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On the Static Deformation of an Earth Model with a Fluid Core

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Summary

The concept of a purely static deformation of an Earth model, although strictly a highly artificial physical situation, nevertheless provides a useful idealization in certain problems. We derive a computational algorithm which allows one to determine any geophysically observable aspect of the purely static deformation of a non-rotating, spherically symmetric Earth model which has an elastic mantle and a compressible fluid core. In particular, we show how to compute the static perturbation to the density, the gravitational potential, and the fluid pressure at every point within the core, as well as every aspect of the deformation in the solid elastic mantle. Because of the artificiality of the concept of a purely static deformation, the static Lagrangian particle displacement in the fluid core is indeterminate.

1. Introduction

There has been considerable discussion and controversy recently (Longman 1963; Jeffreys & Vicente 1966; Smylie & Mansinha 1971; Dahlen 1971, 1973; Israel, Ben-Menahem & Singh 1973; Pekeris & Accad 1972) concerning the static deformation of an Earth model which has a fluid core. Contradictory claims have appeared concerning various aspects of this problem: the nature of the boundary conditions to be imposed at the core–mantle boundary, the determinacy or indeterminacy of the motion in the core, etc. This paper will attempt to clarify this presently confused situation. We will show that most of the confusion has arisen from an insistence on the part of all of the above authors on utilizing linearized equations involving the Lagrangian particle displacement to describe the deformation in the fluid core. Since, in the limit of a purely static (zero frequency) deformation, the Lagrangian displacements of the fluid particles in the core may become arbitrarily large, a linearized Lagrangian description of the core motion will in general be inadequate. The more natural description of possible core motions is in terms of an Eulerian formulation of the equations of motion. We will show that the use of an Eulerian description of the motion in the fluid core leads naturally to a complete solution of the problem of static deformation. This solution is characterized by a complete indeterminacy of the Lagrangian particle displacement in the fluid core.

2. Formulation of the problem

We restrict discussion to models of the Earth which are non-rotating, spherically symmetric, and everywhere in hydrostatic equilibrium. In fact, for simplicity, we only consider models which have a single spherical shell or ‘mantle’ surrounding a spherical fluid 'core'. The discussion may however be readily extended to the consideration of models having several alternating fluid and solid spherical shells (and in particular to the case of a model having a solid inner core).

Consider then a spherically symmetric self-gravitating equilibrium configuration of radius a which has a fluid core of radius b. The core fluid, $0 \leqslant r < b$, is assumed to be inviscid, but compressible and inhomogeneous; the mantle material, $b \leqslant r \leqslant a$, is assumed to be a perfectly elastic and isotropic, but inhomogeneous, solid. Let $\rho_0(r)$, $\phi_0(r)$, and $p_0(r)$ denote, respectively, the density, gravitational potential, and hydrostatic pressure in this equilibrium configuration; the gravitational field is $-g_0(r)\,\hat{\mathbf{r}}$, where $g_0(r) = \partial_r \phi_0(r)$. These quantities are related to each other by Poisson's equation

$$(\partial_r + 2r^{-1})\, g_0(r) = 4\pi G \rho_0(r), \tag{1}$$

and by the momentum equation describing the hydrostatic equilibrium

$$\partial_r p_0(r) + \rho_0(r)\, g_0(r) = 0. \tag{2}$$

In equation (1), G is Newton's gravitational constant. The density $\rho_0(r)$ will be assumed to be piecewise continuously differentiable in $0 \leqslant r \leqslant a$; i.e. jump discontinuities in $\rho_0(r)$ are allowed in both the core and the mantle, as well as at the core–mantle boundary $r = b$. Both $\phi_0(r)$ and $g_0(r)$ are continuous for all r in $0 \leqslant r < \infty$ and vanish at $r = \infty$; in addition, $g_0(r)$ vanishes at $r = 0$. The static pressure $p_0(r)$ is continuous for all r in $0 \leqslant r \leqslant a$, and vanishes at the free surface, $r = a$.

Choose an origin of co-ordinates at the centre of this spherical mass, and let $\mathbf{r}$ denote the position vector of points in an inertial reference frame fixed to this origin. We label material particles in both the mantle and the core by their positions $\mathbf{x}$ in the equilibrium configuration. The subscript or superscript E will be used to denote Eulerian field variables, while the subscript or superscript L will denote the corresponding Lagrangian field variables. We wish to consider the infinitesimal deformations of this Earth model which result, say, from the application of an external body force. Let $\mathbf{f}_E(\mathbf{r}, t)$ be the net externally applied body force per unit volume acting at the location $\mathbf{r}$ at time t. We are primarily interested in the static response to a time-independent imposed force $\mathbf{f}_E(\mathbf{r})$, but it is instructive to consider the more general case.

The response of the Earth model to an arbitrary applied body force $\mathbf{f}_E(\mathbf{r}, t)$ will in general include a change in the shapes of both the core–mantle boundary and the outer free surface. Say that at time t, the core–mantle boundary and the outer free surface have the form, respectively

$$\left.\begin{aligned} r &= b[1+\beta(\hat{\mathbf{r}}, t)] \\ r &= a[1+\alpha(\hat{\mathbf{r}}, t)] \end{aligned}\right\}. \tag{3}$$

Thus, at time t, the solid mantle fills the volume

$$b[1+\beta(\hat{\mathbf{r}}, t)] \leqslant r \leqslant a[1+\alpha(\hat{\mathbf{r}}, t)]$$

and, in order to prevent either cavitation or inter-penetration, the core fluid must occupy $0 \leqslant r < b[1+\beta(\hat{\mathbf{r}}, t)]$; we assume that $|\beta(\hat{\mathbf{r}}, t)| \ll 1$ and $|\alpha(\hat{\mathbf{r}}, t)| \ll 1$.

3. The equations of motion in the mantle

In the mantle, the natural description of the deformation is a Lagrangian description. The discussion will follow that in Dahlen (1972, 1973). Let $\mathbf{r}(\mathbf{x}, t)$ denote the position vector at time t of the material particle $\mathbf{x}$, and define the Lagrangian particle displacement $\mathbf{s}_L(\mathbf{x}, t)$ of particle $\mathbf{x}$ at time t by

$$\mathbf{r}(\mathbf{x}, t) = \mathbf{x} + \mathbf{s}_L(\mathbf{x}, t). \tag{4}$$

Let $\mathbf{T}_L(\mathbf{x}, t)$ denote the non-symmetric Piola–Kirchoff stress tensor (Malvern 1969; Dahlen 1973) and define its incremental part $\tilde{\mathbf{T}}_L(\mathbf{x}, t)$ by

$$\mathbf{T}_L(\mathbf{x}, t) = -p_0(\mathbf{x})\,\mathbf{I}+\tilde{\mathbf{T}}_L(\mathbf{x}, t). \tag{5}$$

It is more natural to utilize in the mantle a fully Lagrangian description of both the perturbation in the density and the perturbation in the gravitational potential, but it is customary to utilize instead an Eulerian description of these quantities. Let $\rho_L(\mathbf{x}, t)$ and $\phi_L(\mathbf{x}, t)$ denote, respectively, the net mass density and the net gravitational potential at the particle $\mathbf{x}$ at time t. We introduce also the Eulerian descriptions of these quantities, defined by

$$\left.\begin{aligned}\rho_E(\mathbf{r}(\mathbf{x}, t), t) &= \rho_L(\mathbf{x}, t)\\ \phi_E(\mathbf{r}(\mathbf{x}, t), t) &= \phi_L(\mathbf{x}, t)\end{aligned}\right\}. \tag{6}$$

We decompose $\rho_L(\mathbf{x}, t)$, $\phi_L(\mathbf{x}, t)$, $\rho_E(\mathbf{r}, t)$, and $\phi_E(\mathbf{r}, t)$ into an equilibrium part plus an incremental part

$$\begin{aligned}\rho_L(\mathbf{x}, t) &= \rho_0(\mathbf{x})+\rho_1{}^L(\mathbf{x}, t)\\ \phi_L(\mathbf{x}, t) &= \phi_0(\mathbf{x})+\phi_1{}^L(\mathbf{x}, t)\\ \rho_E(\mathbf{r}, t) &= \rho_0(\mathbf{r})+\rho_1{}^E(\mathbf{r}, t)\\ \phi_E(\mathbf{r}, t) &= \phi_0(\mathbf{r})+\phi_1{}^E(\mathbf{r}, t).\end{aligned}$$

We similarly define the Lagrangian description of the externally applied body force density $\mathbf{f}_L(\mathbf{x}, t)$ in the usual way by

$$\mathbf{f}_L(\mathbf{x}, t) = \mathbf{f}_E(\mathbf{r}(\mathbf{x}, t), t). \tag{6.5}$$

The exact form of the momentum conservation law in the Lagrangian formulation is

$$\rho_0(\mathbf{x})\, D_t{}^2\, \mathbf{s}_L(\mathbf{x}, t) = \nabla\cdot\mathbf{T}_L(\mathbf{x}, t)-\rho_0(\mathbf{x})\,\nabla\phi_L(\mathbf{x}, t)+\rho_0(\mathbf{x})\,\rho_L{}^{-1}(\mathbf{x}, t)\,\mathbf{f}_L(\mathbf{x}, t), \tag{7}$$

and the exact form of the continuity equation is

$$\rho_L(\mathbf{x}, t)|J\mathbf{r}(\mathbf{x}, t)| = \rho_0(\mathbf{x}), \tag{8}$$

where $J\mathbf{r}(\mathbf{x}, t)$ denotes, for fixed t, the Jacobian of the transformation $\mathbf{r}(\mathbf{x}, t)$.

We are interested only in infinitesimal deformations. Neglecting terms of second order in $\mathbf{s}_L(\mathbf{x}, t)$, equations (6) and (6.5) relating the Eulerian and Lagrangian descriptions of the density, gravitational potential, and external force become

$$\left.\begin{aligned}\rho_1{}^L(\mathbf{x}, t) &= \rho_1{}^E(\mathbf{x}, t)+\mathbf{s}_L(\mathbf{x}, t)\cdot\nabla\rho_0(\mathbf{x})\\ \phi_1{}^L(\mathbf{x}, t) &= \phi_1{}^E(\mathbf{x}, t)+\mathbf{s}_L(\mathbf{x}, t)\cdot\nabla\phi_0(\mathbf{x})\\ \mathbf{f}_L(\mathbf{x}, t) &= \mathbf{f}_E(\mathbf{x}, t).\end{aligned}\right\} \tag{9}$$

We will follow the established convention and utilize the incremental Eulerian quantities $\rho_1{}^E(\mathbf{x}, t)$ and $\phi_1{}^E(\mathbf{x}, t)$ as the first-order field variables, although $\rho_1{}^L(\mathbf{x}, t)$ and $\phi_1{}^L(\mathbf{x}, t)$ are the more natural pair to use in conjunction with the Lagrangian particle displacement $\mathbf{s}_L(\mathbf{x}, t)$.

We obtain the appropriate linearized versions of the conservation laws (7) and (8) by neglecting terms of second order in $\mathbf{s}_L(\mathbf{x}, t)$, and making use of (9)

$$\left.\begin{aligned}\rho_0\, D_t{}^2\, \mathbf{s}_L &= -\rho_0\,\nabla\phi_1{}^E-\rho_0\,\mathbf{s}_L\cdot\nabla(\nabla\phi_0)+\nabla\cdot\tilde{\mathbf{T}}_L+\mathbf{f}_L\\ \rho_1{}^E &= -\nabla\cdot(\rho_0\,\mathbf{s}_L)\end{aligned}\right\}. \tag{10}$$

Note that D_t in (10) denotes a simple partial differentiation with respect to time of the Lagrangian quantity $\mathbf{s}_L(\mathbf{x}, t)$.

The Eulerian first-order perturbation in the gravitational potential $\phi_1^{\mathrm{E}}(\mathbf{x}, t)$ is related to $\mathbf{s}_{\mathrm{L}}(\mathbf{x}, t)$ through the linearized Poisson's equation,

$$\nabla^2 \phi_1^{\mathrm{E}} = 4\pi G \rho_1^{\mathrm{E}}. \tag{11}$$

The equations (10) and (11) must be supplemented by an appropriate constitutive relation relating the incremental Piola–Kirchoff stress $\tilde{\mathbf{T}}_{\mathrm{L}}(\mathbf{x}, t)$ to the Lagrangian displacement $\mathbf{s}_{\mathrm{L}}(\mathbf{x}, t)$. We assume that the mantle material is an isotropic, perfectly elastic solid, and furthermore that the infinitesimal deformation $\mathbf{s}_{\mathrm{L}}(\mathbf{x}, t)$ occurs isentropically. The appropriate linearized constitutive relation is (Dahlen 1972, 1973)

$$\tilde{\mathbf{T}}_{\mathrm{L}} = \kappa(\nabla\cdot\mathbf{s}_{\mathrm{L}})\,\mathbf{I} + 2\mu[\nabla\mathbf{s}_{\mathrm{L}} + (\nabla\mathbf{s}_{\mathrm{L}})^{\mathrm{T}}] - p_0(\nabla\cdot\mathbf{s}_{\mathrm{L}})\,\mathbf{I} + p_0(\nabla\mathbf{s}_{\mathrm{L}})^{\mathrm{T}}, \tag{12}$$

where the superscript T denotes the transpose, and where $\kappa(\mathbf{x})$ and $\mu(\mathbf{x})$ are the *in situ* isentropic bulk modulus and the rigidity of the mantle material. The assumption of spherical symmetry implies that $\kappa(\mathbf{x})$ and $\mu(\mathbf{x})$ are functions only of radius r; we assume also that the rigidity $\mu(r) > 0$ for all r in $b \leqslant r \leqslant a$.

It is convenient to introduce another incremental stress tensor $\mathbf{T}_{\mathrm{L}}(\mathbf{x}, t)$ defined by

$$\mathbf{T}_{\mathrm{L}} = \kappa(\nabla\cdot\mathbf{s}_{\mathrm{L}})\,\mathbf{I} + 2\mu[\nabla\mathbf{s}_{\mathrm{L}} + (\nabla\mathbf{s}_{\mathrm{L}})^{\mathrm{T}}]. \tag{13}$$

To first order in the small displacement $\mathbf{s}_{\mathrm{L}}(\mathbf{x}, t)$, $\mathbf{T}_{\mathrm{L}}(\mathbf{x}, t)$ represents the Lagrangian description of the incremental Cauchy stress; note that $\mathbf{T}_{\mathrm{L}}(\mathbf{x}, t)$ is symmetric whereas the incremental Piola–Kirchoff stress $\tilde{\mathbf{T}}_{\mathrm{L}}(\mathbf{x}, t)$ is not. Written in terms of $\mathbf{T}_{\mathrm{L}}(\mathbf{x}, t)$, the complete system of linearized mantle equations is

$$\left.\begin{aligned}
\rho_0\, D_t^2\, \mathbf{s}_{\mathrm{L}} &= -\rho_0\, \nabla\phi_1^{\mathrm{E}} - \rho_1^{\mathrm{E}}\, \nabla\phi_0 - \nabla(\mathbf{s}_{\mathrm{L}}\cdot\rho_0\,\nabla\phi_0) + \nabla\cdot\mathbf{T}_{\mathrm{L}} + \mathbf{f}_{\mathrm{L}}\\
\rho_1^{\mathrm{E}} &= -\nabla\cdot(\rho_0\,\mathbf{s}_{\mathrm{L}})\\
\nabla^2\phi_1^{\mathrm{E}} &= 4\pi G\rho_1^{\mathrm{E}}\\
\mathbf{T}_{\mathrm{L}} &= \kappa(\nabla\cdot\mathbf{s}_{\mathrm{L}})\,\mathbf{I} + 2\mu[\nabla\mathbf{s}_{\mathrm{L}} + (\nabla\mathbf{s}_{\mathrm{L}})^{\mathrm{T}}].
\end{aligned}\right\} \tag{14}$$

We will refer to this system of equations (14) as the linearized Lagrangian equations in the mantle, even though they are written in terms of the Eulerian field variables $\rho_1^{\mathrm{E}}(\mathbf{x}, t)$ and $\phi_1^{\mathrm{E}}(\mathbf{x}, t)$. The corresponding Lagrangian field variables $\rho_1^{\mathrm{L}}(\mathbf{x}, t)$ and $\phi_1^{\mathrm{L}}(\mathbf{x}, t)$ can of course always be found, using (9).

The equations (14) apply for all times t and at all points $\mathbf{x}$ throughout the undeformed mantle volume $a \leqslant r \leqslant b$. In the event that there are jump discontinuities in any of the mantle properties $\rho_0(r)$, $\kappa(r)$, $\mu(r)$, the differential equations of motion (14) must be supplemented by certain appropriately linearized continuity relations at the undeformed level of the discontinuity. These conditions are:

$$\left.\begin{aligned}
&\mathbf{s}_{\mathrm{L}}(\mathbf{x}, t) \text{ continuous}\\
&\hat{\mathbf{r}}\cdot\mathbf{T}_{\mathrm{L}}(\mathbf{x}, t) \text{ continuous}\\
&\phi_1^{\mathrm{E}}(\mathbf{x}, t) \text{ continuous}\\
&\hat{\mathbf{r}}\cdot[\nabla\phi_1^{\mathrm{E}}(\mathbf{x}, t) + 4\pi G\rho_0(\mathbf{x})\,\mathbf{s}_{\mathrm{L}}(\mathbf{x}, t)] \text{ continuous}
\end{aligned}\right\} \tag{15}$$

In particular at the outer free surface $r = a$ of the mantle, we have the linearized boundary conditions:

$$\left.\begin{aligned}
&\hat{\mathbf{r}}\cdot\mathbf{T}_{\mathrm{L}}(a_-\,\hat{\mathbf{r}}, t) = 0\\
&\phi_1^{\mathrm{E}}(a_-\,\hat{\mathbf{r}}, t) = \phi_1^{\mathrm{E}}(a_+\,\hat{\mathbf{r}}, t)\\
&\hat{\mathbf{r}}\cdot\nabla\phi_1^{\mathrm{E}}(a_-\,\hat{\mathbf{r}}, t) + 4\pi G\rho_0(a)\,\hat{\mathbf{r}}\cdot\mathbf{s}_{\mathrm{L}}(a_-\,\hat{\mathbf{r}}, t) = \hat{\mathbf{r}}\cdot\nabla\phi_1^{\mathrm{E}}(a_+\,\hat{\mathbf{r}}, t)
\end{aligned}\right\} \tag{16}$$

where for any field $q(\mathbf{x}, t)$, the symbols $q(a_\pm\,\hat{\mathbf{r}}, t)$ denote the limits as $\varepsilon > 0$ tends to zero of the quantities $q((a\pm\varepsilon)\,\hat{\mathbf{r}}, t)$.

We will also require a mathematical statement of the fact that the boundaries

$r = a[1+\alpha(\hat{\mathbf{r}}, t)]$ and $r = b[1+\beta(\hat{\mathbf{r}}, t)]$ of the solid mantle must always consist of the same material particles. Consider the outer boundary; simple geometrical considerations make it clear that this condition takes the form

$$a\alpha(\hat{\boldsymbol{\eta}}(\hat{\mathbf{r}}, t), t) = \hat{\mathbf{r}}\cdot\mathbf{s}_L(a_-\,\hat{\mathbf{r}}, t), \tag{17}$$

where the unit vector $\hat{\boldsymbol{\eta}}(\hat{\mathbf{r}}, t)$ is defined by

$$\hat{\boldsymbol{\eta}}(\hat{\mathbf{r}}, t) = \frac{(a\hat{\mathbf{r}}+\mathbf{s}_L(a_-\,\hat{\mathbf{r}}, t))}{|(a\hat{\mathbf{r}}+\mathbf{s}_L(a_-\,\hat{\mathbf{r}}, t))|}. \tag{18}$$

The linearized version of this condition is, neglecting terms of second order in $\mathbf{s}_L(\mathbf{x}, t)$,

$$a\alpha(\hat{\mathbf{r}}, t) = \hat{\mathbf{r}}\cdot\mathbf{s}_L(a_-\,\hat{\mathbf{r}}, t). \tag{19}$$

Likewise, at the core–mantle boundary we must have

$$b\beta(\hat{\mathbf{r}}, t) = \hat{\mathbf{r}}\cdot\mathbf{s}_L(b_+\,\hat{\mathbf{r}}, t). \tag{20}$$

Equations (19) and (20) serve to define $\alpha(\hat{\mathbf{r}}, t)$ and $\beta(\hat{\mathbf{r}}, t)$ in terms of the Lagrangian description of the mantle deformation.

4. The equations of motion in the core

The natural description of the motion in the fluid core is an Eulerian description. We consider the fluid motion at a point $\mathbf{r}$ which is truly within the deformed core volume $0 \leqslant r < b[1+\beta(\hat{\mathbf{r}}, t)]$ for all times t. Let $\mathbf{u}_E(\mathbf{r}, t)$ denote the fluid velocity at the point $\mathbf{r}$ at time t, let $\rho_E(\mathbf{r}, t)$ denote the density of the fluid at the point $\mathbf{r}$ at time t, and let $\phi_E(\mathbf{r}, t)$ denote the net gravitational potential at the point $\mathbf{r}$ at time t. Since the fluid is assumed to be inviscid, the Eulerian or Cauchy stress tensor $\mathbf{T}_E(\mathbf{r}, t)$ at a point $\mathbf{r}$ in the core is always isotropic, i.e.

$$\mathbf{T}_E(\mathbf{r}, t) = -p_E(\mathbf{r}, t)\,\mathbf{I}, \tag{21}$$

where $p_E(\mathbf{r}, t)$ is the Eulerian pressure at the point $\mathbf{r}$ at time t. We decompose $\rho_E(\mathbf{r}, t)$, $\phi_E(\mathbf{r}, t)$ and $p_E(\mathbf{r}, t)$ into an equilibrium part plus an incremental part

$$\left.\begin{aligned} \rho_E(\mathbf{r}, t) &= \rho_0(r)+\rho_1{}^E(\mathbf{r}, t) \\ \phi_E(\mathbf{r}, t) &= \phi_0(\mathbf{r})+\phi_1{}^E(\mathbf{r}, t) \\ p_E(\mathbf{r}, t) &= p_0(\mathbf{r})+p_1{}^E(\mathbf{r}, t) \end{aligned}\right\} \tag{22}$$

The quantity $p_1{}^E(\mathbf{r}, t)$ will be called the incremental Eulerian pressure.

The exact form of the momentum equation in an inviscid fluid is

$$\rho_E(\mathbf{r}, t)\, D_t\, \mathbf{u}_E(\mathbf{r}, t) = -\rho_E(\mathbf{r}, t)\, \nabla\phi_E(\mathbf{r}, t) - \nabla p_E(\mathbf{r}, t) + \mathbf{f}_E(\mathbf{r}, t), \tag{23}$$

and the exact form of the continuity equation is

$$D_t\, \rho_E(\mathbf{r}, t) + \rho_E(\mathbf{r}, t)\, \nabla\cdot\mathbf{u}_E(\mathbf{r}, t) = 0. \tag{24}$$

The appropriate linearized versions of the Eulerian conservation laws are obtained by neglecting terms of second order in the Eulerian velocity $\mathbf{u}_E(\mathbf{r}, t)$

$$\left.\begin{aligned} \rho_0\, \partial_t\, \mathbf{u}_E &= -\nabla p_1{}^E - \rho_0\, \nabla\phi_1{}^E - \rho_1{}^E\, \nabla\phi_0 + \mathbf{f}_E \\ \partial_t\, \rho_1{}^E &= -\nabla\cdot(\rho_0\, \mathbf{u}_E) \end{aligned}\right\} \tag{25}$$

Notice that in the course of this linearization, we have made no possibly unjustified assumptions about the magnitude of the accompanying Lagrangian particle displacements in the fluid core. Poisson's equation in the fluid core is

$$\nabla^2\, \phi_1{}^E = 4\pi G \rho_1{}^E. \tag{26}$$

The linearized equations (25) and (26) must be supplemented by an equation of state for the inviscid fluid. We shall assume that the deformation is isentropic, i.e. that

$$D_t S_E(\mathbf{r}, t) = 0, \tag{27}$$

where $S_E(\mathbf{r}, t)$ is the fluid entropy at the point $\mathbf{r}$ at time t. It is well known (see, e.g. Eckart 1960) that the linearized version of this condition may be written in the form

$$\rho_0(\partial_t p_1{}^E + \mathbf{u}_E \cdot \nabla p_0) = \kappa(\partial_t \rho_1{}^E + \mathbf{u}_E \cdot \nabla \rho_0), \tag{28}$$

where $\kappa(\mathbf{r})$ is the isentropic bulk modulus of the fluid, and where terms of second order in $\mathbf{u}_E(\mathbf{r}, t)$ have been neglected. The assumption of spherical symmetry implies that $\kappa(\mathbf{r})$ is a function only of radius r in $0 \leqslant r < b$.

Summarizing, the linearized Eulerian equations describing the deformation at any time t and at any point $\mathbf{r}$ in the core $0 \leqslant r < b[1+\beta(\hat{\mathbf{r}}, t)]$ are:

$$\left.\begin{aligned} &\rho_0 \partial_t \mathbf{u}_E = -\nabla p_1{}^E - \rho_0 \nabla \phi_1{}^E - \rho_1{}^E \nabla \phi_0 + \mathbf{f}_E \\ &\partial_t \rho_1{}^E = -\nabla \cdot (\rho_0 \mathbf{u}_E) \\ &\nabla^2 \phi_1{}^E = 4\pi G \rho_1{}^E \\ &\partial_t p_1{}^E + \mathbf{u}_E \cdot \nabla p_0 = -\kappa \nabla \cdot \mathbf{u}_E. \end{aligned}\right\} \tag{29}$$

The equation of state (28) has here been rewritten in a more convenient form by an application of the linearized continuity equation.

In the event that there are jump discontinuities in either of $\rho_0(r)$ or $\kappa(r)$, the equations of motion (29) must be supplemented by the appropriate linearized continuity conditions at the undeformed level of the discontinuity. These conditions are:

$$\left.\begin{aligned} &\hat{\mathbf{r}} \cdot \mathbf{u}_E(\mathbf{r}, t) \text{ continuous} \\ &\hat{\mathbf{r}} \cdot [\partial_t p_1{}^E(\mathbf{r}, t) + \rho_0(\mathbf{r}) g_0(\mathbf{r}) \mathbf{u}_E(\mathbf{r}, t)] \text{ continuous} \\ &\phi_1{}^E(\mathbf{r}, t) \text{ continuous} \\ &\hat{\mathbf{r}} \cdot [\partial_t \nabla \phi_1{}^E(\mathbf{r}, t) + 4\pi G \rho_0(\mathbf{r}) \mathbf{u}_E(\mathbf{r}, t)] \text{ continuous} \end{aligned}\right\} \tag{30}$$

Note that continuity of the tangential component of the velocity $\mathbf{u}_E(\mathbf{r}, t)$ is not required, and in fact at any discontinuity in the density $\rho_0(r)$, the tangential velocity will in general be discontinuous.

Consider now the appropriate continuity conditions to be imposed at the core–mantle boundary. We have thus far developed a Lagrangian description of the deformation in the solid mantle and an Eulerian description in the fluid core. In order to connect the two descriptions at the core–mantle boundary, we must at least temporarily employ either a linearized Lagrangian description of the core motion or, alternatively, a linearized Eulerian description of the mantle motion. We choose the latter alternative, since it will be shown that in a static deformation the Lagrangian particle displacements in the fluid core may become arbitrarily large, thus invalidating any linearized Lagrangian treatment. The introduction of first-order or incremental Eulerian field variables $\mathbf{u}_E(\mathbf{r}, t)$, $\rho_1{}^E(\mathbf{r}, t)$, $\phi_1{}^E(\mathbf{r}, t)$, and $\mathbf{T}_E(\mathbf{r}, t)$ into the mantle is, on the other hand, straightforward. To conform to convention, we are already utilizing $\rho_1{}^E(\mathbf{r}, t)$ and $\phi_1{}^E(\mathbf{r}, t)$ as field variables in the mantle. We also have, correct to first-order in $\mathbf{s}_L(\mathbf{x}, t)$

$$\left.\begin{aligned} &\mathbf{u}_E(\mathbf{r}, t) = \partial_t \mathbf{s}_L(\mathbf{r}, t) \\ &\mathbf{T}_E(\mathbf{r}, t) = \mathbf{T}_L(\mathbf{r}, t) + \mathbf{s}_L(\mathbf{r}, t) \cdot \nabla p_0(\mathbf{r}) \mathbf{I}. \end{aligned}\right\} \tag{31}$$

The linearized continuity conditions on the first-order Eulerian field variables at the undeformed core–mantle boundary $r = b$ may be readily expressed in terms of the deformed shape $\beta(\hat{\mathbf{r}}, t)$ of the boundary

$$\left.\begin{aligned}
&\hat{\mathbf{r}}\cdot\mathbf{u}_E(b_-\hat{\mathbf{r}}, t) = \hat{\mathbf{r}}\cdot\mathbf{u}_E(b_+\hat{\mathbf{r}}, t)\\
&-\hat{\mathbf{r}}p_1{}^E(b_-\hat{\mathbf{r}}, t)+\hat{\mathbf{r}}\rho_0(b_-)g_0(b)\,b\beta(\hat{\mathbf{r}}, t) = \hat{\mathbf{r}}\cdot\mathbf{T}_E(b_+\hat{\mathbf{r}}, t)+\hat{\mathbf{r}}\rho_0(b_+)g_0(b)\,b\beta(\hat{\mathbf{r}}, t)\\
&\phi_1{}^E(b_-\hat{\mathbf{r}}, t) = \phi_1{}^E(b_+\hat{\mathbf{r}}, t)\\
&\hat{\mathbf{r}}\cdot\nabla\phi_1{}^E(b_-\hat{\mathbf{r}}, t)+4\pi G\rho_0(b_-)\,b\beta(\hat{\mathbf{r}}, t) = \hat{\mathbf{r}}\cdot\nabla\phi_1{}^E(b_+\hat{\mathbf{r}}, t)+4\pi G\rho_0(b_+)\,b\beta(\hat{\mathbf{r}}, t).
\end{aligned}\right\} \quad (32)$$

We also require a mathematical expression of the condition that the deformed boundary surface $r = b[1+\beta(\hat{\mathbf{r}}, t)]$ always consists of the same fluid particles. This condition is familiar from the theory of surface ocean gravity waves (Lamb 1945; Phillips 1966). Correct to first order in the Eulerian fluid velocity $\mathbf{u}_E(\mathbf{r}, t)$ in the core, it may be written as

$$\hat{\mathbf{r}}\cdot\mathbf{u}_E(b_-\hat{\mathbf{r}}, t) = b\partial_t\,\beta(\hat{\mathbf{r}}, t). \quad (33)$$

Now equations (20), (31) and (33) may be used to convert the strictly Eulerian continuity conditions (32) into the appropriate conditions which connect the Eulerian description of the motion in the core to the Lagrangian description in the mantle. We obtain

$$\left.\begin{aligned}
&\hat{\mathbf{r}}\cdot\mathbf{u}_E(b_-\hat{\mathbf{r}}, t) = \hat{\mathbf{r}}\cdot\partial_t\,\mathbf{s}_L(b_+\hat{\mathbf{r}}, t)\\
&-\hat{\mathbf{r}}p_1{}^E(b_-\hat{\mathbf{r}}, t) = \hat{\mathbf{r}}\cdot\mathbf{T}_L(b_+\hat{\mathbf{r}}, t)-\hat{\mathbf{r}}\rho_0(b_-)g_0(b)[\hat{\mathbf{r}}\cdot\mathbf{s}_L(b_+\hat{\mathbf{r}}, t)]\\
&\phi_1{}^E(b_-\hat{\mathbf{r}}, t) = \phi_1{}^E(b_+\hat{\mathbf{r}}, t)\\
&\hat{\mathbf{r}}\cdot\nabla\phi_1{}^E(b_-\hat{\mathbf{r}}, t) = \hat{\mathbf{r}}\cdot\nabla\phi_1{}^E(b_+\hat{\mathbf{r}}, t)+4\pi G[\rho_0(b_+)-\rho_0(b_-)]\,\hat{\mathbf{r}}\cdot\mathbf{s}_L(b_+\hat{\mathbf{r}}, t).
\end{aligned}\right\} \quad (34)$$

We now have a complete linearized, spherically symmetric mathematical specification of the problem we have posed; namely, the determination of the infinitesimal elastic-gravitational response of a simple Earth model to an arbitrary externally applied body force. We utilize the Eulerian differential equations (29) and the associated continuity conditions (30) at all points $\mathbf{r}$ in the core $0 \leqslant r < b$, and we utilize the Lagrangian differential equations (14) and the associated continuity conditions (15) at all points (i.e. material particles) $\mathbf{x}$ in the mantle $b \leqslant r \leqslant a$. We use the continuity conditions (34) to connect these two systems of equations at the core–mantle boundary $r = b$, and we must satisfy the free surface boundary conditions (16) on the outer free surface $r = a$.

5. Principal applications

We list briefly the most important possible applications of the formalism developed above.

An important application is the determination of the free oscillation eigenfrequencies and eigenfunctions of the Earth model under consideration. If we consider oscillations of angular frequency ω, then the relevant equations are obtained by setting $\mathbf{f}_E(\mathbf{r}, t) = \mathbf{f}_L(\mathbf{x}, t) = 0$, and making the substitution $\partial_t = i\omega$ in the core and $D_t = i\omega$ in the mantle.

Another problem of interest is the response to a harmonic imposed body force $\mathbf{f}_E(\mathbf{r}, t) = \mathbf{f}_E(\mathbf{r}, \omega)\,e^{i\omega t}$ of a fixed angular frequency ω. In this case, we again simply make the substitution $\partial_t = i\omega$ in the core and $D_t = i\omega$ in the mantle. The most important practical application is the determination of the response to the harmonic luni-solar tidal forces. These forces may be derived from a potential, $\mathbf{f}_E(\mathbf{r}, \omega) = -\rho_0(\mathbf{r})\,\Delta\phi_{\text{tidal}}(\mathbf{r}, \omega)$, called the tidal potential; the tidal potential $\phi_{\text{tidal}}(\mathbf{r}, \omega)$ may be expressed throughout $0 \leqslant r \leqslant a$ as an expansion in spherical harmonics $Y_l^m(\hat{\mathbf{r}})$,

$$\phi_{\text{tidal}}(\mathbf{r}, \omega) = \sum_{l=2}^{\infty}\sum_{m=-l}^{l} \phi_l{}^m(\mathbf{r}, \omega)\;Y_l^m(\hat{\mathbf{r}}). \quad (35)$$

The response at the surface $r = a$ of a spherically symmetric Earth model is traditionally and conveniently characterized in terms of the dynamic Love numbers $h_l(\omega)$, $k_l(\omega)$, $l_l(\omega)$, $2 \leqslant l < \infty$ (Munk & MacDonald 1960). The usual method of treating this problem is to take advantage of the fact that $\nabla^2 \phi_{\text{tidal}}(\mathbf{r}, \omega) = 0$ throughout $0 \leqslant r \leqslant a$. We can thus simply incorporate $\phi_{\text{tidal}}(\mathbf{r}, \omega)$ into the definition of $\phi_1{}^{\text{E}}(\mathbf{r}, \omega)$ throughout $0 \leqslant r \leqslant a$, and appropriately alter the boundary conditions involving $\phi_1{}^{\text{E}}(a_+ \hat{\mathbf{r}}, \omega)$ at the free surface $r = a$ (see, e.g. Takeuchi 1950).

A related problem is the response to a harmonic surface mass loading $\sigma(\hat{\mathbf{r}}, t) = \sigma(\hat{\mathbf{r}}, \omega)\, e^{i\omega t}$ of a fixed angular frequency ω (here $\sigma(\hat{\mathbf{r}}, t)$ is the variable mass load per unit area which loads the outer surface $r = a$). If the load $\sigma(\hat{\mathbf{r}}, \omega)$ is expanded in a spherical harmonic expansion

$$\sigma(\hat{\mathbf{r}}, \omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_l{}^m(\omega)\, Y_l{}^m(\hat{\mathbf{r}}), \tag{36}$$

then the response, again at the surface $r = a$, is conveniently characterized in terms of the dynamic load Love numbers $h_l'(\omega)$, $k_l'(\omega)$, $l_l'(\omega)$, $0 \leqslant l < \infty$ (Munk & MacDonald 1960).

A surface mass load $\sigma(\hat{\mathbf{r}}, \omega)$ acts on the Earth model in two distinct ways; it produces a variable gravitational potential $\phi_{\text{load}}(\mathbf{r}, \omega)$ throughout the volume $0 \leqslant r \leqslant a$ of the model, as well as a variable normal traction $p_{\text{load}}(a_+ \hat{\mathbf{r}}, \omega) = -g_0(a)\, \sigma(\hat{\mathbf{r}}, \omega)$ on the outer surface $r = a$. Since $\nabla^2 \phi_{\text{load}}(\mathbf{r}, \omega) = 0$ throughout $0 \leqslant r < a$, the simplest way to treat this problem is again to incorporate the imposed potential $\phi_{\text{load}}(\mathbf{r}, \omega)$ into the definition of $\phi_1{}^{\text{E}}(\mathbf{r}, \omega)$ throughout $0 \leqslant r \leqslant a$, and to then appropriately alter the boundary conditions at the outer boundary $r = a$; in this case the surface $r = a$ is no longer a free surface because of the normal traction $p_{\text{load}}(a_+ \hat{\mathbf{r}}, \omega)$, and the surface stress boundary condition $\hat{\mathbf{r}} \cdot \mathbf{T}_{\text{L}}(a_- \hat{\mathbf{r}}, \omega) = 0$ must be replaced by $\hat{\mathbf{r}} \cdot \mathbf{T}_{\text{L}}(a_- \hat{\mathbf{r}}, \omega) = \hat{\mathbf{r}} p_{\text{load}}(a_+ \hat{\mathbf{r}}, \omega)$; see Longman (1963) and Farrell (1972).

Another problem which could be posed is the response of the Earth model to a transient applied body force, say $\mathbf{f}_{\text{E}}(\mathbf{r}, t) = 0$ for $t < 0$. In this case the most important practical application is the study of the excitation of the free oscillations of the Earth model by a kinematically prescribed dislocation source; the applied body force $\mathbf{f}_{\text{E}}(\mathbf{r}, t)$ is then taken to be the equivalent body force of the dislocation (Burridge & Knopoff 1964; Dahlen 1972). The solution to this problem is customarily obtained by assuming that the eigenfunctions of the Earth model form a complete set in terms of which the Laplace transform $\mathbf{f}_{\text{E}}(\mathbf{r}, p)$ of the applied force may be expanded, for a fixed value of p, the Laplace transform variable. Gilbert (1970) has thus obtained a particularly simple and elegant result for the case of a point dislocation source with a step function time dependence.

The final class of interesting problems concern the static deformation of the Earth model. There are three applications, corresponding to the three above-mentioned dynamical response problems. We may consider the response of the Earth model to a time-independent tidal potential $\phi_{\text{tidal}}(\mathbf{r})$, in which case we would characterize the response by the static Love numbers h_l, k_l, l_l, $2 \leqslant l < \infty$. We may also consider the response of the Earth model to a time-independent surface mass load $\sigma(\hat{\mathbf{r}})$, in which case we would characterize the response by the static load Love numbers h_l', k_l', l_l', $0 \leqslant l < \infty$. Finally, we may consider the static deformation of the Earth model produced by a static elastic dislocation; it is usually convenient to consider instead the response produced by the time-independent equivalent body force $\mathbf{f}_{\text{E}}(\mathbf{r})$. Since earthquakes occur only in the upper few hundred kilometres of the Earth, we are here dealing with a problem in which the applied body force $\mathbf{f}_{\text{E}}(\mathbf{r})$ is zero inside the core, $0 \leqslant r < b$ (in fact, in most applications concerned with the large scale or far field static deformation, it is sufficient to utilize a point dislocation model, in which case $\mathbf{f}_{\text{E}}(\mathbf{r}) = 0$ except at a single point $\mathbf{r}_0$, the earthquake epicentre).

We now consider in more detail two of the above possible applications. In Section 6, we show how the formalism which has been developed above reduces to the familiar equations which govern the free oscillations of the Earth model. In Section 7 we consider the static deformation of an Earth model by a time-independent external body force $\mathbf{f}_E(\mathbf{r})$ which is confined to the mantle (i.e. $\mathbf{f}_E(\mathbf{r}) = 0$ for $0 \leqslant r < b$). The consideration of these two examples should make it clear how to treat all of the above list of possible applications.

6. The free oscillations

Consider a free oscillation of the Earth model with a fixed angular frequency ω. The equations of motion appropriate to this case are obtained by setting $\mathbf{f}_E(\mathbf{r}, t) = \mathbf{f}_L(\mathbf{x}, t) = 0$, and by substituting $\partial_t = i\omega$ in the core and $D_t = i\omega$ in the mantle. Thus, in the core, $0 \leqslant r < b$, the Eulerian variables $\mathbf{u}_E(\mathbf{r}, \omega)$, ${\rho_1}^E(\mathbf{r}, \omega)$, ${\phi_1}^E(\mathbf{r}, \omega)$, and ${p_1}^E(\mathbf{r}, \omega)$ are required to satisfy

$$\left.\begin{aligned} i\omega\rho_0\,\mathbf{u}_E &= -\nabla {p_1}^E - \rho_0\,\nabla{\phi_1}^E - {\rho_1}^E\,\nabla\phi_0 \\ i\omega{\rho_1}^E &= -\nabla\cdot(\rho_0\,\mathbf{u}_E) \\ \nabla^2{\phi_1}^E &= 4\pi G{\rho_1}^E \\ i\omega{p_1}^E + \mathbf{u}_E\cdot\nabla p_0 &= -\kappa\nabla\cdot\mathbf{u}_E \end{aligned}\right\} \tag{37}$$

In the mantle, $b \leqslant r \leqslant a$, the Lagrangian variables $\mathbf{s}_L(\mathbf{x}, \omega)$, $\mathbf{T}_L(\mathbf{x}, \omega)$ and the Eulerian variables ${\rho_1}^E(\mathbf{x}, \omega)$, ${\phi_1}^E(\mathbf{x}, \omega)$ are required to satisfy

$$\left.\begin{aligned} -\omega^2\rho_0\,\mathbf{s}_L &= -\rho_0\,\nabla{\phi_1}^E - {\rho_1}^E\,\nabla\phi_0 - \nabla(\mathbf{s}_L\cdot\rho_0\,\nabla\phi_0) + \nabla\cdot\mathbf{T}_L \\ {\rho_1}^E &= -\nabla\cdot(\rho_0\,\mathbf{s}_L) \\ \nabla^2{\phi_1}^E &= 4\pi G{\rho_1}^E \\ \mathbf{T}_L &= \kappa(\nabla\cdot\mathbf{s}_L)\,\mathbf{I} + 2\mu[\nabla\mathbf{s}_L + (\nabla\mathbf{s}_L)^T] \end{aligned}\right\} \tag{38}$$

The continuity conditions at the core–mantle boundary $r = b$ take the form

$$\left.\begin{aligned} \hat{\mathbf{r}}\cdot\mathbf{u}_E(b_-\,\hat{\mathbf{r}}, \omega) &= i\omega\hat{\mathbf{r}}\cdot\mathbf{s}_L(b_+\,\hat{\mathbf{r}}, \omega) \\ -\hat{\mathbf{r}}{p_1}^E(b_-\,\hat{\mathbf{r}}, \omega) &= \hat{\mathbf{r}}\cdot\mathbf{T}_L(b_+\hat{\mathbf{r}}, \omega) - \hat{\mathbf{r}}\rho_0(b_-)\,g_0(b)[\hat{\mathbf{r}}\cdot\mathbf{s}_L(b_+\,\hat{\mathbf{r}}, \omega)] \\ {\phi_1}^E(b_-\,\hat{\mathbf{r}}, \omega) &= {\phi_1}^E(b_+\,\hat{\mathbf{r}}, \omega) \\ \hat{\mathbf{r}}\cdot\nabla{\phi_1}^E(b_-\,\hat{\mathbf{r}}, \omega) &= \hat{\mathbf{r}}\cdot\nabla{\phi_1}^E(b_+\,\hat{\mathbf{r}}, \omega) + 4\pi G[\rho_0(b_+) - \rho_0(b_-)]\,\hat{\mathbf{r}}\cdot\mathbf{s}_L(b_+\,\hat{\mathbf{r}}, \omega). \end{aligned}\right\} \tag{39}$$

The free surface boundary conditions at the outer boundary $r = a$ are

$$\left.\begin{aligned} \hat{\mathbf{r}}\cdot\mathbf{T}_L(a_-\,\hat{\mathbf{r}}, \omega) &= 0 \\ {\phi_1}^E(a_-\,\hat{\mathbf{r}}, \omega) &= {\phi_1}^E(a_+\,\hat{\mathbf{r}}, \omega) \\ \hat{\mathbf{r}}\cdot\nabla{\phi_1}^E(a_-\,\hat{\mathbf{r}}, \omega) + 4\pi G\rho_0(a_-)\,\hat{\mathbf{r}}\cdot\mathbf{s}_L(a_-\,\hat{\mathbf{r}}, \omega) &= \hat{\mathbf{r}}\cdot\nabla{\phi_1}^E(a_+\,\hat{\mathbf{r}}, \omega) \end{aligned}\right\} \tag{40}$$

These differential equations (37) and (38) and the associated boundary conditions (39) and (40) are all that is required for a complete description of the possible normal mode eigenfrequencies and eigenfunctions of the Earth model. They could be solved in the above form, retaining the full Eulerian formulation in the core, but it is not customary to do so. Instead of proceeding directly to solve the above system of equations, we make use of the observation that, as long as we restrict attention to those normal modes with a non-zero squared eigenfrequency $\omega^2 > 0$, it is not inconsistent with the approximations which have already been made to utilize a linearized Lagrangian formulation in the core as well as in the mantle. This can be shown in the following way. The Lagrangian particle displacement $\mathbf{s}_L(\mathbf{x}, t)$ is defined in the core by

$$D_t\,\mathbf{s}_L(\mathbf{x}, t) = \mathbf{u}_E(\mathbf{x} + \mathbf{s}_L(\mathbf{x}, t), t). \tag{41}$$

Expanding equation (41), we have

$$D_t \mathbf{s}_L(\mathbf{x}, t) = \mathbf{u}_E(\mathbf{x}, t) + \mathbf{s}_L(\mathbf{x}, t) \cdot \nabla \mathbf{u}_E(\mathbf{x}, t), \tag{42}$$

where the higher order terms have been omitted. Now whenever $\mathbf{u}_E(\mathbf{r}, t)$ is of the form $\mathbf{u}_E(\mathbf{r}, t) = \mathbf{u}_E(\mathbf{r}, \omega)\, e^{i\omega t}$ with $\omega^2 > 0$, the Lagrangian particle displacement $\mathbf{s}_L(\mathbf{x}, t)$ is of the same order as $\mathbf{u}_E(\mathbf{x}, t)$, and hence $\mathbf{s}_L(\mathbf{x}, t) \cdot \nabla \mathbf{u}_E(\mathbf{x}, t)$ is of second order and may be neglected. Thus, correct to first order in either $\mathbf{u}_E(\mathbf{x}, \omega)$ or $\mathbf{s}_L(\mathbf{x}, \omega)$, we have

$$i\omega \mathbf{s}_L(\mathbf{x}, \omega) = \mathbf{u}_E(\mathbf{x}, \omega), \tag{43}$$

and we may use a linearized Lagrangian description throughout the fluid core. We continue, as in the mantle equations (40) to use the Eulerian density and gravitational potential perturbations $\rho_1{}^E(\mathbf{x}, t)$ and $\phi_1{}^E(\mathbf{x}, t)$ in these otherwise Lagrangian equations. We do however introduce the incremental Lagrangian pressure $p_1{}^L(\mathbf{x}, t)$ which is defined, correct to first order in $\mathbf{s}_L(\mathbf{x}, t)$, by

$$p_1{}^L(\mathbf{x}, t) = p_1{}^E(\mathbf{x}, t) + \mathbf{s}_L(\mathbf{x}, t) \cdot \nabla p_0(\mathbf{x}). \tag{44}$$

The linearized Lagrangian equations valid at all points (fluid particles) $\mathbf{x}$ in the undeformed core $0 \leqslant r < b$ are obtained by substituting (43) and (44) into (37)

$$\left.\begin{aligned} -\omega^2 \rho_0 \mathbf{s}_L &= -\rho_0 \nabla\phi_1{}^E - \rho_1{}^E \nabla\phi_0 - \nabla(\mathbf{s}_L \cdot \rho_0 \nabla\phi_0) - \nabla p_1{}^L \\ \rho_1{}^E &= -\nabla \cdot (\rho_0 \mathbf{s}_L) \\ \nabla^2 \phi_1{}^E &= 4\pi G \rho_1{}^E \\ p_1{}^L &= -\kappa(\nabla \cdot \mathbf{s}_L). \end{aligned}\right\} \tag{45}$$

The continuity conditions (39) at the core–mantle boundary may be written in terms of the Lagrangian description in the core as

$$\left.\begin{aligned} \hat{\mathbf{r}} \cdot \mathbf{s}_L(b_- \hat{\mathbf{r}}, \omega) &= \hat{\mathbf{r}} \cdot \mathbf{s}_L(b_+ \hat{\mathbf{r}}, \omega) \\ -\hat{\mathbf{r}} p_1{}^L(b_- \hat{\mathbf{r}}, \omega) &= \hat{\mathbf{r}} \cdot \mathbf{T}_L(b_+ \hat{\mathbf{r}}, \omega) \\ \phi_1{}^E(b_- \hat{\mathbf{r}}, \omega) &= \phi_1{}^E(b_+ \hat{\mathbf{r}}, \omega) \\ \hat{\mathbf{r}} \cdot \nabla\phi_1{}^E(b_- \hat{\mathbf{r}}, \omega) + 4\pi G \rho_0(b_-) \hat{\mathbf{r}} \cdot \mathbf{s}_L(b_- \hat{\mathbf{r}}, \omega) & \\ = \hat{\mathbf{r}} \cdot \nabla\phi_1{}^E(b_+ \hat{\mathbf{r}}, \omega) &+ 4\pi G \rho_0(b_+) \hat{\mathbf{r}} \cdot \mathbf{s}_L(b_+ \hat{\mathbf{r}}, \omega). \end{aligned}\right\} \tag{46}$$

The linearized Lagrangian differential equations (45) in the core and (38) in the mantle, taken together with the linearized conditions (46) at the core–mantle boundary $r = b$ and (40) at the free surface $r = a$ constitute the familiar equations which are conventionally used in treatments of the free oscillations of spherically symmetric Earth models. The field variables in these conventional equations are partly Lagrangian quantities (the Lagrangian particle displacement $\mathbf{s}_L(\mathbf{x}, t)$ and the Lagrangian description of the incremental Cauchy stress tensor $\mathbf{T}_L(\mathbf{x}, t)$) and partly Eulerian (the Eulerian descriptions of the perturbations in density $\rho_1{}^E(\mathbf{x}, t)$ and gravitational potential $\phi_1{}^E(\mathbf{x}, t)$). These equations may be readily used for the determination of the eigenfunctions associated with any normal mode which has a non-zero squared eigenfrequency, $\omega^2 > 0$. Precisely these same linearized Lagrangian equations in both the core and the mantle may be used to determine the dynamic Love numbers $h_l(\omega)$, $k_l(\omega)$, $l_l(\omega)$ and the dynamic load Love numbers $h_l'(\omega)$, $k_l'(\omega)$, $l_l'(\omega)$ of any Earth model, provided that the free surface boundary conditions (40) are appropriately altered, and provided that we restrict consideration to angular frequencies $\omega > 0$. The essential point is that the assumption of a harmonic time dependence of the first-order field variables allows the consistent introduction of a linearized Lagrangian description into the fluid core.

The nature of the normal mode solutions to the equations (45), (38), (46) and

(40) is well known. They are of two types. There are toroidal mode multiplets, denoted ${}_nT_l$, characterized by a vanishing perturbation in the density and the gravitational potential, and having Lagrangian particle displacement fields $\mathbf{s}_L(\mathbf{x}, \omega)$ of the form

$$\mathbf{s}_L(\mathbf{x}, \omega) = {}_nW_l(r)[-\hat{\mathbf{r}} \times \nabla_1 \; Y_l^m(\hat{\mathbf{r}})], \qquad -l \leqslant m \leqslant l, \tag{47}$$

where $\nabla_1 = \hat{\theta}\partial_\theta + (\sin\theta)^{-1}\hat{\phi}\partial_\phi$. Provided that the associated $2l+1$-degenerate toroidal eigenfrequency ${}_n\omega_l{}^{\mathrm{T}}$ is non-zero, the Lagrangian particle displacement (47) of any member of the toroidal mode multiplet ${}_nT_l$ vanishes in the fluid core, i.e. ${}_nW_l(r) = 0$ for all r in $0 \leqslant r < b$. For every value of $l > 0$, there are an infinite number of toroidal multiplets ${}_nT_l$ ($n \geqslant 0$ unless $l = 1$ in which case $n \geqslant 1$); the associated $2l+1$-degenerate eigenfrequencies ${}_n\omega_l{}^{\mathrm{T}}$ become arbitrarily large as $n \to \infty$.

There are also poloidal mode multiplets, denoted ${}_nS_l$, characterized by a non-zero perturbation in both the density and the gravitational potential, and having a Lagrangian particle displacement of the form

$$\mathbf{s}_L(\mathbf{x}, \omega) = {}_nU_l(r)\,\hat{\mathbf{r}}\,Y_l^m(\hat{\mathbf{r}}) + {}_nV_l(r)\,\nabla_1 \; Y_l^m(\hat{\mathbf{r}}), \qquad -l \leqslant m \leqslant l. \tag{48}$$

The poloidal mode multiplets ${}_nS_l$ of Earth models which have a fluid core may be conveniently divided further into two classes. One class consists of all those poloidal mode multiplets whose features are controlled primarily by the elasticity of the Earth model, with buoyancy forces and gravitational restoring forces playing a relatively minor role. These normal modes, which we will call the elastic-gravitational (eg) type poloidal modes, have received much attention and are well understood. For every value of $l \geqslant 0$, there are an infinite number of poloidal multiplets ${}_nS_l$ of this type ($n \geqslant 0$ unless $l = 1$, in which case $n \geqslant 1$); the associated $2l+1$-degenerate eigenfrequencies ${}_n\omega_l{}^S$ become arbitrarily large as $n \to \infty$. The gravest eg type poloidal multiplet is in general the ${}_0S_2$ multiplet, whose associated eigenfrequency for any realistic Earth model is on the order of $2\pi/{}_0\omega_2{}^S \approx 54$ minutes.

The second class of poloidal modes consists of all those whose features are controlled primarily by buoyancy forces and by gravitational restoring forces in the fluid core, with the elasticity playing a relatively minor role. These normal modes, which we will call the gravitational-elastic (ge) type modes, have received correspondingly little attention, the reason being that any ge type poloidal mode is characterized by having a Lagrangian particle displacement (48) which is essentially confined to the fluid core; the existence of any such motion is thus virtually impossible to detect at the free surface $r = a$. Since we will have occasion to allude to these ge type poloidal normal modes later (in Section 7), we take this opportunity to discuss briefly and qualitatively a few of their properties.

The ge type poloidal modes of an Earth model with a fluid core are closely analogous to the internal gravity waves which are known to propagate in inviscid stratified fluids. The propagation of internal gravity waves in plane stratified fluids has been studied a great deal, because the phenomenon is important in the dynamics of the oceans and the atmosphere (see, e.g. Eckart 1960; Tolstoy 1963; Phillips 1966; Turner 1973). The complete set of equations (45), (38), (46), and (40) which govern the ge type poloidal modes of an Earth model with a fluid core are more complicated than are the equations which govern internal gravity wave propagation in plane stratified fluids, both because of the spherical symmetry and because of the effects of self-gravitation; the analogy is however sufficiently close that the principal qualitative features of the ge modes may be predicted with some confidence.

We introduce the local Brunt–Väisälä frequency $N(r)$ of the fluid core, defined for all r in $0 \leqslant r < b$ by

$$N^2(r) = -g_0(r)\,\rho_0{}^{-1}(r)\,\partial_r\,\rho_0(r) - g_0{}^2(r)\,\rho_0(r)\,\kappa^{-1}(r). \tag{49}$$

Most of the ge type dynamical characteristics of an Earth model with a fluid core may be conveniently summarized in terms of the local Brunt–Väisälä frequency. In particular, if $N^2(r) > 0$ for all $0 \leqslant r < b$, then the fluid core will be everywhere dynamically stable. Any Earth model with such an everywhere dynamically stable core will have, for every fixed value of $l > 0$, an infinite number of ge type poloidal mode multiplets ${}_nS_l$ $(n \leqslant -1)$ with positive squared eigenfrequencies $({}_n\omega_l^S)^2$. If $N^2(r) < 0$ in any finite region of the core, then the core will be dynamically unstable; this dynamical instability will be pointed up by the fact that any Earth model with such an anywhere unstable core will possess ge type poloidal mode solutions to equations (45), (38), (46) and (40) with associated squared eigenfrequencies ω^2 which are negative. If in fact $N^2(r) < 0$ for all r in $0 \leqslant r < b$, then all of the ge type solutions will be characterized by negative squared eigenfrequencies. If $N^2(r) = 0$ for all r in $0 \leqslant r < b$, then the core is in a state of neutral equilibrium. Such a core model has often been called an Adams–Williamson core model, since in the fluid core the Adams–Williamson equation, familiar in seismology, may be written simply $N^2(r) = 0$. Any Earth model which has an Adams–Williamson core will have no ge type normal modes with a non-zero squared eigenfrequency ω^2.

Earth models with everywhere dynamically stable cores may possibly be those of the most geophysical interest, especially in view of the recent suggestions of Higgins & Kennedy (1971). For every fixed value of $l > 0$, such Earth models will have an infinite number of ge type poloidal modes ${}_nS_l$ $(n \leqslant -1)$; the associated $2l+1$-degenerate eigenfrequencies ${}_n\omega_l^S$ become arbitrarily small as $n \to -\infty$. The highest frequency $(n = -1)$ ge type poloidal multiplet, for any fixed value of $l > 0$ will in general be on the order of $N_{\max}$, where $N_{\max}$ is the maximum value obtained by $N(r)$ in the core, $0 \leqslant r < b$. The ge type normal mode eigenfrequency spectrum becomes dense near zero frequency $(n \ll -1)$ for every value of $l > 0$. The Lagrangian particle displacement eigenfunctions $\mathbf{s}_L(\mathbf{x}, \omega)$ of the high overtone $(n \ll -1)$ ge type normal modes become increasingly confined to the core as $n \to -\infty$; the number of spherical nodal surfaces of both ${}_nU_l(r)$ and ${}_nV_l(r)$ within the core, $0 \leqslant r < b$, tends meanwhile to increase like n. The ratio ${}_nV_l(r)/{}_nU_l(r)$ of the tangential to the radial component of the Lagrangian particle displacement associated with a particular ge type mode ${}_nS_l$ will in general be on the order of $N_{\max}/{}_n\omega_l^S$, which also will become arbitrarily large as $n \to -\infty$. Thus, as $n \to -\infty$ for a fixed value of l, the ge type poloidal modes of an Earth model with a stably stratified core assume increasingly the character of slow $({}_n\omega_l^S \ll N_{\max})$, almost steady, purely tangential flows in the fluid core of the Earth model.

This brief description of the main properties of the ge type poloidal normal modes of an Earth model with a fluid core has been given here, because there is some interest in the relation of the very low frequency $({}_n\omega_l^S \ll N_{\max})$ modes to the problem of static deformation to be treated in the next section. Some of the remarks made above are strictly unproved, but they can be made with some confidence because of the close analogy with the simpler problem of internal gravity wave propagation in a plane, stratified fluid. A more quantitative discussion is planned for the future. The existence of ge type poloidal modes in the case of a dynamically stable core model, and the analogy with the propagation of internal gravity waves was first pointed out to me by Freeman Gilbert (private communication, *ca.* 1967).

7. Static deformation

We now consider the static deformation of a spherically symmetric Earth model with a fluid core; in particular we will treat the case of the static deformation produced by an applied body force $\mathbf{f}_E(\mathbf{r})$ which is confined to the mantle. The real Earth is never in fact deformed by the action of purely static imposed forces, but the concept of a static deformation is a useful approximation in treating certain phenomena.

The most important application is probably the determination of the permanent deformation which is associated with the slip on a seismic fault during an earthquake; this is most conveniently idealized as a purely static problem. It is in this context that much of the confusion and controversy has arisen.

We will show that the Lagrangian particle displacements in the fluid core may become arbitrarily large and are in fact indeterminate in the limit of a purely static (zero-frequency) deformation. This fact limits the usefulness and even the validity of a linearized Lagrangian formulation of the equations of motion in the fluid core; for that reason, we maintain the Eulerian formulation in the core.

The relevant equations governing this case are obtained by setting $D_t = 0$ in the mantle, and by setting $\mathbf{f}_E(\mathbf{r}) = 0$ and $\partial_t = 0$ in the core. Thus, in the core, $0 \leqslant r < b$, the static Eulerian variables $\mathbf{u}_E(\mathbf{r})$, ${\rho_1}^E(\mathbf{r})$, ${\phi_1}^E(\mathbf{r})$, and ${p_1}^E(\mathbf{r})$ are required to satisfy

$$\left.\begin{aligned} -\rho_0 \nabla {\phi_1}^E - {\rho_1}^E \nabla \phi_0 - \nabla {p_1}^E &= 0 \\ \nabla^2 {\phi_1}^E &= 4\pi G {\rho_1}^E \\ \nabla \cdot (\rho_0 \mathbf{u}_E) &= 0 \\ \mathbf{u}_E \cdot \nabla p_0 + \kappa \nabla \cdot \mathbf{u}_E &= 0 \end{aligned}\right\} \tag{50}$$

In the mantle, $b \leqslant r \leqslant a$, the static Lagrangian variables $\mathbf{s}_L(\mathbf{x})$, $\mathbf{T}_L(\mathbf{x})$ and the static Eulerian variables ${\rho_1}^E(\mathbf{x})$, ${\phi_1}^E(\mathbf{x})$ are required to satisfy

$$\left.\begin{aligned} &-\rho_0 \nabla {\phi_1}^E - {\rho_1}^E \nabla \phi_0 - \nabla(\mathbf{s}_L \cdot \rho_0 \nabla \phi_0) + \nabla \cdot \mathbf{T}_L + \mathbf{f}_L = 0 \\ &\nabla^2 {\phi_1}^E = 4\pi G {\rho_1}^E \\ &\quad {\rho_1}^E = -\nabla \cdot (\rho_0 \mathbf{s}_L) \\ &\quad \mathbf{T}_L = \kappa(\nabla \cdot \mathbf{s}_L)\mathbf{I} + 2\mu[\nabla \mathbf{s}_L + (\nabla \mathbf{s}_L)^T]. \end{aligned}\right\} \tag{51}$$

The continuity conditions at the core–mantle boundary $r = b$ become

$$\left.\begin{aligned} \hat{\mathbf{r}} \cdot \mathbf{u}_E(b_- \hat{\mathbf{r}}) &= 0 \\ -\hat{\mathbf{r}} {p_1}^E(b_- \hat{\mathbf{r}}) &= \hat{\mathbf{r}} \cdot \mathbf{T}_L(b_+ \hat{\mathbf{r}}) - \hat{\mathbf{r}} \rho_0(b_-) g_0(b)[\hat{\mathbf{r}} \cdot \mathbf{s}_L(b_+ \hat{\mathbf{r}})] \\ {\phi_1}^E(b_- \hat{\mathbf{r}}) &= {\phi_1}^E(b_+ \hat{\mathbf{r}}) \\ \hat{\mathbf{r}} \cdot \nabla {\phi_1}^E(b_- \hat{\mathbf{r}}) &= \hat{\mathbf{r}} \cdot \nabla {\phi_1}^E(b_+ \hat{\mathbf{r}}) + 4\pi G[\rho_0(b_+) - \rho_0(b_-)]\, \hat{\mathbf{r}} \cdot \mathbf{s}_L(b_+ \hat{\mathbf{r}}). \end{aligned}\right\} \tag{52}$$

The free surface boundary conditions at the outer boundary $r = a$ are in this case

$$\left.\begin{aligned} \hat{\mathbf{r}} \cdot \mathbf{T}_L(a_- \hat{\mathbf{r}}) &= 0 \\ {\phi_1}^E(a_- \hat{\mathbf{r}}) &= {\phi_1}^E(a_+ \hat{\mathbf{r}}) \\ \hat{\mathbf{r}} \cdot \nabla {\phi_1}^E(a_- \hat{\mathbf{r}}) + 4\pi G \rho_0(a_-)\, \hat{\mathbf{r}} \cdot \mathbf{s}_L(a_- \hat{\mathbf{r}}) &= \hat{\mathbf{r}} \cdot \nabla {\phi_1}^E(a_+ \hat{\mathbf{r}}). \end{aligned}\right\} \tag{53}$$

The differential equations (50) and (51) and the associated boundary conditions (52) and (53) are the complete set of equations governing the static response of the Earth model to the imposed body force $\mathbf{f}_L(\mathbf{x})$. Notice that we have not required that the Eulerian velocity $\mathbf{u}_E(\mathbf{r})$ vanish in the core; we allow for the possibility that steady fluid flows with infinitesimally small velocities but arbitrarily large particle displacements might be occurring in the core. Only the last two of the core equations (50) involve the Eulerian velocity $\mathbf{u}_E(\mathbf{r})$; these two equations may be conveniently rewritten in the form

$$\left.\begin{aligned} \mathbf{u}_E \cdot ({\rho_0}^2 \nabla \phi_0 + \kappa \nabla \rho_0) &= 0 \\ \mathbf{u}_E \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \mathbf{u}_E &= 0 \end{aligned}\right\}, \tag{54}$$

or even more succinctly in terms of the local Brunt–Väisälä frequency $N(r)$ defined by (49)

$$\left.\begin{aligned} u_r^{\mathrm{E}} N^2 &= 0 \\ u_r^{\mathrm{E}} \partial_r \rho_0 + \rho_0 \nabla\cdot\mathbf{u}_{\mathrm{E}} &= 0 \end{aligned}\right\}, \tag{55}$$

where $u_r^{\mathrm{E}}(\mathbf{r}) = \hat{\mathbf{r}}\cdot\mathbf{u}_{\mathrm{E}}(\mathbf{r})$. The velocity $\mathbf{u}_{\mathrm{E}}(\mathbf{r})$ appears in these equations (55) only in terms of the scalar quantities $u_r^{\mathrm{E}}(\mathbf{r})$ and $\nabla\cdot\mathbf{u}_{\mathrm{E}}(\mathbf{r})$.

In the mantle we introduce, following closely the notation of Backus (1967), the scalar potential functions $U(\mathbf{x})$, $V(\mathbf{x})$, $W(\mathbf{x})$, $P(\mathbf{x})$, $Q(\mathbf{x})$, $R(\mathbf{x})$, $X(\mathbf{x})$, $Y(\mathbf{x})$, $Z(\mathbf{x})$, defined by

$$\left.\begin{aligned} \mathbf{s}_{\mathrm{L}}(\mathbf{x}) &= \hat{\mathbf{r}}U(\mathbf{x}) + \nabla_1 V(\mathbf{x}) - \hat{\mathbf{r}}\times\nabla_1 W(\mathbf{x}) \\ \hat{\mathbf{r}}\cdot\mathbf{T}_{\mathrm{L}}(\mathbf{x}) &= \hat{\mathbf{r}}P(\mathbf{x}) + \nabla_1 Q(\mathbf{x}) - \hat{\mathbf{r}}\times\nabla_1 R(\mathbf{x}) \\ \mathbf{f}_{\mathrm{L}}(\mathbf{x}) &= \hat{\mathbf{r}}X(\mathbf{x}) + \nabla_1 Y(\mathbf{x}) - \hat{\mathbf{r}}\times\nabla_1 Z(\mathbf{x}) \end{aligned}\right\} \tag{56}$$

We also introduce, again following Backus (1967), the scalar field $g_1^{\mathrm{E}}(\mathbf{x})$, defined by

$$g_1^{\mathrm{E}}(\mathbf{x}) = \hat{\mathbf{r}}\cdot\nabla\phi_1^{\mathrm{E}}(\mathbf{x}) + 4\pi G\rho_0(\mathbf{x})\,U(\mathbf{x}). \tag{57}$$

We make use of a spherical harmonic expansion of each of the scalar fields $\rho_1^{\mathrm{E}}(\mathbf{r})$, $\phi_1^{\mathrm{E}}(\mathbf{r})$, $p_1^{\mathrm{E}}(\mathbf{r})$, $u_r^{\mathrm{E}}(\mathbf{r})$, $\nabla\cdot\mathbf{u}_{\mathrm{E}}(\mathbf{r})$ in the core, and $U(\mathbf{x})$, $V(\mathbf{x})$, $W(\mathbf{x})$, $P(\mathbf{x})$. $Q(\mathbf{x})$, $R(\mathbf{x})$, $X(\mathbf{x})$, $Y(\mathbf{x})$, $Z(\mathbf{x})$, $\phi_1^{\mathrm{E}}(\mathbf{x})$, and $g_1^{\mathrm{E}}(\mathbf{x})$ in the mantle. Thus if $q(\mathbf{r})$ (or $q(\mathbf{x})$) is used to denote any of the listed scalar field variables, we write

$$q(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} q_l^m(r)\, Y_l^m(\hat{\mathbf{r}}), \tag{58}$$

where $Y_l^m(\hat{\mathbf{r}})$ denotes the complex normalized surface spherical harmonic. To achieve a unique potential representation (56), we require that $V_0^0(r)$, $W_0^0(r)$, $Q_0^0(r)$, $R_0^0(r)$, $Y_0^0(r)$, $Z_0^0(r)$ all vanish. In what follows we will for brevity omit the subscript l and superscript m in writing the scalar radial variables $U_l^m(r)$, etc. No confusion should arise, since these variables are functions only of radius r, and not of position vector $\mathbf{r}$ or particle label $\mathbf{x}$; we will always indicate the radial functional dependence explicitly when utilizing any of the spherical harmonic expansion coefficients.

We first dispose of the toroidal part of the static deformation. The toroidal part of the deformation depends only on the toroidal part $Z(r)$ of the scalar potential representation of the imposed force $\mathbf{f}_{\mathrm{L}}(\mathbf{x})$. The fluid core is not affected by any toroidal deformation; such a deformation is characterized by the vanishing of all the scalar field variables except for $W(r)$ and $T(r)$ in the mantle $b \leqslant r \leqslant a$. The boundary conditions (52) and (53) require that both $T(b_+)$ and $T(a_-)$ vanish. For every value of $l > 0$, the second order static ($\omega = 0$) scalar differential equations, given for example by Backus (1967), are easily altered to include a non-homogeneous term $Z(r)$, and the toroidal part of the static deformation may be without difficulty computed.

The confusion in the past has arisen in the determination of the poloidal part of the deformation. It is in this connection that the utilization of an Eulerian formulation of the equations of motion in the fluid core will prove useful. This formulation leads to a treatment of the fluid core which is very similar in spirit to, and which was in fact suggested by, Backus' (1967) construction of a catalogue of all possible static equilibrium stress fields in a slightly aspherical Earth model.

The first of the core equations (50) gives rise to two conditions which must be satisfied, for all r in $0 \leqslant r < b$, by the spherical harmonic expansion coefficients $\rho_1^{\mathrm{E}}(r)$, $\phi_1^{\mathrm{E}}(r)$ and $p_1^{\mathrm{E}}(r)$. First, for all values of $l \geqslant 0$, we must have

$$\partial_r p_1^{\mathrm{E}}(r) + \rho_0(r)\,\partial_r\phi_1^{\mathrm{E}}(r) + \rho_1^{\mathrm{E}}(r)\,g_0(r) = 0. \tag{59}$$

Second, for all values of $l > 0$, but not necessarily for $l = 0$, we must have

$$p_1{}^{E}(r) = -\rho_0(r)\,\phi_1{}^{E}(r). \tag{60}$$

The case $l = 0$ emerges as a special case, which we will consider separately in Section 8. For $l > 0$, by differentiation of (60) and substitution into (59) we obtain another relation

$$\rho_1{}^{E}(r) = g_0{}^{-1}(r)\,\partial_r\,\rho_0(r)\,\phi_1{}^{E}(r). \tag{61}$$

Now upon substitution of (61) into the second of the core equations (50) we obtain, for every value of $l > 0$, a second order, homogeneous, differential equation for the spherical harmonic coefficients $\phi_1{}^{E}(r)$, valid throughout $0 \leqslant r < b$

$$\partial_r{}^2\,\phi_1{}^{E}(r) + 2r^{-1}\,\partial_r\,\phi_1{}^{E}(r) - [l(l+1)\,r^{-2} + 4\pi G g_0{}^{-1}(r)\,\partial_r\,\rho_0(r)]\,\phi_1{}^{E}(r) = 0. \tag{62}$$

Equation (62) may be integrated from $r = 0$ to $r = b_-$ for any specified piecewise continuously differentiable density structure $\rho_0(r)$ in the core, although a preliminary numerical differentiation of $\rho_0(r)$ is required. If $\rho_0(r)$ suffers a jump discontinuity at some radius, say $r = c$, $0 \leqslant c \leqslant b$, then the continuity conditions

$$\left.\begin{aligned} \phi_1{}^{E}(c_+) &= \phi_1{}^{E}(c_-) \\ \partial_r\,\phi_1{}^{E}(c_+) &= \partial_r\,\phi_1{}^{E}(c_-) - 4\pi G g_0{}^{-1}(c)[\rho_0(c_+) - \rho_0(c_-)]\,\phi_1{}^{E}(c_-) \end{aligned}\right\} \tag{63}$$

must be applied at that point. In any case, equation (62) will in general allow a complete specification of the ratio $\partial_r\,\phi_1{}^{E}(b_-)/\phi_1{}^{E}(b_-)$ at the base $r = b_-$ of the core–mantle boundary; since (62) is a homogeneous equation, the actual value of $\phi_1{}^{E}(b_-)$ itself is not determined.

We now consider the specification of the various poloidal spherical harmonic expansion coefficients on the mantle side $r = b_+$ of the core–mantle boundary. Notice that the continuity conditions (52) impose no restrictions on either $U(r)$ or $V(r)$; they may be chosen arbitrarily. For any value of $l > 0$, the continuity conditions (52), together with (57) and (60), yield the conditions

$$\left.\begin{aligned} U(b_+) &= A \\ V(b_+) &= B \\ \phi_1{}^{E}(b_+) &= C \\ P(b_+) &= \rho_0(b_-)\,C + \rho_0(b_-)\,g_0(b)\,A \\ Q(b_+) &= 0 \\ g_1{}^{E}(b_+) &= [\partial_r\,\phi_1{}^{E}(b_-)/\phi_1{}^{E}(b_-)]\,C + 4\pi G\rho_0(b_-)\,A, \end{aligned}\right\} \tag{63}$$

where A, B, C are arbitrary constants. The equations (63) specify, for any value of $l > 0$, the six poloidal spherical harmonic expansion coefficients $U(r)$, $V(r)$, $P(r)$, $Q(r)$, $\phi_1{}^{E}(r)$, $g_1{}^{E}(r)$ at the base $r = b_+$ of the mantle, in terms of three arbitrary constants A, B and C. The usual static ($\omega = 0$), inhomogeneous (modified to include $X(r)$ and $Y(r)$) sixth order poloidal system of differential equations (see, e.g. Backus 1967) may now be integrated from $r = b_+$ to the free surface $r = a_-$; the three arbitrary constants A, B, C may then be determined by means of the three relevant free surface boundary conditions obtained from (53)

$$\left.\begin{aligned} P(a_-) &= 0 \\ Q(a_-) &= 0 \\ g_1{}^{E}(a_-) + (l+1)\,a^{-1}\,\phi_1{}^{E}(a_-) &= 0. \end{aligned}\right\} \tag{64}$$

The above procedure allows a complete and unique determination, for all values of $l > 0$, of the entire poloidal deformation $U(r)$, $V(r)$, $P(r)$, $Q(r)$, $\phi_1^{E}(r)$, $g_1^{E}(r)$ of the mantle, and of the perturbation in the Eulerian gravitational potential $\phi_1^{E}(r)$ in the fluid core. The perturbations in both fluid pressure $p_1^{E}(r)$ and density $\rho_1^{E}(r)$ in the core may be subsequently determined, again for $l > 0$, throughout the fluid core by means of (60) and (61). All these aspects of the static deformation can always be uniquely determined, regardless of the nature of the stratification in the fluid core. We note that the procedure may be readily extended to the computation, for $l > 0$, of the static Love numbers h_l, k_l, l_l and the static load Love numbers h_l', k_l', l_l'. Only the free surface boundary conditions (64) need alteration.

We now consider the restrictions on the Eulerian velocity field $\mathbf{u}_E(\mathbf{r})$ which are imposed by the two remaining core equations (55). We assume for the moment that the local Brunt–Väisälä frequency $N(r)$ is non-zero throughout the core; the case where $N(r) = 0$ in some finite region of the core constitutes a slightly special case, and will be discussed in Section 8. The case of either an Adams–Williamson core or a uniformly stable core, $N^2(r) \geqslant 0$ for all r in $0 \leqslant r < b$, is perhaps of the most geophysical interest. Provided $N(r) \neq 0$ throughout the core, equations (55) lead to

$$\left.\begin{aligned} u_r^{E}(\mathbf{r}) &= 0 \\ \nabla\cdot\mathbf{u}_E(\mathbf{r}) &= 0 \end{aligned}\right\}. \tag{65}$$

The general solution to the pair of equations (65) is any purely toroidal flow $\mathbf{u}_E(\mathbf{r})$, i.e. any $\mathbf{u}_E(\mathbf{r})$ of the form

$$\mathbf{u}_E(\mathbf{r}) = -\hat{\mathbf{r}}\times\nabla_1 F(\mathbf{r}). \tag{66}$$

Here $F(\mathbf{r})$ is an arbitrary, continuously differentiable scalar field, which may of course be expanded in a series of spherical harmonics $Y_l^m(\hat{\mathbf{r}})$. Thus we may say that for the case $N(r) \neq 0$, and for any value of $l > 0$, there can be an arbitrary, steady, incompressible, purely toroidal Eulerian fluid flow occurring throughout the fluid core, without in any way affecting the various aspects of the deformation which have already been uniquely determined. Since there are no toroidal fields of degree $l = 0$, the $l = 0$ or purely radial part of the flow must be identically zero. For any value of $l > 0$, the steady Eulerian velocity field $\mathbf{u}_E(\mathbf{r})$ in the core is however completely unrestricted by the inviscid equations of infinitesimal motion, apart from being required to be purely toroidal. This circumstance clearly renders the static Lagrangian particle displacement in the core quite indeterminate; in fact any purely toroidal Lagrangian particle displacement of arbitrarily large magnitude may occur.

This conclusion may be stated somewhat differently by a consideration of the density change $\rho_L(\mathbf{x}, t)$ which is experienced by some given fluid particle $\mathbf{x}$ which is participating in a steady toroidal flow of the form (66). We restrict attention, for the moment, to the case where the body force $\mathbf{f}_E(r)$ giving rise to the deformation does not have an $l = 0$ spherical harmonic part; there is thus no $l = 0$ deformation. The additional complications introduced by a non-zero term of degree $l = 0$ will be considered in Section 8. Since the flow $\mathbf{u}_E(\mathbf{r})$ is incompressible, $\nabla\cdot\mathbf{u}_E(\mathbf{r}) = 0$ correct to first order in $\mathbf{u}_E(\mathbf{r})$, the exact (not linearized) Eulerian form of the continuity equation (24) requires that, again correct to first order in $\mathbf{u}_E(\mathbf{r})$,

$$D_t\,\rho_E\big(\mathbf{x}+\mathbf{s}_L(\mathbf{x}, t)\big) = D_t\,\rho_L(\mathbf{x}, t) = 0, \tag{67}$$

where $\mathbf{s}_L(\mathbf{x}, t)$ is the toroidal Lagrangian displacement of the particle $\mathbf{x}$. Hence, correct to first order in $\mathbf{u}_E(\mathbf{r})$, the statement that a steady flow of the compressible core fluid is purely toroidal is equivalent to the statement that the fluid is flowing in such a way that the density of every particle $\mathbf{x}$ remains constant. The surfaces of equal density in the fluid core are spherical surfaces before the deformation, but they become distorted by the static deformation. Provided the deformation does not have an $l = 0$ part, the fluid particles are constrained to move in such a way that every distorted equal density surface still consists of the same fluid particles after the deformation, but the Lagrangian fluid particle displacements produced by the imposed body force in the mantle are otherwise unrestricted. The fluid particles which comprise any given surface of constant density before the static deformation behave essentially indistinguishably during the deformation.

It is well known that in any hydrostatic situation (e.g. both before and after the static deformation), the surfaces of constant density, constant gravitational potential, and constant fluid pressure all coincide. Denote the net static radial displacement of the equipotential surface initially at the radius r by $\xi(r\hat{\mathbf{r}})$, and consider its expansion into a series of spherical harmonics $Y_l^m(\hat{\mathbf{r}})$. For any value of $l > 0$, the scalar radial expansion coefficient $\xi(r)$ may be determined uniquely throughout the core $0 \leqslant r < b$ since $\phi_1^{\mathrm{E}}(r)$ is known, and we have, correct to first order in $\phi_1^{\mathrm{E}}(r)$,

$$\xi(r) = -g_0^{-1}(r)\,\phi_1^{\mathrm{E}}(r). \tag{68}$$

If, for all values of $l > 0$, the fluid particles in the core really do remain on the equipotential surfaces on which they originate, then the perturbations in the Eulerian density $\rho_1^{\mathrm{E}}(r)$ and pressure $p_1^{\mathrm{E}}(r)$ should arise only from advection, i.e. for $l > 0$

$$\left.\begin{aligned} \rho_1^{\mathrm{E}}(r) &= -\xi(r)\,\partial_r\,\rho_0(r) \\ p_1^{\mathrm{E}}(r) &= -\xi(r)\,\partial_r\,p_0(r). \end{aligned}\right\} \tag{69}$$

Equations (69) are in full agreement with (60) and (61).

We have thus, at least for the case where $N(r) \neq 0$ and for every value of l except $l = 0$, obtained a complete picture of the role played by the fluid core in any static deformation. The Lagrangian particle displacement in the fluid core is indeterminate, except for the fact that every particle must remain on the equipotential surface on which it originates. The perturbations in density $\rho_1^{\mathrm{E}}(r)$, pressure $p_1^{\mathrm{E}}(r)$, and gravitational potential $\phi_1^{\mathrm{E}}(r)$, as well as the net static vertical displacement $\xi(r)$ of the core equipotential surfaces may, on the other hand, be completely and uniquely determined. The static deformation of the fluid core may be said to be determined to within an arbitrary toroidal Lagrangian displacement. Every aspect of the deformation in the mantle may be uniquely determined.

The complete indeterminacy of the Lagrangian particle displacement in the core produced by a static imposed body force may be more fully understood in the following terms. Note that, for the case where $N(r) \neq 0$ throughout the core, an arbitrary steady toroidal flow $\mathbf{u}_{\mathrm{E}}(\mathbf{r})$ of the form (66) is in fact a perfectly valid solution to the full set of zero-frequency normal mode equations (37), (38), (39), and (40) (set $\omega = 0$ and set all other field variables other than $\mathbf{u}_{\mathrm{E}}(\mathbf{r})$ to zero). There is then, for any value of $l > 0$, an infinitely large class of zero-frequency normal modes of any Earth model with a fluid core. In the case where $N(r) \neq 0$, this infinitely large class of zero-frequency normal modes consists precisely of the class of all steady toroidal flows in the core. A static force $\mathbf{f}_{\mathrm{E}}(\mathbf{r})$ confined to the mantle may be thought of as a resonant excitation at a node of these steady toroidal flow modes (a static gravitational potential such as $\phi_{\mathrm{tidal}}(\mathbf{r})$ or $\phi_{\mathrm{load}}(\mathbf{r})$ may be similarly thought of, in a generalized sense, as acting at a node of the steady toroidal flow modes). The idealized purely harmonic resonant excitation at the node of a normal mode of an arbitrary mechanical system leads in general to an indeterminate response; an arbitrary amount of the resonant mode may be superposed on the otherwise determinate response.

Viewed in this way, it is clear that the indeterminacy of the Lagrangian particle displacement in the fluid core is a consequence of the somewhat idealized concept of a purely static deformation. Any real geophysical process does not occur statically, and in general real geophysical problems are initial value problems. In principle, one could of course compute every detail of the response of the Earth to any realistic geophysical process; for example, one could in principle solve the initial value problem which determines the complete response to a kinematically prescribed dislocation on a fault surface somewhere within the Earth. To compute the permanent deformation after all the faulting has ceased and all the free oscillations of the Earth excited by the faulting process have decayed, it would be necessary to make further assumptions about the constitutive nature of the Earth model. In particular, it would be necessary to make some assumptions regarding the nature of the anelasticity (e.g. the viscosity of the fluid core) which gives rise to the dissipation of the free oscillations excited by the faulting. We have seen that such a procedure may be avoided, as long as one does not wish to know the precise permanent Lagrangian particle displacement in the core. Any aspect of the permanent deformation which is conceivably measurable at the surface of the Earth may be uniquely determined by idealizing the faulting process as a static phenomenon.

We mention finally, for completeness, the existence of the other zero-frequency normal modes of any Earth model, namely the rigid body translations and rotations. The complete catalogue of normal modes of any non-rotating, spherically symmetric Earth model which has a fluid core consists in general of the zero-frequency rigid body translation and rotations, the zero-frequency steady toroidal flows in the core, the toroidal modes ${}_nT_l$, and the eg and ge type poloidal modes ${}_nS_l$. It is of interest to note the way in which the overall characteristics of the ge type poloidal modes ${}_nS_l$, for a fixed $l > 0$, of an Earth model with an everywhere stable core assume increasingly the characteristics of the toroidal steady flow modes as $n \to -\infty$ and ${}_n\omega_l{}^S$ becomes arbitrarily small.

8. Some special cases

We will now point out and consider briefly the special cases which must be treated separately in any computation of the static deformation of an Earth model with a fluid core. The most important such special case is the case of the contribution from the terms of spherical harmonic degree $l = 0$. There are no purely toroidal fields of degree $l = 0$, so we need only consider the poloidal type deformation characterized by $l = 0$. The case $l = 0$ arises naturally as a special case in the poloidal analysis, because of the fact that equation (50) leads directly to equation (60) only for $l > 0$. We will show that the $l = 0$ portion of the poloidal deformation produced by an arbitrary body force $\mathbf{f}_E(\mathbf{r})$ in the mantle may, contrary to the situation for all values of $l > 0$, be completely determined throughout the entire Earth model. In particular the $l = 0$ part of the Lagrangian particle displacement may be determined even in the fluid core. This is not unreasonable, since we observed in Section 7 that the indeterminacy for $l > 0$ was associated with the nodal resonance excitation of the zero-frequency toroidal flow normal modes. There are no such steady toroidal flow normal modes for $l = 0$, since purely toroidal fields of degree $l = 0$ do not exist.

We will consider first the case of a purely $l = 0$ poloidal deformation; i.e. a deformation produced by a body force $\mathbf{f}_E(\mathbf{r})$ containing only an $l = 0$ part $\big(X(r) = 0$ for $l > 0$, $Y(r) = 0$ for $l \geqslant 0\big)$. The more general case of a body force $\mathbf{f}_E(\mathbf{r})$ containing spherical harmonic terms of all degrees $l \geqslant 0$ may then be treated by superposition. The Lagrangian particle displacement associated with a purely $l = 0$ deformation is purely radial, even in the fluid core. A purely radial Lagrangian particle displacement can never become arbitrarily large, and this allows the use of a linearized Lagrangian formulation of the $l = 0$ equations of motion even in the core. We may use the $l = 0$ scalar radial variables $U(r)$, $P(r)$, $\phi_1{}^E(r)$, and $g_1{}^E(r)$ throughout the entire Earth model, in the core as well as in the mantle ($V(r)$ and $Q(r)$ are identically zero everywhere, and $P(r) = -p_1{}^L(r) = -p_1{}^E(r) + \rho_0(r)\, g_0(r)\, U(r)$ in the core. For $l = 0$, we may also show that $g_1{}^E(r) = 0$, and it is well known that the usual sixth order poloidal system degenerates to a second order system involving

$U(r)$, $P(r)$ and the inhomogeneous term $X(r)$, together with a quadrature for $\phi_1^E(r)$. These equations allow a complete determination of the $l = 0$ part of any static deformation. Such a complete determination of an $l = 0$ static deformation was first obtained by Longman (1963); Longman's particular application was to the computation of the degree $l = 0$ load Love numbers.

A purely $l = 0$ static deformation does not in general share the common property of all $l > 0$ deformations that the fluid particles in the core remain on the equipotential surface on which they originate. That is, the purely radial Lagrangian particle displacement $U(r)$, $0 \leqslant r < b$ is not in general equal to the purely radial displacement of the core equipotential surfaces $\xi(r) = -g_0^{-1}(r)\phi_1^E(r)$, $0 \leqslant r < b$. This is of course entirely consistent with the fact any purely radial deformation must in general cause either a net increase or a net decrease in core volume; the associated perturbation in the Lagrangian density at any fluid particle $\mathbf{x}$ must under such circumstances be non-zero. The fluid particles thus cannot remain on the equal density surface on which they originate.

It is now easy to describe the static deformation produced by an arbitrary imposed body force $\mathbf{f}_E(\mathbf{r})$. We consider the deformation to be a two-stage process. We compute first the $l = 0$ part of the deformation. During this part of the deformation every fluid particle in the core (as well as every particle in the mantle) is displaced purely radially to some new level in the Earth model. The equipotential surfaces in the fluid core (as well as in the mantle) also suffer a purely radial deformation. The $l = 0$ part of the deformation thus uniquely associates every fluid particle in the core with a new spherical equipotential surface. Now we allow the remainder (i.e. all terms of spherical harmonic degree $l > 0$) of the imposed body force to act. Every fluid particle in the core must now remain on the equipotential surface on which it resides as a result of the $l = 0$ part of the deformation. The Lagrangian particle displacement in the fluid core is indeterminate to within an arbitrary toroidal displacement. Every other aspect of the deformation is uniquely determined.

The spherical harmonic terms of degree $l = 1$ also constitute a slightly special case, although it was not explicitly pointed out in Section 7. The reason is that in the special case $l = 1$, a purely static deformation corresponds in general not only to a nodal resonance excitation of the toroidal steady flow modes in the core, but also to a resonance excitation of the $l = 1$ zero-frequency normal modes corresponding to rigid body translation and rotation. The resolution of this difficulty has been amply discussed in the geophysical literature. In the case of the static response to an arbitrary body force $\mathbf{f}_E(\mathbf{r})$, it is clear that static equilibrium cannot be expected unless the imposed body force exerts neither a net force nor a net torque on the Earth model (the body forces equivalent to a static elastic dislocation satisfy this criterion). Such a body force cannot give rise to either a net translation or rotation; this extra information can be used to determine the $l = 1$ response (see, e.g. Ben-Menahem & Singh 1968). In the case of the static response to an $l = 1$ surface mass load, the extra information required for a complete solution is the fact that the centre of mass of the Earth model plus the mass load must not move in space (see, e.g. Farrell 1972). The reason that the case $l = 1$ is a special case has nothing to do with the existence of a fluid core. The treatment of the fluid core is essentially unaltered, and, in general, one can determine the $l = 1$ response to within an arbitrary $l = 1$ Lagrangian particle displacement within the core.

One further special case arose in the treatment in Section 7, and that is the case where the fluid core is neutrally stratified $(N(r) = 0)$ throughout some finite region of the core $0 \leqslant r < b$. This affects only the determination of the possible steady Eulerian flow fields $\mathbf{u}_E(r)$ which can occur in that region. If $N(r) = 0$, then the set of equations (55) together with the core–mantle boundary conditions (52) lead only to the condition that

$$\left.\begin{aligned} u_r^{\mathrm{E}}\,\partial_r\rho_0+\rho_0\,\nabla\cdot\mathbf{u}_{\mathrm{E}} &= 0, \qquad 0\leqslant r<b \\ u_r^{\mathrm{E}} &= 0, \qquad r=b, \end{aligned}\right\} \tag{70}$$

instead of to the condition (65) that both $u_r^{\mathrm{E}}(\mathbf{r})$ and $\nabla\cdot\mathbf{u}_{\mathrm{E}}(\mathbf{r})$ vanish simultaneously throughout $0\leqslant r\leqslant b$. Equations (70) place even fewer restrictions on the class of compatible flows $\mathbf{u}_{\mathrm{E}}(\mathbf{r})$ than do the equations (65). In particular, a steady flow with a non-zero vertical component $u_r^{\mathrm{E}}(\mathbf{r})$ is not prohibited in any neutrally stable region of the core. The fluid particles in any such region of the core are thus not even required to remain on the equipotential surface on which they originate. This is of course not unexpected; it is essentially what is meant by neutral stability.

Note that the treatment of an everywhere neutrally stratified core ($N(r)=0$ for all r in $0\leqslant r<b$) is, from a numerical point of view, slightly more convenient than the more general case. This is because numerical differentiation of the core density profile $\rho_0(r)$ is not required prior to the solution of (62). The second order homogeneous equation for $\phi_1^{\mathrm{E}}(r)$ in the core is, in this case

$$\partial_r^2\,\phi_1^{\mathrm{E}}(r)+2r^{-1}\,\partial_r\phi_1^{\mathrm{E}}(r)+[4\pi G\rho_0^2(r)\,\kappa^{-1}(r)-l(l+1)\,r^{-2}]\,\phi_1^{\mathrm{E}}(r)=0. \tag{71}$$

Equation (71) follows from (62) upon substitution of the Adams–Williamson condition that $N(r)=0$. The treatment of an Adams–Williamson core, for the case $l>0$, thus involves the numerical integration of (71) throughout the core, followed by the imposition of the continuity conditions (63) at the core–mantle boundary $r=b$.

9. Comparison with previous treatments

In an attempt to truly clarify the confusion which prevails in this subject, we will now discuss the relation of the above static treatment of the fluid core to a few of the earlier treatments of this problem. We have shown in Section 7 that the use of a linearized Eulerian formulation of the equations of motion in the fluid core enables one to obtain a complete and consistent picture of the role played by the core in any static deformation. The chief difficulty with all (to my knowledge) previous treatments of this problem has stemmed ultimately from the fact that they have all employed, without any justification, a linearized Lagrangian formulation of the equations of motion in the fluid core.

We will compare briefly the methods used to treat the fluid core in the work of Takeuchi (1950), Longman (1963), Jeffreys & Vicente (1966), Smylie & Mansinha (1971), Dahlen (1971), Farrell (1972), Dahlen (1973), Pekeris & Accad (1972), and Israel *et al.* (1973). Most of these papers are not concerned primarily with the rather academic question of how to treat the fluid core in static problems; instead they are for the most part examining some geophysically interesting problem whose solution requires a method of treating the core. The main goal of most of these papers is thus the computation of some geophysical quantity which is observable at the surface of the Earth. It turns out that, even though most of these papers employ an inadequate or an internally inconsistent treatment of the fluid core, they almost without exception do compute the desired geophysical observable correctly. In almost all cases, the physics is faulty or perhaps a bit unclear, but the computational algorithm turns out to be correct. We will concentrate our discussion of the individual papers primarily on the extent to which there is mathematical agreement with the treatment in Section 7, rather than dwell heavily on the sometimes peculiar physical interpretations.

First we make some general comments which apply to all of the earlier treatments. Typically, the equations of motion employed in the core were obtained by simply setting the rigidity $\mu(r)=0$ in the linearized Lagrangian equations of motion (51). When the scalar potential variables (56) are introduced and the spherical harmonic resolution performed, one obtains certain scalar equations in the core;

one of these equations takes the form, for every value of $l > 0$,

$$[\partial_r U(r) + 2r^{-1} U(r) - l(l+1) r^{-1} V(r)] N^2(r) = 0. \qquad (72)$$

Equation (72) plays a leading role in all of the earlier treatments. If the Earth model under consideration has an Adams–Williamson core, then equation (72) would be satisfied identically. If on the other hand $N(r) \neq 0$ throughout the core, then equation (72) seems to imply that the Lagrangian dilation is required (for $l > 0$) to vanish.

The first paper which deals with the static deformation of a reasonably realistic Earth model with a fluid core is that of Takeuchi (1950); Takeuchi was interested in the determination of the static $l = 2$ Love numbers h_2, k_2, l_2. He utilized a Lagrangian formulation in the core and arrived at equation (72), although he did not express it in exactly the same notation. He states that equation (72) is in fact an identity; he seems to have been under the impression that the condition $N(r) = 0$ is a necessary condition for the static equilibrium of a fluid (since the 1936 Bullen core density model he utilized for numerical calculations was obtained by employing the Adams–Williamson equation, it turns out that equation (72) was in fact satisfied identically in his case). Takeuchi showed that the rest of the relevant Lagrangian equations in the core led to the homogeneous second order equation (62), and it was this equation which he integrated throughout the core. The boundary conditions which he employed at the core–mantle boundary were equivalent to the conditions (63). Takeuchi (1950) thus utilizes a computational procedure which, for an Adams–Williamson core, is mathematically equivalent to that in Section 7, in so far as any aspect of the deformation in the solid mantle is concerned. In particular, his algorithm for computing the surface expression of the static tidal deformation, namely the Love numbers h_2, k_2, l_2, is valid.

Longman (1963) was interested in the determination of the static load Love numbers h_l', k_l', l_l' for all values of the spherical harmonic degree $l \leqslant 40$. His treatment of the case $l = 0$ is exactly the same as that given in Section 8. In the case $l = 0$, a Lagrangian formulation may be employed in the core, and the purely radial $l = 0$ Lagrangian particle displacement may be uniquely determined. He did not treat the case $l = 1$, seemingly assuming it to be of no interest, or to lead to no observable effects. For the case $l \geqslant 2$, he too arrived at equation (72), and he too states that the condition $N(r) = 0$ 'is really implicit in our assumption of elastic equilibrium under gravity'. He accordingly modified the bulk modulus $\kappa(r)$ in the Gutenberg Earth model he used for numerical computations, so that the Adams–Williamson equation was satisfied throughout the core. He used the same second order differential equation for $\phi_1^{\mathrm{E}}(r)$ in the core as did Takeuchi (1950), except that he wrote it in the form (71) instead of (62). He too thus utilized a computational algorithm which leads to a correct computation of all aspects of the deformation in the solid mantle.

Jeffreys & Vicente (1966) seem to have been the first to point out explicitly that equation (72) can only be considered an identity for a very particular class of core models, namely those in uniform neutral equilibrium. They offer the opinion that Longman's (1963) treatment is correct for that special case, and that for any other case, no solution may be obtained. The reason for this, from a mathematical point of view, is that in the case $N(r) \neq 0$, equation (72) provides an additional relation between the scalars $U(r)$ and $V(r)$ in the core, and this has the undesirable effect of eliminating one of the three necessary arbitrary constants in the core–mantle boundary conditions (63).

The next important development was made by Smylie & Mansinha (1971). They wished to determine the change in the inertia tensor of the Earth produced by the action of a point static elastic dislocation situated in the mantle. The motivation behind such a computation was their desire to subject to a test their hypothesis that seismic activity was responsible for the observed excitation of the Chandler wobble. Smylie & Mansinha point out quite correctly that the fluid core of the real Earth is unlikely to be exactly in a state of neutral equilibrium at all points, and that it would be desirable to be able to treat the problem of static deformation for an arbitrary core model. Their physical picture of the role played by the core in a static deformation is in many respects similar to the one presented in Section 7, but their derivation of the relevant core equations seems somewhat *ad hoc*. They simply assume, without any apparent justification, that one can employ the linearized Lagrangian equations in the core, with the proviso that, for $l > 0$, the scalar radial variable $U(r)$ is to be interpreted not as the radial Lagrangian particle displacement, but rather as the radial displacement $\xi(r)$ of the equipotential surface of radius r. This argument leads, it turns out, to exactly the second order differential equation (62) in the core and the boundary conditions (63) at the core–mantle boundary. Thus, the procedure of Smylie & Mansinha (1971) leads to a correct computational algorithm for $\phi_1{}^{E}(r)$ in the core and for all aspects of the deformation in the mantle, regardless of the nature of the core stratification. The limitations of their argument do not become apparent until the next step in the computation, namely the evaluation of the volume integrals which define the change in the inertia tensor. Both the mantle and the core contribute to the change in the inertia tensor. Smylie & Mansinha utilize a certain expression (their equation (44)) to compute the contribution of the core to the change in the inertia tensor. Their expression (44) is obtained by making use of a purely Lagrangian description of the core deformation, but then arbitrarily setting the tangential displacement equal to zero, and taking the vertical displacement to be that suffered by equipotential surfaces in the core (Smylie 1973, private communication). This somewhat *ad hoc* argument yields an expression which is not consistent with the treatment described here.

Dahlen (1971) criticized the *ad hoc* nature of Smylie & Mansinha's (1971) arguments, and agreed with the view of Jeffreys & Vicente (1966) that the static deformation problem was indeterminate, except for the case of a core in uniformly neutral equilibrium. He conjectured that, in the case of a stably stratified core model, a physical explanation of this could perhaps be in some way connected with the near resonance excitation of the very low frequency ge type poloidal modes. It has been shown above that this is not the case at all. The situation may be more properly viewed as an exactly resonance nodal excitation of the zero-frequency toroidal flow modes, and it is only to within such an arbitrary steady toroidal flow that the problem is at all indeterminate.

Farrell (1972) has computed the load Love numbers of a realistic Earth model with a fluid core, up to values of spherical harmonic degree $l = 10\,000$, and has used these to form the Green function for a point mass load. He simply avoided most of the problems associated with the static deformation by computing not the static load Love numbers h_l', k_l', l_l', but rather the dynamic load Love numbers $h_l'(\omega)$, $k_l'(\omega)$, $l_l'(\omega)$, choosing ω to be the semi-diurnal tidal frequency. The special case $l = 1$, however, he treated statically, using an Adams–Williamson core model. Farrell's procedure for treating the special case $l = 1$ is readily extended to allow for an arbitrary core model by simply utilizing (62) instead of (71) in the core.

Dahlen (1973), in a correction to a previously published numerically inaccurate calculation, was concerned with the same geophysical application as Smylie & Mansinha (1971). Dahlen (1973) used an Earth model with an Adams–Williamson core, since he did not know how to perform the calculation for any other type of Earth model. He used equation (71) in the core and the boundary conditions (63) at the core–mantle boundary. In evaluating the volume integrals to compute the change in the inertia tensor, he took advantage of the Lagrangian formulation in the mantle, and computed the mantle contribution by means of an integration over the undeformed mantle volume $b \leqslant r \leqslant a$. In the fluid core he utilized the Eulerian description of the perturbation in the density $\rho_1{}^{E}(r)$ (given in terms of $\phi_1{}^{E}(r)$ by equation (61)) and performed an integration over the deformed core volume. This procedure is entirely consistent with the treatment in Section 7, and it could now easily be extended to deal with any arbitrary core model. From a purely mathematical point of view, the only difference (apart from the different Earth models employed) between the treatments of Smylie & Mansinha (1971) and Dahlen (1973) is in this final step, i.e. the form of the expression which gives the core contribution to the change in the inertia tensor (compare equation (44) of Smylie & Mansinha (1971) with equation (23) of Dahlen (1973)). Dahlen (1973) actually performed this step of the calculation using both procedures, and found that the final answer was affected by less than 10 per cent.

Pekeris & Accad (1973) have made a lengthy analysis of the general problem of the low frequency and the zero frequency response of spherically symmetric Earth models which have a fluid core. They take as the starting point of their analysis the linearized Lagrangian formulation of the equations of motion throughout the entire Earth model, in the fluid core as well as in the mantle. They define throughout the core a dimensionless stability parameter $\beta(r)$, which is related to the local Brunt–Väisälä frequency $N(r)$ by $\beta(r) = -\kappa(r)\rho_0{}^{-1}(r)g_0{}^{-2}(r)N^2(r)$, and they restrict consideration to core models characterized by a constant value of $\beta(r)$. They obtain, as did Smylie & Mansinha (1971), the second order homogeneous equation (62) for the perturbation in the gravitational potential $\phi_1{}^{E}(r)$ in the core; they express this equation in terms of the constant parameter $\beta(r) = \beta$. For the case of a uniformly unstable core model ($\beta > 0$), they utilize a complicated asymptotic boundary layer theory to obtain the proper form of the boundary conditions to be employed at the core–mantle boundary in the case of a purely static deformation. For that case ($\beta > 0$), this procedure does, it turns out, yield boundary conditions which are equivalent to the conditions (63). Their procedure thus leads to a correct computational algorithm for $\phi_1{}^{E}(r)$ in the core and for all parameters of the deformation in the mantle. If the core model is uniformly stable ($\beta < 0$), the existence of the ge type poloidal modes of the Earth model prevents them from utilizing asymptotic boundary layer arguments, and they are unable to suggest a static computational procedure for such models. For the other case however ($\beta > 0$), Pekeris & Accad (1971) do seem to have devised a method whereby the linearized Lagrangian equations of motion, although not strictly valid in the static limit, may be employed to obtain a computational algorithm which is valid in that limit. Their method, besides being rather complicated and only applicable in the case of an unstable core model, seems also to have other limitations, since Pekeris and Accad claim that it provides a unique determination of the static Lagrangian particle displacement throughout the core. It is clear from the analysis in Section 7 that in fact the displacement field which they obtain is only one of an infinitely large class of possible displacement fields, each of which is completely compatible with the more properly formulated Eulerian equations of motion.

Israel *et al.* (1973) discuss in some detail the static response of a spherically symmetric Earth model to a point static elastic dislocation in the mantle. They perform numerical computations only for the case $l = 2$, since they are primarily interested in the change in the inertia tensor. Following Pekeris & Accad (1973), they consider two Earth models, one with an Adams–Williamson core ($\beta = 0$) and one with a highly unstable core ($\beta = 0{\cdot}2$). In both cases, their computational algorithm is completely consistent with that developed in Section 7, and in the first case identical to that employed by Dahlen (1973). Unlike Smylie & Mansinha (1971), they compute the change in the inertia tensor in a way which is entirely consistent with the treatment in Section 7, employing a Lagrangian integration over the undeformed mantle volume and an Eulerian integration over the deformed core volume. They find that the two

Earth models, one with an Adams–Williamson core and one with a highly unstable core, have changes in the inertia tensor which agree to within 0·1 per cent. Dahlen (1973) has shown that his corrected numerical results agree very well with those of Israel *et al.* (1973).

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