

# On the Stanley-Wilf conjecture for the number of permutations avoiding a given pattern

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ABSTRACT. Consider, for a permutation  $\sigma \in \mathcal{S}_k$ , the number  $F(n, \sigma)$  of permutations in  $\mathcal{S}_n$  which avoid  $\sigma$  as a subpattern. The conjecture of Stanley and Wilf is that for every  $\sigma$  there is a constant  $c(\sigma) < \infty$  such that for all  $n$ ,  $F(n, \sigma) \leq c(\sigma)^n$ . All the recent work on this problem also mentions the “stronger conjecture” that for every  $\sigma$ , the limit of  $F(n, \sigma)^{1/n}$  exists and is finite. In this short note we prove that the two versions of the conjecture are equivalent, with a simple argument involving subadditivity.

We also discuss  $n$ -permutations, containing all  $\sigma \in \mathcal{S}_k$  as subpatterns. We prove that this can be achieved with  $n = k^2$ , we conjecture that asymptotically  $n \sim (k/e)^2$  is the best achievable, and we present Noga Alon’s conjecture that  $n \sim (k/2)^2$  is the threshold for random permutations.

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## 1. INTRODUCTION

Consider, for a permutation  $\sigma \in \mathcal{S}_k$ , the set  $\mathcal{A}(n, \sigma)$  of permutations  $\tau \in \mathcal{S}_n$  which avoid  $\sigma$  as a subpattern, and its cardinality,  $F(n, \sigma) := |\mathcal{A}(n, \sigma)|$ . Recall that “ $\tau$  contains  $\sigma$ ” as a subpattern means that there exist  $1 \leq x_1 < x_2 < \cdots < x_k \leq n$  such that for  $1 \leq i, j \leq k$ ,

$$(1) \quad \tau(x_i) < \tau(x_j) \quad \text{if and only if} \quad \sigma(i) < \sigma(j).$$

An outstanding conjecture is that for every  $\sigma$  there is a finite constant  $c(\sigma)$  such that for all  $n$ ,  $F(n, \sigma) \leq c(\sigma)^n$ . In the 1997 Ph.D. thesis of Bóna [2], supervised by

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Stanley, this conjecture is attributed to “Wilf and Stanley [oral communication] from 1990.” All the recent work on this problem also mentions the “stronger conjecture” that for every  $\sigma$ , the limit of  $F(n, \sigma)^{1/n}$  exists and is finite. According to Wilf (private communication, 1999) the original conjecture was of this latter form.

In this short note we give, as Theorem 1, a simple argument, involving subadditivity, which shows that the two versions of the conjecture are equivalent.

Here is some background information on the current status of the Stanley-Wilf conjecture. Represent  $\sigma \in \mathcal{S}_k$  by the word  $\sigma(1)\sigma(2)\cdots\sigma(k)$ . For the case of the increasing pattern, i.e the identity permutation,  $\sigma = 12\cdots k$ , the upper bound  $F(n, \sigma) \leq ((k-1)^2)^n$  is well known, and follows from the Robinson-Schensted-Knuth correspondence; also Regev [7] gives the asymptotics

$$F(n, 12\cdots k) \sim \lambda_k \frac{(k-1)^{2n}}{n^{k(k-2)/2}},$$

with an explicit constant  $\lambda_k$ . Simion and Schmidt [8] give a bijective proof that for each  $\sigma \in \mathcal{S}_3$ ,  $F(n, \sigma) = \frac{1}{n+1} \binom{2n}{n}$ ; see also Knuth [6], section 2.2.1, exercises.

For  $\sigma = 1342$ , Bóna [2] finds the explicit generating function for  $F(n, \sigma)$ , showing that for all  $n$ ,  $F(n, 1342) < 8^n$ , and  $\lim F(n, 1342)^{1/n} = 8$ . Note in contrast that  $\lim F(n, 1234)^{1/n} = 9$ . Bóna observes that indeed, in all cases for which  $\lim F(n, \sigma)^{1/n}$  is known explicitly, it is an integer! For the special class of “layered patterns,” such as  $\sigma = 6734512$ , Bóna [3] has shown that  $\sup_n F(n, \sigma)^{1/n}$  is finite. Alon and Friedgut [1] prove an upper bound for the general case which is tantalizingly close to the goal; they relate the problem to a result on generalized Davenport-Schinzel sequences from Klazar [5], and show that for every  $\sigma \in \mathcal{S}_k$  there exists  $c(\sigma) < \infty$  such that for all  $n$ ,  $F(n, \sigma) \leq c(\sigma)^{n\gamma^*(n)}$ , where  $\gamma^*(n)$  is an extremely slowly growing function, given explicitly in terms of the inverse of the Ackermann function.

**Theorem 1.** *For every  $k \geq 2$  and  $\sigma \in \mathcal{S}_k$ , for every  $m, n \geq 1$ ,*

$$(2) \quad F(m+n, \sigma) \geq F(m, \sigma) F(n, \sigma)$$

and  $F(n, \sigma) \geq 1$ ; hence by Fekete’s lemma on subadditive sequences,

$$(3) \quad c(\sigma) := \lim_{n \rightarrow \infty} F(n, \sigma)^{1/n} \in [1, \infty] \text{ exists,}$$

and  $\forall n \geq 1, \quad F(n, \sigma) \leq c(\sigma)^n$ .

*Proof.* First we will show (2) by constructing, from an  $m$ -permutation and an  $n$ -permutation which avoid  $\tau$ , an  $(m+n)$ -permutation which avoids  $\tau$ , injectively.

Without loss of generality, we may assume that  $k$  precedes 1 in  $\sigma$  (since, with  $(\cdot)^r$  to denote the left-right reverse of a permutation,  $\tau$  avoids  $\sigma$  iff  $\tau^r$  avoids  $\sigma^r$ , and hence for all  $n$ ,  $F(n, \sigma) = F(n, \sigma^r)$ .)

Let  $\tau' \in \mathcal{S}_m$  and  $\tau'' \in \mathcal{S}_n$ , where each of  $\tau'$  and  $\tau''$  avoids  $\sigma$ . Let  $\tau'''$  be the result of adding  $m$  to each symbol in the word for  $\tau''$ , so that  $\tau'''$  is a word in which each of the symbols  $m+1, \dots, m+n$  appears exactly once.

Consider the concatenation  $\tau$  of  $\tau'$  with  $\tau'''$  as a permutation,  $\tau \in \mathcal{S}_{m+n}$ . Clearly,  $\tau$  avoids  $\sigma$ , establishing (2).

[In detail, suppose to the contrary that  $\tau$  contains  $\sigma$ , say at the  $k$ -tuple of positions given by  $1 \leq x_1 < x_2 < \dots < x_k \leq m+n$ . Recall that  $k$  precedes 1 in  $\sigma$ ; say that  $\sigma(a) = 1$  and  $\sigma(b) = k$  with  $1 \leq b < a \leq k$ , so that by (1), for  $1 \leq i \leq k$ ,  $\tau(x_a) \leq \tau(x_i) \leq \tau(x_b)$ . If  $x_k \leq m$  then  $\tau'$  contains  $\sigma$ , and if  $x_1 > m$  then  $\tau''$  contains  $\sigma$ . If neither of these, then the  $x_1 \leq m$  so that  $\tau(x_1) \leq m$ , hence  $\tau(x_a) \leq \tau(x_1) \leq m$  and therefore  $x_a \leq m$ ; similarly  $x_k > m$  so that  $\tau(x_k) > m$ , hence  $\tau(x_b) \geq \tau(x_k) > m$  and therefore  $x_b > m$ , contradicting  $b < a$ .]

Recalling that  $k$  precedes 1 in  $\sigma$ , the identity permutation in  $\mathcal{S}_n$  avoids  $\sigma$  and demonstrates that  $F(n, \sigma) \geq 1$  for every  $n \geq 1$ . Fekete's lemma [4], see also [9], is that if  $a_1, a_2, \dots \in \mathbb{R}$  satisfy for all  $m, n \geq 1$ ,  $a_m + a_n \leq a_{m+n}$ , then  $\lim_{n \rightarrow \infty} a_n/n = \inf_{n \geq 1} a_n/n \in [-\infty, \infty)$ . Applying this with  $a_n := -\log F(n, \sigma)$  completes our proof.  $\square$

There exist [10] examples with  $\sigma, \sigma' \in \mathcal{S}_k$ , with  $\sigma'$  the identity permutation, and  $F(n, \sigma) > F(n, \sigma')$ , and Bóna [2], Theorem 4 shows that for all  $n \geq 7$ ,  $F(n, 1324) > F(n, 1234)$ . Nevertheless, it is possible that for every  $k$ , the largest exponential growth rate is the  $(k-1)^2$  achieved by the identity permutation.

**Conjecture 1.** (*\$100.00*) For all  $\sigma \in \mathcal{S}_k$  and  $n \geq 1$ ,  $F(n, \sigma) \leq (k-1)^{2n}$ .

### The problem of the shortest common superpattern.

Define  $G(n, k)$  to be the number of permutations  $\tau \in \mathcal{S}_n$  which avoid *at least one* permutation in  $\mathcal{S}_k$ , i.e.

$$G(n, k) := |\cup_{\sigma \in \mathcal{S}_k} \mathcal{A}(n, \sigma)|, \text{ where } F(n, \sigma) := |\mathcal{A}(n, \sigma)|.$$

Simion and Schmidt [8], p. 398, give a formula for  $n! - G(n, 3)$ , the number of  $n$ -permutations which contain all six patterns of length 3. In considering  $G(n, k)$ , it is natural to consider the length  $m(k)$  of the shortest permutation which contains every  $\sigma \in \mathcal{S}_k$  as a subpattern, i.e. to consider

$$m(k) := \min\{n: G(n, k) < n!\} = \min\{n: \cup_{\sigma \in \mathcal{S}_k} \mathcal{A}(n, \sigma) \neq \mathcal{S}_n\}.$$

For a trivial lower bound on  $m(k)$ , since  $\tau \in \mathcal{S}_n$  contains at most  $\binom{n}{k}$  subpatterns, to contain every subpattern requires  $\binom{n}{k} \geq k!$ , hence  $\liminf_k m(k)/k^2 \geq 1/e^2$ .

**Theorem 2.** *There exists an  $n$ -permutation, with  $n = k^2$ , containing every  $k$ -permutation as a subpattern; i.e.  $m(k) \leq k^2$ .*

*Proof.* Consider the lexicographic order on  $[k]^2$  as a one-to-one map specifying the ranks of the ordered pairs, i.e. let  $r : [k]^2 \rightarrow [k^2]$ , with  $(i, j) \mapsto (i-1)k + j$ . Also consider the transposed lexicographic order  $t : [k]^2 \rightarrow [k^2]$  given by  $t(i, j) := r(j, i)$ . Consider the permutation  $\tau \in \mathcal{S}_{k^2}$  given by  $\tau = r \circ t^{-1}$ ; for example, with  $k = 3$ , this is  $\tau = 147258369$ . Then, clearly,  $\tau$  contains every  $\sigma \in \mathcal{S}_k$  as a subpattern. In detail, with the positions  $x_1 := t(\sigma(1), 1), \dots, x_k := t(\sigma(k), k)$  we have  $x_1 < \dots < x_k$  and for  $m = 1$  to  $k$ ,  $\tau(x_m) = (r \circ t^{-1})(t(\sigma(m), m)) = r(\sigma(m), m)$  so that  $\tau(x_a) < \tau(x_b)$  iff  $\sigma(a) < \sigma(b)$ .  $\square$

**Conjecture 2.** As  $k \rightarrow \infty$ ,  $m(k) \sim (k/e)^2$ .

In contrast, from the known behavior of the length  $L_n$  of the longest increasing subsequence,  $L_n \sim 2\sqrt{n}$  with high probability, one cannot hope to use *random* permutations to show that  $\liminf m(k)/k^2 \leq (1/e)^2$ . The probability that a random  $n$ -permutation does *not* contain every  $\sigma \in \mathcal{S}_k$  as a subpattern is  $G(n, k)/n!$ . Define the threshold  $t(k)$  by  $t(k) = \min\{n : G(n, k)/n! \leq 1/2\}$ , so that trivially  $m(k) \leq t(k)$ , and hence  $\liminf t(k)/k^2 \geq 1/4$ .

**Conjecture 3.** (Noga Alon) The threshold length  $t(k)$ , for a random permutation to contain all  $k$ -permutations with substantial probability, has  $t(k) \sim (k/2)^2$ .

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