ON THE STANLEY DEPTH OF WEAKLY POLYMATROIDAL IDEALS

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Abstract. Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over the field $\mathbb{K}$. In this paper, it is shown that Stanley's conjecture holds for I and S/I if I is a product of monomial prime ideals or I is a high enough power of a polymatroidal or a stable ideal generated in a single degree.

1. Introduction

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over the field $\mathbb{K}$. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element and $Z \subseteq \{x_1, \dots, x_n\}$. The $\mathbb{K}$-subspace $u\mathbb{K}[Z]$ generated by all elements uv with $v \in \mathbb{K}[Z]$ is called a *Stanley space* of dimension $|Z|$ if it is a free $\mathbb{K}[\mathbb{Z}]$-module. Here, as usual, $|Z|$ denotes the number of elements of Z. A decomposition $\mathcal{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimum dimension of a Stanley space in $\mathcal{D}$ is called *Stanley depth* of $\mathcal{D}$ and is denoted by $\mathrm{sdepth}(\mathcal{D})$. The quantity

$$\mathrm{sdepth}(M) := \max\left\{\mathrm{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\right\}$$

is called *Stanley depth* of M. Stanley [10] conjectured that

$$\mathrm{depth}(M) \leq \mathrm{sdepth}(M)$$

for all $\mathbb{Z}^n$-graded S-modules M. For a reader friendly introduction to the Stanley depth, we refer the reader to [8].

Let I be a monomial ideal of S whose Rees algebra is $\mathcal{R}(I)$ and let $\mathfrak{m} = (x_1, \dots, x_n)$ be the graded maximal ideal of S. Then the $\mathbb{K}$-algebra $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ is called the *fibre ring*, and its Krull dimension is called the *analytic spread* of I and is denoted by $\ell(I)$. This invariant is a measure for the growth of the number of generators of the powers of I. Indeed, for $k \gg 0$, the Hilbert function $H(\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I), \mathbb{K}, k) = \dim_{\mathbb{K}}(I^k/\mathfrak{m}I^k)$, which counts the number of generators of the powers of I, is a polynomial function of degree $\ell(I) - 1$.

Let I be a weakly polymatroidal ideal of S which is generated in a single degree and $\ell(I)$ its analytic spread. In this paper, we show that $\mathrm{sdepth}(I) \geq n - \ell(I) + 1$

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and $\operatorname{sdepth}(S/I) \geq n - \ell(I)$ (see Theorem 2.5) and we conclude that if I is a product of monomial prime ideals of S, then I and S/I satisfy Stanley's conjecture. We also show that if either I is a polymatroidal ideal or it is a stable ideal of S which is generated in a single degree, then I^k and S/I^k satisfy Stanley's conjecture for $k \gg 0$ (see Corollaries 2.7 and 2.11).

2. The results

In this paper, we deal with polymatroidal ideals. They were introduced in [4] and represent a natural generalization of matroidal ideals. In the following, we define polymatroidal ideals, and for more detailed information, we refer the reader to [4, 5, 6].

Definition 2.1. Let I be a monomial ideal of $S = \mathbb{K}[x_1, \dots, x_n]$ which is generated in a single degree, and assume that $G(I)$ is the set of minimal monomial generators of I. The ideal I is called *polymatroidal* if the following exchange condition is satisfied: For monomials $u = x_1^{a_1} \dots x_n^{a_n}$ and $v = x_1^{b_1} \dots x_n^{b_n}$ belonging to $G(I)$ and for every i with $a_i > b_i$, one has j with $a_j < b_j$ such that $x_j(u/x_i) \in G(I)$.

Weakly polymatroidal ideals are generalizations of polymatroidal ideals, and they are defined as follows.

Definition 2.2 ([5], Definition 12.7.1)**.** A monomial ideal I of $S = \mathbb{K}[x_1, \dots, x_n]$ is called *weakly polymatroidal* if for every two monomials $u = x_1^{a_1} \dots x_n^{a_n}$ and $v = x_1^{b_1} \dots x_n^{b_n}$ in $G(I)$ such that $a_1 = b_1, \dots, a_{t-1} = b_{t-1}$ and $a_t > b_t$ for some t, there exists $j > t$ such that $x_t(v/x_j) \in I$.

It is clear from the above definition that every polymatroidal ideal is weakly polymatroidal.

Lemma 2.3. *Let I be a monomial ideal of $S = \mathbb{K}[x_1, \dots, x_n]$ which is generated in a single degree. Then for every $1 \leq i \leq n$, we have $\ell((I : x_i)) \leq \ell(I)$.*

Proof. It is enough to show that for every integer $k \geq 1$, $\mu(I^k) \geq \mu((I : x_i)^k)$, where $\mu(I)$ denotes the number of minimal generators of I. Now assume that I is generated in degree p and $G(I) = \{u_1, \dots, u_s\}$ is the set of minimal monomial generators of I. Without loss of generality, we may assume that there exists $0 \leq t \leq s$ such that $u_1, \dots, u_t$ are divisible by x_i and $u_{t+1}, \dots, u_s$ are not divisible by x_i. Let $u'_j = u_j/x_i$ $(1 \leq j \leq t)$.

For every integer $k \geq 1$, we define an injective map f from $G((I : x_i)^k)$ to $G(I^k)$, and this completes the proof. In order to do this, let $u \in G((I : x_i)^k)$. Then we may write u as below, where $0 \leq q \leq k$:

$$u = u'_{i_1} \dots u'_{i_q} u_{i_{q+1}} \dots u_{i_k}.$$

Note that $q = kp - \deg(u)$, and therefore q is independent from the above representation. Therefore, we may define

$$f(u) := x_i^q u = u_{i_1} \dots u_{i_q} u_{i_{q+1}} \dots u_{i_k} \in I^k.$$

Since I^k is generated in degree pk, $f(u) \in G(I^k)$. We now prove that f is injective. Assume that there exist $u, v \in G((I : x_i)^k)$ such that $f(u) = f(v)$. Then by definition of f, for every $j \neq i$, we have $\deg_{x_j}(u) = \deg_{x_j}(v)$. Hence, if $\deg_{x_i}(u) > \deg_{x_i}(v)$, then $v|u$ and if $\deg_{x_i}(v) > \deg_{x_i}(u)$, then $u|v$ and in the both cases we derive a contradiction because $u, v \in G((I : x_i)^k)$. Therefore, $\deg_{x_i}(u) = \deg_{x_i}(v)$ and so $u = v$, which implies that f is injective. □

For proving our main result, we need the following lemma.

Lemma 2.4. *Let I be a weakly polymatroidal ideal of $S = \mathbb{K}[x_1, \ldots, x_n]$ which is generated in a single degree. Then $(I : x_1)$ satisfies the same property.*

Proof. It is clear from the definition that $(I : x_1)$ is a weakly polymatroidal ideal. Therefore, we prove that it is generated in a single degree. Suppose that $G(I) = \{u_1, \ldots, u_s\}$ is the set of minimal monomial generators of I, and let $\deg(u_i) = k$. Without loss of generality, we may assume that $u_1, \ldots, u_t$ are divisible by x_1 and $u_{t+1}, \ldots, u_s$ are not divisible by x_1, where $1 \leq t \leq s$. Let $v_i = u_i/x_1$ $(1 \leq i \leq t)$. We claim that $(I : x_1)$ is generated by $v_1, \ldots, v_t$. In order to prove the claim, let $v \in (I : x_1)$ be a monomial. Then $x_1 v \in I$ and so there exists $1 \leq i \leq s$ in such a way that u_i divides $x_1 v$. If $1 \leq i \leq t$, then v is divisible by v_i and therefore $v \in (v_1, \ldots, v_t)$. Therefore, we may assume that $i \geq t+1$. Now u_i is not divisible by x_1 and so $u_i|v$. By Definition 2.2, there exists $j \geq 2$ such that $x_1 u_i/x_j \in I$. Since $\deg(x_1 u_i/x_j) = k$, there exists $1 \leq p \leq t$ such that $u_p = x_1 u_i/x_j$ and hence $v_p = u_i/x_j$ divides v and therefore $v \in (v_1, \ldots v_t)$. This proves the claim and completes the proof of the lemma. □

We are now in the position to state and prove our main result.

Theorem 2.5. *Let I be a weakly polymatroidal ideal of $S = \mathbb{K}[x_1, \ldots, x_n]$ which is generated in a single degree. Then we have the following assertions:*

(i) $\mathrm{sdepth}(I) \geq n - \ell(I) + 1$ *and* $\mathrm{sdepth}(S/I) \geq n - \ell(I)$.

(ii) $\mathrm{depth}(S/I) \geq n - \ell(I)$.

Proof. We prove (i) and (ii) simultaneously by induction on n and k, where k is the degree of generators of I. Let $G(I) = \{u_1, \ldots, u_s\}$ be the set of minimal monomial generators of I, and let $\deg(u_i) = k$. If $n = 1$, then I is a principal ideal, and so we have $\ell(I) = 1$, $\mathrm{sdepth}(I) = 1$, and $\mathrm{depth}(S/I) = \mathrm{sdepth}(S/I) = 0$. Therefore, in this case, the inequalities in (i) and (ii) are trivial. If $k = 1$, then I is a complete intersection and so $\ell(I) = s$. In this case, the inequality in (ii) is trivial, and the inequalities in (i) follow from [9, Theorem 1.1] and [7, Proposition 3.4]. We now consider $n \geq 2$ and $k \geq 2$. Assume that there exists a variable x_j such that

$$x_j \notin \bigcup_{i=1}^{s} \mathrm{Supp}(u_i),$$

where for a monomial $u \in S$, $\mathrm{Supp}(u)$ is the set of variables which divide u. Hence, x_j is regular over S/I and so $\mathrm{depth}(S/I) = \mathrm{depth}(S_j/IS_j) + 1$, where S_j is the polynomial ring obtained from S by deleting the variable x_j. Therefore, the induction hypothesis on n implies that $\mathrm{depth}(S/I) \geq n - \ell(I)$. On the other hand, by [9, Theorem 1.1] and [7, Lemma 3.6], we conclude that $\mathrm{sdepth}(S/I) = \mathrm{sdepth}(S_j/IS_j)+1$ and $\mathrm{sdepth}(I) = \mathrm{sdepth}(IS_j) + 1$. Therefore, using the induction hypothesis on n, we conclude that $\mathrm{sdepth}(I) \geq n - \ell(I) + 1$ and $\mathrm{sdepth}(S/I) \geq n - \ell(I)$. Therefore, we may assume that

$$\bigcup_{i=1}^{s} \mathrm{Supp}(u_i) = \{x_1, \ldots, x_n\}.$$

Let $S' = \mathbb{K}[x_2, \ldots, x_n]$, and consider $I' = I \cap S'$ and $I'' = (I : x_1)$. Now $I = I'S' \oplus x_1 I''S$ and $S/I = (S'/I'S') \oplus x_1(S/I''S)$ and therefore by the definition of Stanley depth we have

$$\mathrm{sdepth}(I) \geq \min\{\mathrm{sdepth}_{S'}(I'S'), \mathrm{sdepth}_S(I'')\} \tag{1}$$

and

$$\mathrm{sdepth}(S/I) \geq \min\{\mathrm{sdepth}_{S'}(S'/I'S'), \mathrm{sdepth}_S(S/I'')\}. \tag{2}$$

On the other hand, by applying the depth lemma on the exact sequence

$$0 \longrightarrow S/(I : x_1) \longrightarrow S/I \longrightarrow S/(I, x_1) \longrightarrow 0,$$

we conclude that

$$\mathrm{depth}(S/I) \geq \min\{\mathrm{depth}_{S'}(S'/I'S'), \mathrm{depth}_S(S/I'')\}. \tag{3}$$

Using Lemmas 2.3 and 2.4 and the induction hypothesis on k, we now conclude that $\mathrm{depth}_S(S/I'') \geq n - \ell(I)$, $\mathrm{sdepth}_S(I'') \geq n - \ell(I) + 1$, and $\mathrm{sdepth}_S(S/I'') \geq n - \ell(I)$.

Note that $I'S'$ is a weakly polymatroidal ideal of S' which is generated in a single degree. Since

$$x_1 \in \bigcup_{i=1}^{s} \mathrm{Supp}(u_i)$$

and the generators of $I'S$ are not divisible by x_1, using [5, Lemma 10.3.19], we conclude that $\ell(I'S') \leq \ell(I) - 1$, and therefore, by our induction hypothesis on n, we conclude that

$$\mathrm{sdepth}_{S'}(I'S') \geq (n-1) - \ell(I'S') + 1 \geq (n-1) - (\ell(I) - 1) + 1 = n - \ell(I) + 1$$

and similarly $\mathrm{sdepth}_{S'}(S'/I'S') \geq n - \ell(I)$ and $\mathrm{depth}_{S'}(S'/I'S') \geq n - \ell(I)$. Now the inequalities (1), (2), and (3) complete the proof of the theorem. □

It is known and easy to prove that $\mathrm{ht}(I) \leq \ell(I)$ for every monomial ideal I. In the following corollary, we give a stronger lower bound for the analytic spread of a weakly polymatroidal ideal which is generated in a single degree.

Corollary 2.6. *Let I be a weakly polymatroidal ideal of $S = \mathbb{K}[x_1, \ldots, x_n]$ which is generated in a single degree. Then*

$$\max\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}(S/I)\} \leq \ell(I).$$

Proof. Let $\mathfrak{p} \in \mathrm{Ass}(S/I)$ be given. By [2, Proposition 1.2.13] we have $\mathrm{depth}(S/I) \leq n - \mathrm{ht}(\mathfrak{p})$, while by Theorem 2.5 we have $\mathrm{depth}(S/I) \geq n - \ell(I)$. This implies that $\mathrm{ht}(\mathfrak{p}) \leq \ell(I)$ for every $\mathfrak{p} \in \mathrm{Ass}(S/I)$ and completes the proof of the corollary. $\square$

Let I be a monomial ideal of $S = \mathbb{K}[x_1 \dots, x_n]$. A classical result by Burch [3] says that

$$\min_t \mathrm{depth}(S/I^t) \leq n - \ell(I).$$

By a theorem of Brodmann [1], the quantity $\mathrm{depth}(S/I^t)$ is constant for large t. We call this constant value the *limit depth* of I, and we denote it by $\lim_{t\to\infty} \mathrm{depth}(S/I^t)$. Brodmann improved the Burch's inequality by showing that

$$\lim_{t\to\infty} \mathrm{depth}(S/I^t) \leq n - \ell(I).$$

Corollary 2.7. *Let I be a polymatroidal ideal of $S = \mathbb{K}[x_1, \dots, x_n]$. Then there exists an integer $k_0 \geq 1$ such that for every $k \geq k_0$, I^k and S/I^k satisfy Stanley's conjecture.*

Proof. Note that by [5, Theorem 12.6.3], every power of a polymatroidal ideal is again a polymatroidal ideal. Since every polymatroidal ideal is a weakly polymatroidal ideal which is generated in a single degree, Theorem 2.5 implies that for every $k \geq 1$, $\mathrm{sdepth}(I^k) \geq n - \ell(I^k) + 1 = n - \ell(I) + 1$ and $\mathrm{sdepth}(S/I^k) \geq n - \ell(I^k) = n - \ell(I)$. Now applying Burch's inequality completes the proof. $\square$

Definition 2.8. Let F be a nonempty subset of $[n]$. We denote by P_F the monomial prime ideal $(x_i \mid i \in F)$. A *transversal polymatroidal ideal* is an ideal I of the form

$$I = P_{F_1} P_{F_2} \dots P_{F_r},$$

where $F_1, \dots, F_r$ is a collection of nonempty subsets of $[n]$ with $r \geq 1$.

It follows from the above definition that the product of transversal polymatroidal ideals is again a transversal polymatroidal ideal and that every transversal polymatroidal ideal is a polymatroidal ideal.

Corollary 2.9. *If I is a transversal polymatroidal ideal of $S = \mathbb{K}[x_1, \dots, x_n]$, then I and S/I satisfy Stanley's conjecture.*

Proof. Note that by Theorem 2.5, we have $\mathrm{sdepth}(I) \geq n - \ell(I) + 1$ and $\mathrm{sdepth}(S/I) \geq n - \ell(I)$. Also, [6, Corollary 3.14] implies that $\mathrm{depth}(S/I) = n - \ell(I)$. Therefore, I and S/I satisfy Stanley's conjecture. $\square$

One should note that Corollary 2.9 essentially says that if I is a product of some monomial primes, then I and S/I satisfy Stanley's conjecture.

Definition 2.10. Let u be a monomial in $S = \mathbb{K}[x_1, \dots, x_n]$. We denote by $m(u)$ the maximum number j such that $x_j | u$. Then a monomial ideal I of S is called a *stable ideal* if for all monomials $u \in I$ and all $i < m(u)$ one has $x_i(u/x_{m(u)}) \in I$.

It is clear from the above definition that every stable ideal is weakly polymatroidal.

Corollary 2.11. *Let I be a stable ideal of $S = \mathbb{K}[x_1, \dots, x_n]$ which is generated in a single degree. Then there exists an integer $k_0 \geq 1$ such that for every $k \geq k_0$, I^k and S/I^k satisfy Stanley's conjecture.*

Proof. Since every power of a stable ideal is again stable, Theorem 2.5 implies that for every $k \geq 1$, $\operatorname{sdepth}(I^k) \geq n - \ell(I^k) + 1 = n - \ell(I) + 1$ and $\operatorname{sdepth}(S/I^k) \geq n - \ell(I^k) = n - \ell(I)$. Now applying Burch's inequality completes the proof. $\square$

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