

## ON THE STABLE JAMES NUMBERS OF COMPLEX PROJECTIVE SPACES

HIDEAKI ŌSHIMA

(Received November 14, 1973)

### 1. Introduction

For a pointed finite CW-pair  $i: A \subset X$  where  $A$  is a connected oriented topological manifold, a (stable) map  $f: X \rightarrow A$  is of type  $r$  if the composite  $A \xrightarrow{i} X \xrightarrow{f} A$  has degree  $r$ .  $j(X, A)$  and  $j_s(X, A)$  denote the sets of integers  $r$  for which there exists a map  $f: X \rightarrow A$  of type  $r$  and a stable map of type  $r$  respectively. When  $j(X, A)$  forms an ideal  $(k(X, A))$  in the ring of integers  $Z$ —here  $k(X, A)$  denotes the non-negative generator, we call  $k(X, A)$  the James number of the pair  $(X, A)$ . In the stable case  $j_s(X, A)$  is always an ideal of  $Z$ . So we may define the stable James number  $k_s(X, A)$ .

James [3] has posed the problem of determining  $j(SP^m(S^n), S^n)$ , where  $SP^m(S^n)$  is the  $m$ -fold symmetric product of an  $n$ -sphere  $S^n$  with a base point  $x_0$  and  $i: S^n \rightarrow SP^m(S^n)$  is the axial embedding  $x \rightarrow [x, x_0, \dots, x_0]$ . James showed for example  $j(SP^m(S^n), S^n)$  forms an ideal of  $Z$  and, for an even dimensional sphere  $S^{2n}$ ,  $k(SP^m(S^{2n}), S^{2n}) = 0$ . On the contrary  $k_s(SP^m(S^n), S^n) \neq 0$  for any positive integers  $m$  and  $n$ . From now on we introduce the notation  $k_s^{m,n}$  instead of  $k_s(SP^m(S^n), S^n)$ .

In this note we give lower bounds and an upper bound of  $k_s^{m,2}$ . That is, we prove

**Theorem.** For positive integers  $m$  and  $n$

- (1)  $k_s^{m,n} \neq 0$ ;
- (2)  $k_s^{m+1,n}$  is a multiple of  $k_s^{m,n}$ ;
- (3)  $k_s^{m,2}$  is divisible by all the integers  $m, m-1, \dots, 2$ ;
- (4)  $k_s^{2^m-1,2}$  is divisible by  $2^m$  for  $m \geq 2$ ;
- (5)  $k_s^{m,2}$  is a divisor of  $m!(m-1)! \cdots 2!$ , in particular none of the prime factors of  $k_s^{m,2}$  is greater than  $m$ .

**Corollary.** The above lower estimates (3) and (4) are best possible for  $m \leq 4$ . That is

$$k_s^{1,2} = 1, k_s^{2,2} = 2, k_s^{3,2} = k_s^{4,2} = 12.$$

There is a homeomorphism  $SP^m(S^2) \simeq CP^m$ , the  $m$ -dimensional complex projective space. Under this identification the natural inclusions  $S^2 \subset SP^m(S^2) \subset SP^{m+1}(S^2)$  become the standard ones  $CP^1 \subset CP^m \subset CP^{m+1}$ .  $k_s^{m,2}$  is just the same as Conner-Smith's  $d(m)$  [1], Example 4.

The author wishes to express his thanks to Professor S. Araki for his kind advices.

**2. Proofs of (1) and (2)**

Using the group multiplication of  $S^1$ , we know  $k_s^{m,1} = 1$ . So we assume  $n \geq 2$ . First we prove (1). Consider the stable Puppe exact sequence

$$\cdots \rightarrow \{SP^m(S^n), S^n\} \xrightarrow{i^*} \{S^n, S^n\} \rightarrow \{SP^m(S^n)/S^n, S^{n+1}\} \rightarrow \cdots,$$

here  $\{X, Y\}$  denotes the set of stable homotopy classes of stable maps  $X \rightarrow Y$ . Since  $\{S^n, S^n\} \cong Z$ , if  $i^*$  is non-trivial, then  $k_s^{m,n} = \text{index of image } i^* \neq 0$ . So, for our purpose, it suffices to show that  $\{SP^m(S^n)/S^n, S^{n+1}\}$  is finite. Notice that  $\{SP^m(S^n)/S^n, S^{n+1}\} = \pi_s^{n+1}(SP^m(S^n)/S^n)$  is the reduced framed cobordism group. Let  $E_2^{u,v} = \hat{H}^u(SP^m(S^n)/S^n; G_{-v}) \Rightarrow \pi_s^*(SP^m(S^n)/S^n)$  be the Atiyah-Hirzebruch spectral sequence for  $SP^m(S^n)/S^n$ , where  $G_k$  is the stable  $k$ -stem of spheres. Since  $\hat{H}^u(SP^m(S^n)/S^n; Z) = 0$  for  $u \leq n+1$ ,  $\sum_{u+v=n+1} E_2^{u,v}$  is finite. Then  $\sum_{u+v=n+1} E_\infty^{u,v}$  and hence  $\pi_s^{n+1}(SP^m(S^n)/S^n)$  are finite. This implies (1).

From the equality  $k_s^{m,n} = \text{index of image } i^*$ , (2) is obvious. Thus (1) and (2) follow.

**3. Proof of (3)**

We use the complex  $K$ -theory. Let  $\eta_m$  be the canonical complex line bundle over  $CP^m$  and  $g_C = \eta_1 - 1 \in \tilde{K}(S^2)$  be the Bott generator. Then  $K(CP^m)$  is the truncated polynomial ring with generator  $\eta_m - 1$  and the relation  $(\eta_m - 1)^{m+1} = 0$ . Choose  $f \in \{CP^m, S^2\}$  such that  $i^*(f) = k_s^{m,2} \iota$ , where  $\iota$  denotes the identity map of  $S^2$ . Let  $f^*: K^*(S^2) \rightarrow K^*(CP^m)$  and  $f^*: H^*(S^2; Q) \rightarrow H^*(CP^m; Q)$  be the induced homomorphisms. Put

$$f^*(g_C) = \sum_{j=1}^m a_j (\eta_m - 1)^j$$

where  $a_j \in Z$ . Since  $i^*(f) = k_s^{m,2} \iota$ , we have  $a_1 = k_s^{m,2}$ . Let  $t \in H^2(CP^m; Z)$  be the first Chern class of  $\eta_m$ . We apply the Chern character,  $ch$ , for  $\tilde{K}^*(CP^m)$  and  $\tilde{K}^*(S^2)$ . Then

$$a_1 t = k_s^{m,2} t = (f^* \circ ch)(g_C) = (ch \circ f^*)(g_C) = \sum_{j=1}^m a^j (\exp(t) - 1)^j$$

that is

$$t = \sum_{j=1}^m (a_j/a_1)(\exp(t)-1)^j$$

in  $H^*(CP^m; Q)$ , where we used the fact that  $ch$  is “stable”. On the other hand

$$t = \log(1+(\exp(t)-1)) = \sum_{j=1}^{\infty} ((-1)^{j+1}/j)(\exp(t)-1)^j.$$

Hence

$$k_s^{m,2} = a_1 = (-1)^{j+1}j a_j; j = 1, 2, \dots, m.$$

This implies (3).

#### 4. Proof of (4)

In this section we use  $KO$ -theory. We introduce the following notations:  $\eta_H$  = the canonical symplectic line bundle over  $S^4$ ;  $g_H = \eta_H - 1 \in \widetilde{KSp}(S^4)$ ;  $g_R = g_H \wedge g_H \in \widetilde{KO}(S^6)$ ;  $\rho: K^*( ) \rightarrow KO^*( )$ , the real restriction;  $\varepsilon: KO^*( ) \rightarrow K^*( )$ , the complexification;  $\mu_3 = \rho(g_C^3 \wedge (\eta_m - 1)) \in \widetilde{KO}^{-6}(CP^m)$ ;  $\mu_0 = \rho(\eta_m - 1) \in \widetilde{KO}(CP^m)$ . We require the following theorem of Fujii [2]:

$\widetilde{KO}^{-6}(CP^m)$  is the free module with basis  $\mu_3, \mu_3\mu_0, \dots, \mu_3\mu_0^{u-1}$ , and also, in case  $m$  is odd,  $\mu_3\mu_0^u$  (if  $m \equiv 3 \pmod{4}$ ) or  $\tau$  (if  $m \equiv 1 \pmod{4}$ ), where  $2\tau = \mu_3\mu_0^u$  and  $u = [m/2]$  ( $[ ]$  is the Gauss notation).

Choose  $f \in \{CP^m, S^2\}$  such that  $i^*(f) = k_s^{m,2}i$ . Let  $f^*: KO^*(S^2) \rightarrow KO^*(CP^m)$  be the induced homomorphism. By Fujii's theorem we may write

$$f^*(g_R) = \begin{cases} \sum_{j=0}^{\binom{m/2}{2}-1} a_j \mu_3 \mu_0^j & \text{if } m \equiv 0 \pmod{2} \\ \sum_{j=0}^{\lfloor m/2 \rfloor} a_j \mu_3 \mu_0^j & \text{if } m \equiv 3 \pmod{4} \\ \sum_{j=0}^{\lfloor m/2 \rfloor - 1} a_j \mu_3 \mu_0^j + a_{\lfloor m/2 \rfloor} \tau & \text{if } m \equiv 1 \pmod{4} \end{cases}$$

where  $a_j \in Z$ . In case  $m=1$ , we have  $\mu_3 = 2g_R \in \widetilde{KO}^{-6}(S^2)$ . This and  $i^*(f) = k_s^{m,2}i$  imply  $2a_0 = k_s^{m,2}$ . We write  $ch$  for  $ch \circ \varepsilon$ . Then we have

$$ch(\mu_3) = \exp(t) - \exp(-t) = 2 \sinh(t)$$

and

$$ch(\mu_0) = \exp(t) + \exp(-t) - 2 = 2(\cosh(t) - 1).$$

Since

$$2a_0 t = k_s^{m,2} t = (f^* \circ ch)(g_R) = (ch \circ f^*)(g_R),$$

we obtain

$$(\#) a_0 t = \begin{cases} \sinh(t) \sum_{j=0}^{(m/2)-1} 2^j a_j (\cosh(t)-1)^j & \text{if } m \equiv 0 \pmod 2 \\ \sinh(t) \sum_{j=0}^{\lfloor m/2 \rfloor} 2^j a_j (\cosh(t)-1)^j & \text{if } m \equiv 3 \pmod 4 \\ \sinh(t) \left\{ \sum_{j=0}^{\lfloor m/2 \rfloor - 1} 2^j a_j (\cosh(t)-1)^j \right. \\ \quad \left. + 2^{\lfloor m/2 \rfloor - 1} a_{\lfloor m/2 \rfloor} (\cosh(t)-1)^{\lfloor m/2 \rfloor} \right\}, & \text{if } m \equiv 1 \pmod 4 \end{cases}$$

in  $H^*(CP^m; Q)$ . In case  $m \equiv 0 \pmod 2$ , if we differentiate the two sides of (#) by  $t$ , then we have

$$j! a_0 = (-1)^j 2^j \cdot 3 \cdot 5 \cdots (2j+1) a_j \quad \text{for } 1 \leq j \leq m/2 - 1,$$

and elementary calculation shows that we obtain the same information on  $k_s^{m,2} = 2a_0$  as (3). This and (#) imply that we obtain the same information about  $k_s^{4j+1,2}$  and  $k_s^{4j+1,2}$  for  $j \geq 1$ . Hence, in case  $m \equiv 1 \pmod 4$ , we obtain nothing more than (3). In case  $m \equiv 3 \pmod 4$ , that is  $m = 4j - 1$  for some  $j$ , we have the same information about  $k_s^{4j-1,2}$  and  $k_s^{4j,2}$ . If  $j$  is a power of two,  $2^q$ , from (3) we see that  $2^{q+1}$  divides  $k_s^{2^{q+2}-1,2}$ , but the aboves imply that  $2^{q+2}$  divides  $k_s^{2^{q+2}-1,2}$ . Thus (4) follows. Remark, in case  $j$  is not a power of two, we obtain nothing more than (3).

### 5. Proofs of (5) and Corollary

Choose  $f_{m-1} \in \{CP^{m-1}, S^2\}$  such that  $i^*(f_{m-1}) = k_s^{m-1,2} \iota$ . Let  $p_{m-1}: S^{2m-1} \rightarrow CP^{m-1}$  be the canonical fibration and  $\text{ord}(p_{m-1})$  be its order as a stable map. The composite  $(\text{ord}(p_{m-1}))\iota \circ f_{m-1} \circ p_{m-1}$  is null homotopic. Hence there exists  $f \in \{CP^m, S^2\}$  such that  $f \circ j = (\text{ord}(p_{m-1}))\iota \circ f_{m-1} \in \{CP^{m-1}, S^2\}$ , where  $j: CP^{m-1} \subset CP^m$ . This implies that  $k_s^{m,2}$  is a divisor of  $\text{ord}(p_{m-1}) \cdot k_s^{m-1,2}$ . Inductively we know that  $k_s^{m,2}$  is a divisor of  $\text{ord}(p_{m-1}) \cdot \text{ord}(p_{m-2}) \cdots \text{ord}(p_1) k_s^{1,2}$ . Obviously  $k_s^{1,2} = 1$ . By Toda [4], page 1103,  $\text{ord}(p_{m-1})$  is a divisor of  $m!$ . Thus (5) follows. And we complete the proof of Theorem.

We prove Corollary. For  $m \leq 3$ , the estimates (3), (4) and (5) imply that  $k_s^{1,2} = 1$ ,  $k_s^{2,2} = 2$  and  $k_s^{3,2} = 12$ . We show  $k_s^{4,2} = 12$ . Choose  $f_3 \in \{CP^3, S^2\}$  such that  $i^*(f_3) = 12\iota$ . The composite  $f_3 \circ p_3: S^7 \rightarrow S^2$  represents an element of  $G_5$ , five-stem of spheres. It is well known that  $G_5 = 0$ . Hence there exists  $f \in \{CP^4, S^2\}$  such that the composite  $CP^3 \subset CP^4 \xrightarrow{f} S^2$  coincides with  $f_3$ . This implies that  $k_s^{4,2}$  is a divisor of 12. By (2)  $k_s^{4,2}$  is a multiple of  $k_s^{3,2} = 12$ . Therefore  $k_s^{4,2} = 12$ . This completes the proof of Corollary.

### 6. Addendum

The same technique is applicable to the stable James number  $d_H(m) = k_s$

$(HP^m, S^4)$  of the pair of symplectic projective spaces. Using the complex  $K$ -theory, a lower bound of  $d_H(m)$  can be obtained from

$$a_1 t = \sum_{j=1}^m 2^j a_j (\sum_{k=1}^{\infty} t^k / (2k)!)^j, \quad a_1 = d_H(m),$$

where  $t \in H^4(HP^m; Z)$  is a generator and  $a_j \in Z$ . For example, we have  $12 | d_H(2)$ . Since the order of the canonical fibration  $S^7 \rightarrow S^4$  as a stable map is 24, we have  $d_H(2) = 24$ . So that this estimate is not best possible.

The unstable James numbers of the pairs  $(RP^m, S^1)$ ,  $(CP^m, S^2)$ ,  $(HP^m, S^4)$  and the stable James number of  $(RP^m, S^1)$  are all zero for  $m \geq 2$ , where  $RP^m$  denotes the  $m$ -dimensional real projective space.

OSAKA CITY UNIVERSITY

**References**

[1] P.E. Conner and L. Smith: *On the complex bordism of finite complexes*, II, J. Differential Geometry **6** (1971), 135–174.  
 [2] M. Fujii:  *$K_0$ -groups of projective spaces*, Osaka J. Math. **4** (1967), 141–149.  
 [3] I.M. James: *Symmetric functions of several variables, whose range and domain is a sphere*, Bol. Soc. Mat. Mexicana. **1** (1956), 85–88.  
 [4] H. Toda: *On unstable homotopy of spheres and classical groups*, Proc. Nat. Acad. Sci. USA **46** (1960), 1102–1105.

*Added in proof.* After completed this manuscript, the author has found a paper of J. Ucci “Symmetric maps of spheres of least positive James number, Indiana Univ. Math. J. (1972), 709–714” which gives an upper bound of unstable James numbers  $k^{m,n} = k(SP^m(S^n), S^n)$ . Combining his estimate with ours, we obtain

**Theorem A.**

- (i)  $\beta_2(m) \leq \nu_2(k_s^{m,2}) \leq 2\beta_2(m)$ ,
- (ii)  $m \leq \nu_2(k_s^{2^m-1,2}) \leq 2m-2$  for  $m \geq 2$ ,
- (iii)  $\nu_p(k_s^{m,2}) = \beta_p(m)$  for an odd prime  $p$ ,

where  $\nu_p(n)$  denotes the exponent of  $p$  in the prime factorization of  $n$  and  $\beta_p(m)$  is defined by  $p^{\beta_p(m)} \leq m < p^{\beta_p(m)+1}$ .

**Proof.** Identifying  $S(S^n)$  with  $S^{n+1}$ ,  $S(SP^m(S^n))$  can be embedded in  $SP^m(S^{n+1})$  so that the inclusion  $S^{n+1} \xrightarrow{i} SP^m(S^{n+1})$  factorizes as the composition  $S(S^n) \xrightarrow{S(i)} S(SP^m(S^n)) \subset SP^m(S^{n+1})$ , where  $S(X)$  denotes the reduced suspension of a pointed space  $X$ . This implies that  $k_s^{m,n}$  is a factor of  $k_s^{m,n+1}$ . By definition,  $k_s^{m,n}$  is a factor of  $k^{m,n}$  for odd  $n$ . So, in particular,  $k_s^{m,2}$  is a factor of  $k^{m,3}$ . Ucci's

estimates of  $k^{m,3}$  are  $\nu_2(k^{m,3}) \leq 2\beta_2(m)$  and  $\nu_p(k^{m,3}) = \beta_p(m)$  for an odd prime  $p$ . Therefore we have  $\nu_2(k_s^{m,2}) \leq 2\beta_2(m)$  and  $\nu_p(k_s^{m,2}) \leq \beta_p(m)$  for an odd prime  $p$ . On the other hand the estimates (3) and (4) imply that  $\beta_p(m) \leq \nu_p(k_s^{m,2})$  for a prime  $p$  and  $n \leq \nu_2(k_s^{2^n-1,2})$  for  $n \geq 2$ . Thus Theorem A follows.