such that

$$(2.20)\qquad PT^{-1} = \begin{pmatrix} p_1 & 0 & \cdots 0 & 0 \\ 0 & p_2 & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots p_s & 0 \\ 0 & 0 & \cdots 0 & M \end{pmatrix},$$

where p_i is a 1 by h_i row and M is $m-s$ by $n-h$. The determinant of $KGK^{-1}-\lambda I$ is now readily evaluated by means of Laplace expansions using minors from appropriate columns. Thus, if we expand by minors from the first h_1 columns along with the $(n+1)$st column we find that only one of these is nonzero, namely, that using the first h_1 rows and the $(n+1)$st row. The complementary minor will be a determinant array with exactly similar block structure to that of $KGK^{-1}-\lambda I$. It may thus be expanded similarly by using the corresponding columns, namely, those which appear in $KGK^{-1}-\lambda I$ as the second h_2 columns along with the $(n+2)$nd. Continuing in this way we may write

$$(2.21)\quad \det(KGK^{-1} - \lambda I) = \begin{vmatrix} L - \lambda I & 0 \\ M & Q^* - \lambda I \end{vmatrix} \cdot \prod_{i=1}^{s} \begin{vmatrix} C_i - \lambda I & v_i \\ p_i & q_i - \lambda \end{vmatrix},$$

where $Q^* = \operatorname{diag}(q_{s+1}, \cdots, q_m)$ and, of course, each appearance of I denotes an identity of appropriate size.

First we observe that Q^* may be chosen so that the spectrum of G contains any prescribed set S_{m-s}. We next examine the determinants in the product factor in (2.21). Each of these has the same structure and may readily be evaluated by a Laplace expansion using minors from the last two rows. The result is

$$\begin{vmatrix} C_i - \lambda I & v_i \\ p_i & q_i - \lambda \end{vmatrix} = (-\lambda)^{h_i+1} + (q_i - \alpha_{ih_i})(-\lambda)^{h_i} + \sum_{j=2}^{h_i} (\alpha_{i,j-1} + q_i\alpha_{ij} - p_{ij})(-\lambda)^{j-1} + (q_i\alpha_{i1} - p_{i1}),$$

where the p_{ij}, $j=1, 2, \cdots, h_i$, are the components of the row p_i. In this form it is evident that we may determine q_i and the p_{ij}, $j=1, 2, \cdots, h_i$, so that the coefficients of this polynomial are any we desire. Thus q_i and p_i may be determined so that this factor of $\det(KGK^{-1}-\lambda I)$ has any given collection of h_i+1 roots. This is true for each i so using this and the fact mentioned earlier regarding the choice of Q^* it is clear that we may specify P and Q so that G contains in its spectrum any given nonempty set of $m-s+\sum_{i=1}^{s}(h_i+1)$ $=h+m$ or less complex numbers. Thus if $h \geqq r-m$ we see that for any S_r there exist P and Q such that G contains S_r in its spectrum.

COROLLARY 1. *The condition $h=n$ is necessary and sufficient that for each S_{n+m} there exist P and Q such that G has S_{n+m} as its spectrum.*

PROOF. In any case $h \leqq n$, so if $r=n+m$, then the condition $h \geqq r-m$ is equivalent to $h=n$.

COROLLARY 2. *Condition* (1.3) *is necessary and sufficient that* (A, b) *be stabilizable.*

PROOF. This is Corollary 1 in the case $m=1$.

REMARK. Even in case $h<n$ there may exist P and Q such that all characteristic roots of G have negative real parts. It is clear from (2.21) that this is the case provided the characteristic roots of L have negative real parts which, in turn, is related to how the linear subspace complementary to that spanned by the columns of H is associated with those characteristic roots of A which have negative real parts. In any case this question does not appear to be directly answerable merely in terms of the rank of H.

3. In this section as an application of the results in §2 we point out the relevancy of condition (1.3) to the behavior of a more complicated system of differential equations than (1.1). Thus we consider the system

$$\begin{aligned} \frac{dx}{dt} &= Ax + bu, \\ \frac{d^{r+1}u}{dt^{r+1}} &= px + \sum_{k=0}^{r} q_k \frac{d^k u}{dt^k}, \end{aligned} \tag{3.1}$$

where A, b, x, u, p are as before and q_k, $k=0, 1, 2, \cdots, r$, are scalars.

THEOREM 2. *For any integer $r \geqq 0$, if condition* (1.3) *holds, then there exist p, q_k, $k=0, 1, \cdots, r$, such that, for every solution of* (3.1), *$x(t) \to 0$ and $u(t) \to 0$ as $t \to +\infty$.*

PROOF. Introduce variables u_k by the relations $u_0=u$,

$$(3.2)\qquad u_k = \frac{du_{k-1}}{dt}, \qquad k = 1, 2, \cdots, r.$$

Then (3.1) may be written in the equivalent matrix form

$$(3.3)\quad \frac{d}{dt}\begin{pmatrix} x \\ u_0 \\ \cdot \\ \cdot \\ \cdot \\ u_{r-2} \\ u_{r-1}\end{pmatrix} = \begin{pmatrix} A & b & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}\begin{pmatrix} x \\ u_0 \\ \cdot \\ \cdot \\ \cdot \\ u_{r-2} \\ u_{r-1}\end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1\end{pmatrix} u_r,$$

$$\frac{du_r}{dt} = (p, q_0, \cdots, q_{r-1})(x', u_0, \cdots, u_{r-1})' + e_r u_r.$$

This is the form of (1.1) with $(x', u_0, \cdots, u_{r-1})'$ playing the role of x' there, u_r the role of u, $(p, q_0, \cdots, q_{r-1})$ the role of p and

$$(3.4)\qquad A^* = \begin{pmatrix} A & b & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & \cdot & & & & & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \quad \text{and} \quad b^* = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1\end{pmatrix}$$

playing the roles of A and b, respectively. From Corollary 2 $p, q_0, \cdots, q_r$ exist such that $x(t)\to 0$, $u_k(t)\to 0$, $k=0, 1, \cdots, r$, as $t\to +\infty$ if $b^*, A^*b^*, A^{*2}b^*, \cdots, A^{*n+r-1}b^*$ are linearly independent. From the form of A^* and b^* as given in (3.4) it is easy to verify that this is true if condition (1.3) holds.

Analogous applications of Corollary 1 may be made.

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