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On the Static and Spherically Symmetric Solutions of the Yang-Mills Field

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The ***b*** field equations proposed by Yang and Mills are investigated from a classical point of view, and the general solution is obtained under the condition of the static and spherically symmetric field. It is found that the solution is reducible to the "canonical form" by means of the isotopic gauge transformation. From this canonical form it is very likely that the mass of the ***b*** quantum is zero in spite of the non-linearity of the field equations.

§ 1. Introduction

It was pointed out by Yang and Mills[1] some years ago that the conventional formalism of charge independence hypothesis is not consistent with the concept of the localized field that underlies usual physical theories. They explored the possibility of admitting the arbitrary orientations of the isospin axes at all space-time points. Such an arbitrary way of choosing the orientation of the isospin axes is called isotopic gauge.

The isotopic gauge bears a close analogy with the electromagnetic gauge which represents an arbitrary way of choosing the complex phase factor of a charged field at all space-time points. In electrodynamics the introduction of the electromagnetic field enables us to counteract the gauge transformation of charged fields. In quite a similar manner Yang and Mills introduced a physical field of a new kind to ensure the invariance of the theory under the isotopic gauge transformation. This new field is called a ***b*** (or B) field, which is represented by twelve components $\boldsymbol{b}_\mu$. Here the index μ ($\mu=1, 2, 3, 4$) refers to the space-time coordinates and the bold face letter ***b*** indicates that it is a vector-like quantity in an isospace.

Since the proposal of Yang and Mills, various investigations have been made[2)-10)] concerning the isotopic gauge invariance and the properties of the ***b*** field from the standpoint of the classical as well as that of the quantized theory of fields. In spite of these studies, however, it is still difficult to draw any definite conclusion about their quantum properties on account of the non-linear character of the $\boldsymbol{b}$ field equations. As to the mass of $\boldsymbol{b}$ quantum, for example, the two opposite possibilities are equally probable: i) The mass of the $\boldsymbol{b}$ quantum is zero because of the absence of the mass term $\mu^2\boldsymbol{b}^2$ in the Lagrangian, and ii) it is finite (not zero) because of the presence of self-interactions due to the non-linear character of the field equations. This problem is of special interest to the present authors, because it has a close connection with the possibility of the experimental detection of the $\boldsymbol{b}$ quantum. From a theoretical point of view, it is also connected with a unified description of elementary particle interactions as proposed by Sakurai.[7] Indeed, if the mass of the $\boldsymbol{b}$ quantum is zero, it has nothing to do with strong interactions.

In view of these circumstances it will be of physical importance to investigate the $\boldsymbol{b}$ field equations from a purely classical point of view. Along this line of thought the present authors made a brief mention of a static solution with spherical symmetry of the Yang-Mills equations in a paper previously published.[4] In the present paper we shall look for the most general form of the static and spherically symmetric solutions and investigate their physical implications. The authors believe that such an approach will be useful in clarifying the physical properties of the $\boldsymbol{b}$ fields. We obtain the general form of a spherically symmetric $\boldsymbol{b}_\mu$ in § 2, and the field equations of Yang and Mills are solved in § 3. In § 4 we study the reduction of the solutions by means of isotopic gauge transformations. The final section is devoted to the discussions of the physical properties of the solutions. The transformation formula of the $\boldsymbol{b}$ field under isotopic gauge transformations is given in Appendix.

§ 2. General form of the spherically symmetric $\boldsymbol{b}_\mu$

We shall determine in this section the general form of the spherically symmetric $\boldsymbol{b}_\mu$.

The condition for $\boldsymbol{b}_\mu$ to be spherically symmetric is as follows: Let us consider a system of inertia and let the centre of symmetry be the spatial origin of that system. When we perform an arbitrary rotation around the origin

$$\begin{aligned} &'x^i = a_j{}^i x^j, \quad 'x^4 = x^4; \\ &a_j{}^i a_k{}^i = \delta_{jk}, \quad (i, j, k = 1, 2, 3), \end{aligned} \tag{2·1}$$

$\boldsymbol{b}_\mu(x)$ will be transformed into $'\boldsymbol{b}_\mu('x)$. If the functional form of $'\boldsymbol{b}_\mu('x)$ remains the same as that of $\boldsymbol{b}_\mu(x)$, that is, if

$$'\boldsymbol{b}_\mu(x) = \boldsymbol{b}_\mu(x), \tag{2·2}$$

$\boldsymbol{b}_\mu$ is said to be spherically symmetric.

Now we can easily define the Lie derivative[11] of $\boldsymbol{b}_\mu$ with respect to the transformation (2·1). One way to formulate the spherical symmetry condition of $\boldsymbol{b}_\mu$ mentioned above is to put the Lie derivative equal to zero. The formula thus obtained is a set of partial differential equations of the first order, and by solving it we shall be able to determine the general form of the spherically symmetric $\boldsymbol{b}_\mu$. Practically, however, calculations are somewhat troublesome, so we here adopt an alternative method. This is an extension of the method used by Papapetrou[12] in finding static solutions with spherical symmetry in the relativistic theory of non-symmetric field.

Consider the values of the components of $\boldsymbol{b}_\mu$ at the point $(0, 0, r, ict)$ which lies on the x^3-axis. When we perform the rotation

$$'x^1=-x^2,\quad 'x^2=x^1,\quad 'x^3=x^3,\quad 'x^4=x^4, \tag{2·3}$$

$\boldsymbol{b}_\mu$ is transformed into $'\boldsymbol{b}_\mu$ given by

$$'\boldsymbol{b}_1=-\boldsymbol{b}_2,\quad '\boldsymbol{b}_2=\boldsymbol{b}_1,\quad '\boldsymbol{b}_3=\boldsymbol{b}_3,\quad '\boldsymbol{b}_4=\boldsymbol{b}_4. \tag{2·4}$$

Since for the points on the x^3-axis we have $'x^\mu=x^\mu$ under (2·3), by using the assumption of spherical symmetry (2·2) we get

$$'\boldsymbol{b}_\mu=\boldsymbol{b}_\mu \quad \text{at} \quad x^\mu=(0, 0, r, ict). \tag{2·5}$$

From (2·4) and (2·5) we obtain

$$\boldsymbol{b}_1=\boldsymbol{b}_2=0 \quad \text{at} \quad x^\mu=(0, 0, r, ict),$$

and the only non-vanishing components are $\boldsymbol{b}_3$ and $\boldsymbol{b}_4$.

We next perform another rotation which transforms the point $(0, 0, r, ict)$ into (x^1, x^2, x^3, ict) with

$$(x^1)^2+(x^2)^2+(x^3)^2=r^2.$$

Then $\boldsymbol{b}_\mu$ is transformed into

$$\boldsymbol{b}_i=(x^i/r)\boldsymbol{f}(r, t),\quad \boldsymbol{b}_4=i\boldsymbol{g}(r, t), \tag{2·6}$$

where $\boldsymbol{f}(r, t)$ and $\boldsymbol{g}(r, t)$ are functions of r and t. (2·6) with arbitrary $\boldsymbol{f}$ and $\boldsymbol{g}$ clearly satisfies the condition of spherical symmetry, hence it gives the general form of the spherically symmetric $\boldsymbol{b}_\mu$.

Instead of expressing in Cartesian coordinates, we can express (2·6) in polar coordinates as

$$\boldsymbol{b}_r=\boldsymbol{f}(r, t),\quad \boldsymbol{b}_\theta=\boldsymbol{b}_\varphi=0,\quad \boldsymbol{b}_4=i\boldsymbol{g}(r, t). \tag{2·7}$$

This form of $\boldsymbol{b}_\mu$ seems simple. But the equations of Yang-Mills field take much simpler forms in a Cartesian than in a polar coordinate system, because $\boldsymbol{b}_\mu$ are not scalar quantities and the first order derivatives of the Christoffel symbols appear in the equations. For this reason we shall not use (2·7) in the subsequent sections.

§ 3. Static Yang-Mills field with spherical symmetry

In this section we shall investigate the static and spherically symmetric solution of the free field equations proposed by Yang and Mills.[1]

The equations to be solved are:

$$\partial_\mu \boldsymbol{b}_\mu = 0, \tag{3·1}$$

$$\partial_\nu \boldsymbol{f}_{\mu\nu} + 2\epsilon \boldsymbol{b}_\nu \times \boldsymbol{f}_{\mu\nu} = 0, \tag{3·2}$$

$$\boldsymbol{f}_{\mu\nu} = \partial_\nu \boldsymbol{b}_\mu - \partial_\mu \boldsymbol{b}_\nu - 2\epsilon \boldsymbol{b}_\mu \times \boldsymbol{b}_\nu . \tag{3·3}$$

Substituting (3·3) in (3·2) and making use of (3·1), we have

$$\Box \boldsymbol{b}_\mu + 2\epsilon \boldsymbol{b}_\nu \times (2\partial_\nu \boldsymbol{b}_\mu - \partial_\mu \boldsymbol{b}_\nu - 2\epsilon \boldsymbol{b}_\mu \times \boldsymbol{b}_\nu) = 0. \tag{3·4}$$

From the results obtained in the preceding section, the general form of the static $\boldsymbol{b}_\mu$ with spherical symmetry is given by

$$\boldsymbol{b}_i = (x^i/r)\boldsymbol{f}(r), \quad \boldsymbol{b}_4 = i\boldsymbol{g}(r). \tag{3·5}$$

Equations (3·1) and (3·4) can now be written as

$$\boldsymbol{f}' + 2\boldsymbol{f}/r = 0, \tag{3·6}$$

$$\Delta \boldsymbol{f} - 2\boldsymbol{f}/r^2 + 2\epsilon[\boldsymbol{f}\times\boldsymbol{f}' + \boldsymbol{g}\times\boldsymbol{g}' + 2\epsilon\boldsymbol{g}\times(\boldsymbol{f}\times\boldsymbol{g})] = 0, \tag{3·7}$$

$$\Delta \boldsymbol{g} + 4\epsilon\boldsymbol{f}\times[\boldsymbol{g}' + \epsilon\boldsymbol{f}\times\boldsymbol{g}] = 0, \tag{3·8}$$

where a prime means the differentiation with respect to r.

Integrating (3·6), we easily obtain

$$\boldsymbol{f} = \boldsymbol{a}/r^2, \quad (\boldsymbol{a} = \text{integration constants}). \tag{3·9}$$

Then, from (3·7) and (3·8) we have the following equations for $\boldsymbol{g}$:

$$\boldsymbol{g}\times[\boldsymbol{g}' + 2\epsilon\boldsymbol{a}\times\boldsymbol{g}/r^2] = 0, \tag{3·10}$$

$$(r^2\boldsymbol{g}')' + 4\epsilon\boldsymbol{a}\times[\boldsymbol{g}' + \epsilon\boldsymbol{a}\times\boldsymbol{g}/r^2] = 0. \tag{3·11}$$

$\boldsymbol{g}=0$ is a trivial solution of these two equations. In the following we shall proceed with the assumption $\boldsymbol{g} \neq 0$.

From (3·10) we can see that there is such a scalar function $\lambda(r)$ that

$$r^2\boldsymbol{g}' = -2\epsilon\boldsymbol{a}\times\boldsymbol{g} + \lambda\boldsymbol{g}. \tag{3·12}$$

When we substitute (3·12) in (3·11) we get

$$(\lambda' + \lambda^2/r^2)\,\boldsymbol{g} = 0,$$

which gives

$$\lambda = k(1-k/r)^{-1}, \quad (k = \text{integration constant}). \tag{3·13}$$

When $\lambda \neq 0$, Eq. (3·12) reduces to

$$r^2(\lambda\boldsymbol{g})' = -2\epsilon\boldsymbol{a}\times(\lambda\boldsymbol{g}).$$

We can easily obtain the general solution of this equation, i.e.

$$\boldsymbol{g} = (1-k/r)[\bar{\boldsymbol{a}} + \bar{\boldsymbol{b}}\cos(2\epsilon|\boldsymbol{a}|/r) + \bar{\boldsymbol{c}}\sin(2\epsilon|\boldsymbol{a}|/r)]. \tag{3·14}$$

Here $\bar{\boldsymbol{a}}$, $\bar{\boldsymbol{b}}$ and $\bar{\boldsymbol{c}}$ are integration constants with the conditions

$$\boldsymbol{a}\times\bar{\boldsymbol{a}}=0, \quad |\boldsymbol{a}|\bar{\boldsymbol{c}}=\boldsymbol{a}\times\bar{\boldsymbol{b}}, \quad |\boldsymbol{a}|\bar{\boldsymbol{b}}=-\boldsymbol{a}\times\bar{\boldsymbol{c}}. \tag{3·15}$$

This means that i) $\bar{\boldsymbol{a}}$ is parallel to $\boldsymbol{a}$, ii) $\boldsymbol{a}$, $\bar{\boldsymbol{b}}$ and $\bar{\boldsymbol{c}}$ form a right-handed orthogonal system and iii) $|\bar{\boldsymbol{b}}|=|\bar{\boldsymbol{c}}|$. When $\lambda=0$ (i.e. $k=0$) the general solution of (3·12) is given by (3·14) with the proviso $k=0$.

In case $\boldsymbol{a}\times\boldsymbol{g}\neq 0$ the solution (3·14) has the following property. When we travel from one point in space to another and investigate the vector $\boldsymbol{g}$ at each point, it seems as if $\boldsymbol{g}$ were "rotating" about the $\boldsymbol{a}$ axis, the angle Ψ between $\boldsymbol{a}$ and $\boldsymbol{g}$ being constant everywhere. The "angular velocity" ω of this rotation is obtained in the following manner: From (3·12) the "velocity" component of the point $\boldsymbol{g}$ in the direction of $\boldsymbol{a}\times\boldsymbol{g}$ is

$$(\boldsymbol{a}\times\boldsymbol{g})\cdot\boldsymbol{g}'/|\boldsymbol{a}\times\boldsymbol{g}|=-2\epsilon|\boldsymbol{a}\times\boldsymbol{g}|/r^2=-2\epsilon|\boldsymbol{a}|\cdot|\boldsymbol{g}|\sin\Psi/r^2.$$

If we divide this by the distance of the point $\boldsymbol{g}$ from the $\boldsymbol{a}$ axis (i.e. $|\boldsymbol{g}|\sin\Psi$), we obtain

$$\omega=-2\epsilon|\boldsymbol{a}|/r^2. \tag{3·16}$$

This is the expression of the required "angular velocity".

Summarizing the above, we know that in the static and spherically symmetric case the general solution of the free field equations is given by (3·5) with (3·9), (3·14) and (3·15). A particular solution

$$\begin{aligned}&\boldsymbol{b}_i=0, \quad \boldsymbol{b}_4=i(\boldsymbol{c}_1/r+\boldsymbol{c}_2)\,;\\&\boldsymbol{c}_1=(0,0,c_1), \quad \boldsymbol{c}_2=(0,0,c_2), \quad (c_1, c_2=\text{constants})\end{aligned} \tag{3·17}$$

is important for later purposes, and we shall call it the "canonical" solution. This may be interpreted as the neutral $\boldsymbol{b}$ field.

§ 4. Reduction of the solutions

As mentioned in the Introduction, the $\boldsymbol{b}_\mu$ is a vector with respect to Lorentz transformations, but it has a complicated transformation character with respect to isotopic gauge transformations. That is, if we introduce a matrix field B_μ through

$$B_\mu=2\boldsymbol{b}_\mu\cdot\boldsymbol{T}, \tag{4·1}$$

$\boldsymbol{T}$ denoting isospin matrices, we have

$$B_\mu'=S^{-1}B_\mu S+(i/\epsilon)\,S^{-1}\partial_\mu S, \tag{4·2}$$

where S is the matrix of the isotopic gauge transformation. The manner of transformation of $\boldsymbol{b}_\mu$ is obtained from (4·2) and the explicit expression is given in the Appendix. In this section we proceed to show that the general solution obtained in the preceding section is reducible to the canonical form (3·17) by means of the isotopic gauge transformation.

Consider two (unprimed and primed) isotopic gauges and denote respectively by B_μ and $\bar{B}_\mu'$ the B fields in the respective isotopic gauges. From (4·2), in order that they may be reducible to each other by means of the isotopic gauge transformation, there should exist an S which satisfies

$$\partial_\mu S = i\epsilon (B_\mu S - S\bar{B}_\mu'), \tag{4·3}$$

that is, the system of partial differential equations (4·3) must be integrable.

Differentiating (4·3) with respect to x^λ and antisymmetrizing the resulting expression with respect to λ and μ, we obtain

$$S\bar{F}_{\lambda\mu}' = F_{\lambda\mu} S, \tag{4·4a}$$

where

$$F_{\lambda\mu} = \partial_\mu B_\lambda - \partial_\lambda B_\mu + i\epsilon (B_\lambda B_\mu - B_\mu B_\lambda). \tag{4·5}$$

If we further differentiate (4·4a) with respect to x's repeatedly and use (4·3) and (4·4a), we have

$$S\bar{\nabla}_\nu' \bar{F}_{\lambda\mu}' = \nabla_\nu F_{\lambda\mu} \cdot S, \tag{4·4b}$$

$$S\bar{\nabla}_\rho' \bar{\nabla}_\nu' \bar{F}_{\lambda\mu}' = \nabla_\rho \nabla_\nu F_{\lambda\mu} \cdot S, \tag{4·4c}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

where the operators ∇_ν and $\bar{\nabla}_\nu'$ are defined respectively by

$$\nabla_\nu = \partial_\nu - i\epsilon [B_\nu, \quad] \quad \text{and} \quad \bar{\nabla}_\nu' = \partial_\nu - i\epsilon [\bar{B}_\nu', \quad]. \tag{4·6}$$

According to the theory of differential equations, the integrability condition of (4·3) is as follows: The necessary and sufficient condition is that there exists a positive integer N such that the first N equations of (4·4a), (4·4b), $\cdots$ are compatible and that the $(N+1)$th equation is satisfied because of those equations.

We now apply the above results to the reduction of the general solution obtained in the preceding section. For this purpose we first calculate B_μ's and $F_{\lambda\mu}$'s for the general and the canonical solution, and the results are as follows:

Canonical: $B_i = 0$, $\quad B_4 = 2i(\mathbf{c}_1/r + \mathbf{c}_2) \cdot \boldsymbol{T}$;

General: $B_i = 2(x^i/r^3)\, \boldsymbol{a} \cdot \boldsymbol{T}$, $\quad B_4 = 2i\boldsymbol{g} \cdot \boldsymbol{T}$;

and

Canonical: $F_{i4} = 2i(x^i/r^3)\, \mathbf{c}_1 \cdot \boldsymbol{T}$, $\quad F_{ij} = 0$;

General: $F_{i4} = -2i\lambda (x^i/r^3)\, \boldsymbol{g} \cdot \boldsymbol{T}$, $\quad F_{ij} = 0$,

where $\boldsymbol{g}$ is given by (3·14) and (3·15). Then we have from (4·6)

Canonical: $\nabla_j F_{i4} = 2i\partial_j (x^i/r^3)\, \mathbf{c}_1 \cdot \boldsymbol{T}$, $\quad$ other $\nabla_\nu F_{\lambda\mu} = 0$;

General: $\nabla_j F_{i4} = -2i\lambda\partial_j(x^i/r^3)\boldsymbol{g}\cdot\boldsymbol{T}$, other $\nabla_\nu F_{\lambda\mu}=0$.

The above results for the general solution are obtained by utilizing Eqs. (3·12) and (3·13) together with the relation $[\boldsymbol{a}\cdot\boldsymbol{T},\ \boldsymbol{g}\cdot\boldsymbol{T}]=i\ (\boldsymbol{a}\times\boldsymbol{g})\cdot\boldsymbol{T}$.

Now we consider that the canonical and the general solutions are given in a primed and in an unprimed isotopic gauge respectively. Then (4·4a) is equivalent to

$$\boldsymbol{c}_1\cdot\boldsymbol{T} = -S^{-1}\lambda\boldsymbol{g}\cdot\boldsymbol{T}S, \tag{4·7}$$

$\boldsymbol{c}_1\cdot\boldsymbol{T}$ and $-\lambda\boldsymbol{g}\cdot\boldsymbol{T}$ are isospin components in the directions of $\boldsymbol{c}_1$ and $-\lambda\boldsymbol{g}$, each multiplied by $|\boldsymbol{c}_1|$ and $|\lambda\boldsymbol{g}|$ respectively. Since $|\lambda\boldsymbol{g}|=\text{const.}(\equiv k')$ from (3·13), (3·14) and (3·15), there always exists an isotopic gauge transformation which transforms $-\lambda\boldsymbol{g}$ into $\boldsymbol{c}_1$ (with $|\boldsymbol{c}_1|=k'$). If we denote the corresponding representation matrix by S, it satisfies (4·7), hence (4·4a). It is clear that (4·4b) is also equivalent to (4·7), hence (4·4b) is satisfied in consequence of (4·4a). Thus the general solution is reducible to the canonical form (3·17) by means of the isotopic gauge transformation.

§ 5. Discussions

i) Among the solutions obtained in preceding sections, the present authors are most interested in the canonical solution which represents the neutral $\boldsymbol{b}$ field. In spite of the complicated non-linear character of the original $\boldsymbol{b}$ field equations, this solution is quite similar in form to the Coulomb potential in the classical theory of electromagnetic fields. From a classical point of view we may conclude that, like the photons, the $\boldsymbol{b}$ field quanta have the vanishing rest mass. If we hold the field quanta responsible to the strong interactions, as was suggested by Sakurai,[7] it must be massive enough to be compatible with the short-range character of strong interactions. From the results obtained in this paper, however, the non-linearity of the field equations cannot be considered a sufficient reason to make the field quanta massive from a classical point of view. If there is any reason to make the field quanta massive, it must be some quantum effects which are not considered throughout the present paper. Therefore, the quantization of the $\boldsymbol{b}$ field from a more fundamental point of view remains an urgent problem.

ii) It should be noticed that the general solution obtained in this paper has a singularity only at the spatial origin, although we have solved the free field equations without regard to the singularity. This is in a striking contrast to the case of Einstein's gravitational equations, where the Schwarzschild solution has a singular spherical surface (at $r=2m$) in addition to a singularity at the spatial origin.

iii) In the region where the source of the $\boldsymbol{b}$ field is present, the free field will have a singularity and Eq. (3·2) has to be modified as follows:[1]

$$\partial_\nu \boldsymbol{f}_{\mu\nu} = -\boldsymbol{J}_\mu, \tag{5·1}$$

$$\boldsymbol{J}_\mu = i\epsilon\bar{\psi}\gamma_\mu \boldsymbol{\tau}\psi + 2\epsilon \boldsymbol{b}_\nu \times \boldsymbol{f}_{\mu\nu}. \tag{5·2}$$

Here $\boldsymbol{J}_\mu$ are the isospin and current densities of the system; the first term of (5·2) refers to the source, while the second does to the field. On the boundary of the source, the solution of (3·2) should be joined to some solution of (5·1) under an appropriate condition.

If we derive the integral form of (3·2) and (5·1) with $\mu=4$ through the well-known method, we get at once the following interpretation of the integration constants of the solutions:

Canonical: $\quad 4\pi \boldsymbol{c}_1 = -i\int \boldsymbol{J}_4 d^3x,$

General: $\quad 4\pi k(\bar{\boldsymbol{a}}+\bar{\boldsymbol{b}}) = i\int \boldsymbol{J}_4 d^3x.$

The integration constants of the solution must satisfy these equations as well as the boundary condition at $r=\infty$.

iv) We have found that the general solution is reducible to the canonical one by means of the isotopic gauge transformation. This does not mean, however, that the canonical solution is sufficient for physical considerations, for there is no reason at present to persist in a specific choice of the isotopic gauge. The situation reminds us of the case of the electromagnetic fields, where we have various ways of choosing the electromagnetic gauge in accordance with the problem under consideration. It will be of some interest, therefore, to put to a further examination the physical implications of the general solution.

Appendix

We shall give the explicit transformation rule of the $\boldsymbol{b}$ field under the isotopic gauge transformation.

The relation between the $\boldsymbol{b}$ and the B field is given by (4·1), and the transformation rule for the latter is (4·2). The transformation formula of the $\boldsymbol{b}$ field can now be derived from these two equations. Since the $\boldsymbol{b}$ field is independent of the representation of the isotopic gauge transformation, it is sufficient to consider the specific type of S, e.g. a 3×3 orthogonal matrix:

$$S=(S_b{}^a), \quad (S^{-1})_b{}^a = S_a{}^b, \quad (a, b, \cdots = 1, 2, 3).$$

In this case $\boldsymbol{T}$ or T_i $(i=1, 2, 3)$ are also 3×3 matrices whose elements are given by $(T_i)_b{}^a = -i\epsilon_{iab}$. Writing $\boldsymbol{b}_\mu$ in terms of its components $b_\mu{}^i$ and substituting (4·1) in (4·2), we have

$$b_\mu{}^{i\prime}\epsilon_{iab} = S_a{}^c b_\mu{}^i \epsilon_{icd} S_b{}^d - (1/2\,\epsilon)\, S_a{}^c \partial_\mu S_b{}^c.$$

Multiplying this equation by ϵ_{jab} and using the relations

$$\epsilon_{iab}\,\epsilon_{jab}=2\delta_{ij}, \quad \det(S_q{}^p)\,\epsilon_{ijk}=\epsilon_{mnr}S_i{}^m S_j{}^n S_k{}^r,$$

we obtain

$$b_\mu{}^{i\prime}=\varepsilon(S)\,S_i{}^a b_\mu{}^a-(1/4\,\epsilon)\,\epsilon_{ijk}S_j{}^l\partial_\mu S_k{}^l, \qquad (\mathrm{A}\cdot 1)$$

where $\varepsilon(S)$ is equal to $+1$ or -1 according as the isotopic gauge transformation in question is proper or improper respectively. (A·1) is the transformation formula of the $\boldsymbol{b}$ field.

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