ON THE STEADY-STATE CONTINUOUS CASTING STEFAN PROBLEM WITH NONLINEAR COOLING*

By

MICHEL CHIPOT (*Brown University* and *Université de Nancy I*)

AND

JOSÉ-FRANCISCO RODRIGUES (*Lisbon University*)

Abstract. A steady-state one-phase Stefan problem corresponding to the solidification process of an ingot of pure metal by continuous casting with nonlinear lateral cooling is considered via the weak formulation introduced in [5] for the dam problem. Two existence results are obtained, for a general nonlinear flux and for a maximal monotone flux. Comparison results and the regularity of the free boundary are discussed. An uniqueness theorem is given for the monotone case.

0. Introduction. In this paper we study the one-phase model of the solidification of a pure metal in continuous casting undergoing nonlinear lateral cooling.

In the liquid phase we assume that the metal is at the melting temperature, which is zero after a normalization. In the solid phase the temperature θ satisfies the heat equation. The ingot is extracted with constant velocity b, and the liquid-solid interface (the free boundary) is unknown but steady with respect to a fixed system of coordinates of R^3 in which our problem will be studied. Assuming that the free boundary Φ is representable by a surface $z = \phi(x, y)$, the steady Stefan condition is

$$\theta_z - \theta_x \phi_x - \theta_y \phi_y = \lambda b, \qquad \text{for } z = \phi(x, y) \tag{0.1}$$

where λ is a positive constant representing the heat of melting.

In the lateral boundary one specifies a nonlinear flux condition

$$-\partial\theta/\partial n = G(\theta) \tag{0.2}$$

which expresses the law of cooling, and may be quite general. Here we shall consider a maximal monotone graph G, which may include a cooling process with climatization, as in Chapter 1 of [9].

This model has been considered in a particular case by Rubinstein [16] and, with a linear flux condition of Newtonian type, by Briére [6] and Rodrigues [15], via variational inequalities after a transformation of Baiocchi's type. However, this approach doesn't work with nonlinear cooling.

Since this problem has some similarities with the dam problem, we formulate it in Sec. 1 using the method of Brézis, Kinderlehrer and Stampacchia [5]. In Secs. 2 and 3 we

* Received March 10, 1982. This research has been supported in part by the Air Force Office of Scientific Research under Contract #AF-AFOSR 81-0198 and in part by the U. S. Army Research Office under contract #ARO-DAAG-29-79-C-0161.

prove the existence theorems, first using compactness arguments and next combining compactness and monotonicity techniques for the maximal monotone case.

In Sec. 4 we discuss comparison properties which show that when the extraction velocity b is small the ingot solidifies immediately and there is no free boundary. For some types of cooling and for a high enough velocity b one can show the existence of a free boundary. In this case it is shown, in Sec. 5, that the free boundary is an analytic surface and the weak solution is also a classic one, as in the linear case of [15].

To conclude this paper we give in Sec. 6 an uniqueness theorem for the monotone case, using the technique of Carrillo and Chipot [8].

1. Mathematical formulation. Let Ω denote a cylindric domain in R^3, in the form $\Omega = \Gamma \times]0, H[$, where $\Gamma \subset R^2$ is a bounded domain with Lipschitz boundary $\partial\Gamma$ representing a section of the ingot and $H > 0$ its height. We denote by $\Gamma_i = \Gamma \times \{i\}$, for $i = 0, H$, the bottom and the top of the ingot respectively, and by $\Gamma_1 = \partial\Gamma \times]0, H[$ its lateral boundary. We have $\partial\Omega = \Gamma_0 U \bar{\Gamma}_1 U \Gamma_H$.

Considering **z** the direction of extraction, we can formulate our problem in its classical form:

Problem (C): Find a couple (θ, ϕ), such that

$$\theta \geq 0 \text{ in } \Omega \quad \text{and} \quad \theta = 0 \quad \text{for } 0 \leq z \leq \phi(x, y) < H, \tag{1.1}$$

$$\Delta\theta = b\theta_z \quad \text{for } 0 \leq \phi(x, y) < z < H, \tag{1.2}$$

$$\theta = 0 \text{ on } \Gamma_0, \qquad \theta = h(x, y) > 0 \text{ on } \Gamma_H, \tag{1.3}$$

$$-\partial\theta/\partial n = g(\theta) \text{ on } \Gamma_1, \tag{1.4}$$

$$\theta_z - \theta_x\phi_x - \theta_y\phi_y = \lambda b, \qquad \text{if } z = \phi(x, y) > 0, \tag{1.5}$$

$$\theta_z \geq \lambda b, \qquad \text{if } z = \phi(x, y) = 0. \tag{1.5'}$$

In this formulation b and λ are positive constants, h is a given function, and g will be specified in the next two sections. The reader will note that the condition (1.5′) is a degeneration of the Stefan condition (1.5) in the case when the free boundary Φ can touch the known boundary Γ_0, where the melting condition $\theta = 0$ is assumed by (1.3).

Let us remark that by the maximum principle it must be $\theta > 0$ for $z > \phi(x, y)$. Denoting by χ^+ the characteristic function of the set $\Omega_+ = \{\theta > 0\}$ and integrating formally by parts, for every regular function ζ such that $\zeta = 0$ on Γ_H and $\zeta \geq 0$ on Γ_0, from Problem (C) one has

$$\begin{aligned}
\int_\Omega (\nabla\theta \cdot \nabla\zeta + b\theta_z\zeta - \lambda b_\chi + \zeta_z) &= \int_{\Omega_+} (\nabla\theta \cdot \nabla\zeta + b\theta_z\zeta - \lambda b\zeta_z) \\
&= \int_{\Omega_+} (-\Delta\theta + b\theta_z)\zeta + \int_{\Gamma_1 \cup \Phi \cup \Gamma_0} \frac{\partial\theta}{\partial n}\zeta + \lambda b \int_{\Phi \cup \Gamma_0} l\zeta \\
&= -\int_{\Gamma_1} g(\theta)\zeta + \int_{\Gamma_0} l\zeta(\lambda b - \theta_z) + \int_\Phi l\zeta(\theta_x\phi_x + \theta_y\phi_y - \theta_z + \lambda b) \\
&\leq -\int_{\Gamma_1} g(\theta)\zeta,
\end{aligned}$$

where $l^{-2} = \phi_x^2 + \phi_y^2 + 1$. Therefore, following [5], we introduce the weak formulation of Problem (C):

Problem (P): Find a couple $(\theta, \chi) \in H^1(\Omega) \times L^\infty(\Omega)$ such that

$$\theta \geq 0 \text{ a.e. in } \Omega, \qquad \theta = 0 \text{ on } \Gamma_0 \quad \text{and} \quad \theta = h \text{ on } \Gamma_H; \tag{1.6}$$

$$0 \leq \chi \leq 1 \text{ a.e. in } \Omega \quad \text{and} \quad \chi = 1 \quad \text{where} \quad \theta > 0; \tag{1.7}$$

$$\int_\Omega (\nabla\theta \cdot \nabla\zeta + b\theta_z \zeta - \lambda b_\chi \zeta_z) + \int_{\Gamma_1} g(\theta)\zeta \leq 0, \tag{1.8}$$

for every $\zeta \in H^1(\Omega)$ such that $\zeta \geq 0$ on Γ_0 and $\zeta = 0$ on Γ_H.

If we consider a more restrictive class of test functions one can introduce a more general formulation, which we call Problem (P′), if we replace (1.8) by

$$\int_\Omega (\nabla\theta \cdot \nabla\zeta + b\theta_z \zeta - \lambda b\chi\zeta_z) + \int_{\Gamma_1} g(\theta)\zeta = 0, \qquad \forall\zeta \in H^1(\Omega)\colon \zeta = 0 \text{ on } \Gamma_0 \cup \Gamma_H. \tag{1.9}$$

It is clear that every solution of Problem (P) satisfies (1.9) but that Problem (P′) has more solutions than Problem (P). In particular, if

Problem (P_1): Find θ satisfying (1.6) and

$$\int_\Omega (\nabla\theta \cdot \nabla\zeta + b\theta_z \zeta) + \int_{\Gamma_1} g(\theta)\zeta = 0, \qquad \forall\zeta \in H^1(\Omega);\ \zeta = 0 \text{ on } \Gamma_0 \cup \Gamma_H \tag{1.10}$$

has a solution $\theta \geq 0$, by the maximum principle, one has $\theta > 0$ in Ω and $(\theta, 1)$ is a solution to Problem (P′) which may not satisfy (1.5′) (see Proposition 4).

2. Existence of a weak solution. In this section we assume the lateral cooling given by

$$-\frac{\partial\theta}{\partial n}(X) = g(X, \rho(X), \theta(X)), \qquad X \in \Gamma_1 \tag{2.1}$$

where $\rho \geq 0$ is a given function representing the cooling temperature, and

$$g(X, \rho, \theta) \quad \text{is a bounded Carathe\'odory function,} \tag{2.2}$$

i.e., is continuous in $\theta \in R$, a.e. $(X, \rho) \in \Gamma_1 \times R_+$, measurable in (X, ρ) for all θ, and maps bounded sets of $\Gamma_1 \times R_+ \times R$ in bounded sets of R.

Since the cooling process is determined by ρ, we shall assume that

$$g(X, \rho, \theta) \leq 0, \qquad \text{a.e. } (X, \rho, \theta) \in \Gamma_1 \times R_+ \times R \tag{2.3}$$

$$g(X, \rho, \theta) = 0 \quad \text{for} \quad |\theta| \geq \rho, \qquad \text{a.e. } X \in \Gamma_1. \tag{2.4}$$

Consider a parameterized family of functions $\chi_\varepsilon \in C^\infty(R)$ such that

$$\begin{aligned} \chi_\varepsilon(t) &= 0 && \text{for } t \leq 0, \\ &= 0 \leq \chi_\varepsilon(t) \leq 1 && \text{for } 0 \leq t \leq \varepsilon, \\ &= 1 && \text{for } t \geq \varepsilon, \end{aligned} \tag{2.5}$$

and so it approaches the Heaviside function when $\varepsilon \searrow 0$. Introduce now the following penalized problem, where for the sake of simplicity we denote $g(X, \rho(X), \theta(X))$ by $g(\theta)$:

Problem (P_ε): Find $\theta^\varepsilon \in H^1(\Omega) \cap C^0(\bar{\Omega})$ such that

$$\theta^\varepsilon = 0 \text{ on } \Gamma_0, \qquad \theta^\varepsilon = h \text{ on } \Gamma_H, \tag{2.6}$$

$$\int_\Omega [\nabla\theta^\varepsilon \cdot \nabla\zeta + b\theta^\varepsilon_z \zeta - \lambda b \chi_\varepsilon(\theta^\varepsilon)\zeta_z] + \int_{\Gamma_1} g(\theta^\varepsilon)\zeta = 0, \quad \forall \zeta \in H^1(\Omega)\colon \zeta = 0 \text{ on } \Gamma_0 \cup \Gamma_H. \tag{2.7}$$

Assuming the functions h and ρ satisfy

$$0 < h(x, y) \leq M, \qquad \text{a.e. } (x, y) \in \Gamma_H, \tag{2.8}$$

$$0 \leq \rho(X) \leq M, \qquad \text{a.e. } X \in \Gamma_1, \tag{2.9}$$

one can prove the following "a priori" estimate:

LEMMA 1. If θ^ε is a solution to Problem (P_ε) with assumptions (2.2–4) and (2.8–9), one has

$$0 \leq \theta^\varepsilon(X) \leq M, \qquad \text{for all } X \in \bar{\Omega} \quad \text{and} \quad 0 < \varepsilon \leq M. \tag{2.10}$$

Proof. Let $\zeta = [\theta^\varepsilon]^-$ in (2.7). One has

$$\begin{aligned} 0 &= \int_\Omega \{\nabla\theta^\varepsilon \cdot \nabla[\theta^\varepsilon]^- + b\theta^\varepsilon_z[\theta^\varepsilon]^- - \lambda b\chi_\varepsilon(\theta^\varepsilon)[\theta^\varepsilon]^-_z\} + \int_\Gamma g(\theta^\varepsilon)[\theta^\varepsilon]^- \\ &\leq -\int_\Omega \{|\nabla[\theta^\varepsilon]^-|^2 + b[\theta^\varepsilon]^-_z[\theta^\varepsilon]^-\} = -\int_\Omega |\nabla[\theta^\varepsilon]^-|^2 \end{aligned}$$

from which it follows that $[\theta^\varepsilon]^- = 0$ and $\theta^\varepsilon \geq 0$.

From (2.4), (2.9) and (2.5), one has respectively

$$g(\theta^\varepsilon)[\theta^\varepsilon - M]^+ = 0, \qquad \chi_\varepsilon(\theta^\varepsilon)[\theta^\varepsilon - M]^+_z = [\theta^\varepsilon - M]^+_z \quad \text{for} \quad 0 < \varepsilon \leq M.$$

Then $\zeta = [\theta^\varepsilon - M]^+$ in (2.7) implies

$$\begin{aligned} 0 &= \int_\Omega \{\nabla\theta^\varepsilon \cdot \nabla[\theta^\varepsilon - M]^+ + b\theta^\varepsilon_z[\theta^\varepsilon - M]^+ - \lambda b[\theta^\varepsilon - M]^+_z\} \\ &= \int_\Omega |\nabla[\theta^\varepsilon - M]^+|^2, \end{aligned}$$

and therefore $[\theta^\varepsilon - M]^+ = 0$. The lemma is proved.

We shall need the L^∞ and Hölder estimates due to Stampacchia [14] for the following elliptic problem with mixed boundary conditions:

$$-\Delta u + bu_z = f \text{ in } \Omega, \qquad \partial u/\partial n = g \text{ on } \Gamma_1, \qquad u = h \text{ on } \Gamma_0 \cup \Gamma_H. \tag{2.11}$$

LEMMA 2 [14]. The unique solution of (2.11) satisfies

$$\|u\|_{L^\infty(\Omega)} \leq C_1(\|f\|_{W^{-1,p}(\Omega)} + \|g\|_{L^q(\Gamma_1)} + \|h\|_{L^\infty(\Gamma_0 \cup \Gamma_H)}), \tag{2.12}$$

$$\|u\|_{C^{0,\alpha}(\bar{\Omega})} \leq C_2(\|f\|_{W^{-1,p}(\Omega)} + \|g\|_{L^q(\Gamma_1)} + \|h\|_{C^{0,1}(\bar{\Gamma}_0 \cup \bar{\Gamma}_H)}) \tag{2.13}$$

for all $p > 3$ and $q > 2$ and for some constants C_1, $C_2 > 0$ and $0 < \alpha < 1$ which are independent of f, g, h and u.

Proof. See the results of Sec. 5 of [14] or a more explicit result extended to variational inequalities in Sec. 2 of [13].

Now we can state an existence result for the penalized problem, from which we shall construct a sequence of functions converging to a solution of Problem (P).

PROPOSITION 1. Under the assumptions of Lemma 1 and if

$$h \in C^{0,1}(\bar{\Gamma}_H) \tag{2.14}$$

then there exists a solution θ^ε to Problem (P_ε) for all $0 < \varepsilon \leq M$ satisfying the estimate

$$\| \theta^\varepsilon \|_{H^1(\Omega)} + \| \theta^\varepsilon \|_{C^{0,\alpha}(\bar{\Omega})} \leq C, \tag{2.15}$$

where the constants $C > 0$ and $0 < \alpha < 1$ are independent of ε.

Proof. For $\tau \in B_R = \{\tau \in C^0(\bar{\Omega}) \colon \| \tau \|_{C^0(\bar{\Omega})} \leq R\}$ $(R > 0)$, define

$$\theta = S_\varepsilon(\tau)$$

as the unique solution of the following mixed linear problem:

$$\theta = 0 \text{ on } \Gamma_0, \qquad \theta = h \text{ on } \Gamma_H,$$

$$\int_\Omega (\nabla\theta \cdot \nabla\zeta + b\theta_z \zeta) = \lambda b \int_\Omega \chi_\varepsilon(\tau)\zeta_z - \int_{\Gamma_1} g(\tau)\zeta,$$

$$\forall \zeta \in H^1(\Omega) \colon \zeta = 0 \text{ on } \Gamma_0 \cup \Gamma_H.$$

Since, by definition, $0 \leq \chi_\varepsilon \leq 1$ and g is bounded independently of τ (for $|\tau(X)| \geq M \geq \rho(X)$ one has $g(X, \rho(X), \tau(X)) = 0$) by (2.4)), one can apply Stampacchia's estimate (2.13). Therefore, there exists $C > 0$ and $0 < \alpha < 1$, independent of τ and ε, such that

$$\| \theta \|_{C^{0,\alpha}(\bar{\Omega})} \leq C_2(\lambda b + \| g \|_{L^\infty} + \| h \|_{C^{0,1}}) \leq C$$

and for $R \geq C$ one has $S_\varepsilon(B_R) \subset B_R$.

From the compactness of the imbedding $C^{0,\alpha}(\bar{\Omega}) \subsetneq C^0(\bar{\Omega})$ one finds that S_ε is a continuous and compact mapping of B_R into itself. By the Schauder fixed-point theorem there exists a function $\theta^\varepsilon \in B_R$ satisfying $\theta^\varepsilon = S_\varepsilon(\theta^\varepsilon)$, which is clearly a solution to Problem (P_ε).

The estimate in $H^1(\Omega)$ is classical, since χ_ε and $g(\theta^\varepsilon)$ are bounded independently of ε.

THEOREM 1. Assuming (2.2, 3, 4) and (2.8, 9, 14), there exists a solution $(\theta, \chi) \in [H^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega})] \times L^\infty(\Omega)$ to Problem (P).

Proof. By (2.15) one can consider a sequence of solutions θ^ε of Problem (P_ε) such that, when $\varepsilon \downarrow 0$,

$$\theta^\varepsilon \rightharpoonup \theta \text{ in } H^1(\Omega) - \text{weak} \tag{2.16}$$

$$\theta^\varepsilon(X) \to \theta(X) \text{ uniformly in } X = (x, y, z) \in \bar{\Omega} \tag{2.17}$$

$$\chi_\varepsilon(\theta^\varepsilon) \rightharpoonup \chi \text{ in } L^\infty(\Omega) - \text{weak} *, \tag{2.18}$$

where θ is some function belonging to $H^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$ satisfying (2.10) and $0 \leq \chi \leq 1$. Moreover, in the open set $\{\theta > 0\}$ one has $\chi_\varepsilon(\theta^\varepsilon) \to 1$ and therefore $\chi = 1$ a.e. in $\{\theta > 0\}$.

Let $\zeta \in H^1(\Omega)$, $\zeta \geq 0$ on Γ_0 and $\zeta = 0$ on Γ_H. By Green's formula and since $\partial\theta^\varepsilon/\partial n \leq 0$ on Γ_0, one has

$$\int_\Omega [\nabla\theta^\varepsilon \cdot \nabla\zeta + b\theta^\varepsilon_z \zeta - \lambda b \chi_\varepsilon(\theta^\varepsilon)\zeta_z] + \int_{\Gamma_1} g(\theta^\varepsilon)\zeta = \int_{\Gamma_0} \frac{\partial\theta^\varepsilon}{\partial n} \zeta \leq 0$$

and in the limit we obtain (1.8). The proof is complete.

3. The case of maximal monotone cooling. In this section we consider the existence of a weak solution with lateral cooling:

$$-\partial\theta/\partial n \in G(\theta) \text{ on } \Gamma_1, \tag{3.1}$$

where G denotes a maximal monotone graph, that is, G is a multivalued function whose graph is a continuous monotone increasing curve in R^2 (see [4]). We shall assume

$$G(0) \subset]-\infty, 0], \tag{3.2}$$

$$[0, +\infty[\subset \mathrm{Dom}(G) \equiv \{x \in R \mid G(x) \neq \emptyset\}. \tag{3.3}$$

The weak formulation of the corresponding problem now takes the following form:

Problem $(\tilde{\mathrm{P}})$: Find $(\theta, \chi, g) \in H^1(\Omega) \times L^\infty(\Omega) \times L^2(\Gamma_1)$ such that

$$\theta \geq 0 \text{ a.e. in } \Omega, \qquad \theta = 0 \text{ on } \Gamma_0 \text{ and } \theta = h \text{ on } \Gamma_H; \tag{3.4}$$

$$0 \leq \chi \leq 1 \text{ a.e. in } \Omega, \qquad \chi = 1 \text{ if } \theta > 0; \tag{3.5}$$

$$\int_\Omega (\nabla\theta \cdot \nabla\zeta + b\theta_z\zeta - \lambda b\chi\zeta_z) + \int_{\Gamma_1} g\zeta \leq 0,$$
$$\forall\zeta \in H^1(\Omega)\colon \zeta \geq 0 \text{ on } \Gamma_0, \qquad \zeta = 0 \text{ on } \Gamma_H; \tag{3.6}$$

$$g(X) \in G(\theta(X)) \text{ a.e. } X \in \Gamma_1. \tag{3.7}$$

We shall obtain a solution to Problem $(\tilde{\mathrm{P}})$ as the limit of a sequence of solutions to Problem (P) with a nonlinear cooling given by a function g satisfying:

$$g \text{ is monotone increasing, Lipschitz and such that } g(0) \leq 0. \tag{3.8}$$

THEOREM 2. Assume (3.8) and let $h \in H^{1/2}(\Gamma_H)$, $h > 0$. Then Problem (P) has a solution.

Proof. The proof follows the lines of that in Theorem 1, by considering the penalized problem (P_ε) with g satisfying (3.8). The fixed point is now constructed in $L^2(\Omega)$ by means of the mapping

$$L^2(\Omega) \ni \tau \mapsto \xi = T_\varepsilon(\tau) \in V,$$

where $V = \{v \in H^1(\Omega)\colon v = 0 \text{ on } \Gamma_0\}$ and ξ is the unique solution of the following problem:

$$\begin{gathered} \xi \in V, \qquad \xi = h \text{ on } \Gamma_H, \\ \int_\Omega (\nabla\xi \cdot \nabla\zeta + b\xi_z\zeta) + \int_{\Gamma_1} g(\xi)\zeta = \lambda b \int_\Omega \chi_\varepsilon(\tau)\zeta_z, \qquad \forall\zeta \in V\colon \zeta = 0 \text{ on } \Gamma_H \end{gathered} \tag{3.9}$$

which is a coercive and (strictly) monotone problem in V by assumption (3.8) (see [12]). Denoting by $\tilde{h}$ some function in V whose trace on Γ_H is h, and letting $\zeta = \xi - \tilde{h}$ in (3.9), one easily finds

$$\|\xi\|_{H^1(\Omega)} \leq C = C(\tilde{h}),$$

where C is a constant independent of τ and ε.

Since the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the Schauder fixed-point theorem assures the existence of a solution θ^ε to Problem (P_ε). As in Lemma 1, one finds that $\theta^\varepsilon \geq 0$, since g is monotone increasing and $g(0) \leq 0$, and therefore one has $g(\theta^\varepsilon) \cdot [\theta^\varepsilon]^- \leq 0$.

The passage to the limit as $\varepsilon \downarrow 0$ is straightforward since $\theta^\varepsilon \rightharpoonup \theta$ in $H^1(\Omega)$-weak and g is a Lipschitz function.

Remark 1. Since g is Lipschitz, by Sobolev imbeddings one has $g(\theta) \in H^{1/2}(\Gamma_1) \hookrightarrow L^4(\Gamma_1)$ (see [1, p. 218]) and therefore by applying Lemma 2 it follows that

$$\text{if} \quad h \in L^\infty(\Gamma_H), \qquad \text{then } \theta \in L^\infty(\Omega); \tag{i}$$

$$\text{if} \quad h \in C^{0,1}(\bar{\Gamma}_H), \qquad \text{then } \theta \in C^{0,\alpha}(\bar{\Omega}) \text{ for some } 0 < \alpha < 1. \tag{ii}$$

Since G is a maximal monotone operator, one can introduce the Yosida regularization defined by

$$g_\delta = \frac{1}{\delta}(I - J_\delta) \qquad \text{for } \delta > 0,$$

where $J_\delta = (I + \delta G)^{-1}$ is the resolvent of G. Consider $\tau = J_\delta(0)$, that is $0 \in (I + \delta G)(\tau)$. From the monotonicity of $I + \delta G$ and using assumptions (3.2), one finds $\tau \geq 0$. Therefore $g_\delta(0) = -J_\delta(0)/\delta \leq 0$, which means that, for each $\delta > 0$, the Yosida regularization g_δ satisfies the condition (3.8) (see [4]). So we may apply Theorem 2 to conclude the existence of a solution $(\theta^\delta, \chi^\delta) \in H^1(\Omega) \times L^\infty(\Omega)$ to Problem (P) with lateral cooling given by g_δ. We shall obtain a solution to Problem $(\tilde{\mathrm{P}})$ by considering a subsequence $\delta \downarrow 0$.

THEOREM 3. Problem $(\tilde{\mathrm{P}})$ with a maximal monotone graph G satisfying (3.2) and (3.3) and with $h \in H^{1/2}(\Gamma_H) \cap L^\infty(\Gamma_H)$ has a solution $(\theta, \chi, g) \in [H^1(\Omega) \cap L^\infty(\Omega)] \times L^\infty(\Omega) \times L^\infty(\Gamma_1)$. Moreover, if $h \in C^{0,1}(\bar{\Gamma}_H)$ one has $\theta \in C^{0,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$.

Proof. Consider the (unique) solution θ^0 of the following mixed problem:

$$\begin{gathered} \theta^0 \in H^1(\Omega), \qquad \theta^0 = 0 \text{ on } \Gamma_0, \qquad \theta^0 = h \text{ on } \Gamma_H, \\ \int_\Omega (\nabla\theta^0 \cdot \nabla\zeta + b\theta^0_z \zeta) + \int_{\Gamma_1} g^0(0)\zeta = 0, \\ \forall \zeta \in H^1(\Omega), \qquad \zeta = 0 \text{ on } \Gamma_0 \cup \Gamma_H, \end{gathered} \tag{3.10}$$

where $g^0(t) = \mathrm{Proj}_{G(t)} 0$ is the smallest (in norm) number of $G(t)$. Since $g^0(0) \leq 0$, it is easy to show that $\theta^0 \geq 0$. Since $h \in L^\infty(\Gamma_H)$, one has $\theta^0 \in L^\infty(\Omega)$ by (2.12), and we assume that $\theta^0 \leq M^0 = M^0(h, g^0(0))$.

Then, for every solution θ^δ to Problem (P) with g_δ, we have

$$0 \leq \theta^\delta \leq \theta^0 \leq M^0. \tag{3.11}$$

Indeed, (3.11) follows by a comparison argument: take $\zeta = [\theta^\delta - \theta^0]^+$ in $(1\text{-}8)_\delta$ and in (3.10); one has

$$\int_\Omega |\nabla[\theta^\delta - \theta^0]^+|^2 + \lambda b \int_\Omega \chi^\delta[\theta^\delta - \theta^0]^+_z + \int_{\Gamma_1} [g_\delta(\theta^\delta) - g^0(0)][\theta^\delta - \theta^0]^+ \leq 0 \tag{3.12}$$

Since $\theta^0 \geq 0$ and $\chi^\delta = 1$ in $\{\theta^\delta > 0\}$, the middle term in (3.12) vanishes; using $g_\delta(0) \leq 0$, together with

$$|g_\delta(t)| \leq |g^0(t)| \tag{3.13}$$

(see [4], p. 28) in order to deduce the chain

$$g_\delta(\theta^\delta) \geq g_\delta(\theta^0) \geq g_\delta(0) \geq g^0(0),$$

one finds that the last term in (3.12) is non-negative, which proves (3.11).

Again using (3.13), by (3.11) one has

$$|g_\delta(\theta^\delta)| \leq |g^0(\theta^\delta)| \leq \max[|g^0(0)|, |g^0(M^0)|] \equiv l. \tag{3.14}$$

from which we easily conclude

$$\| \theta^\delta \|_{H^1(\Omega)} \leq C(= \text{constant independent of } \delta).$$

It follows that there exists a subsequence $\delta \downarrow 0$ such that

$$\theta^\delta \rightharpoonup \theta \text{ in } H^1(\Omega)\text{-weak}, \qquad \text{and } 0 \leq \theta \leq M^0, \tag{3.15}$$

$$\chi^\delta \rightharpoonup \chi \text{ in } L^\infty(\Omega)\text{-weak} *, \qquad 0 \leq \chi \leq 1 \tag{3.16}$$

$$g_\delta(\theta^\delta) \rightharpoonup g \text{ in } L^\infty(\Gamma_1)\text{-weak} *, \qquad \text{with } \| g \|_{L^\infty} \leq l. \tag{3.17}$$

Since one can also consider $\theta^\delta \to \theta$ uniformly in each compact subset $K \subset \Omega$, one has $\chi = 1$ in the open set $\{\theta > 0\}$.

Using the compactness of the trace mapping, one can consider $\theta^\delta \to \theta$ in $L^2(\Gamma_1)$-strong and from (3.3) $J_\delta(\theta^\delta) \to \theta$ in $L^2(\Gamma_1)$. Since $g_\delta(\theta^\delta) \in G(J_\delta(\theta^\delta))$, it follows, by a classical argument ([4], p. 27), that $g \in G(\theta)$.

If we assume $h \in C^{0,1}(\bar{\Gamma}_H)$, by Lemma 2 one easily concludes that $\theta \in C^{0,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$. The proof is complete.

Remark 2. Assuming that there exists some $v \geq 0$ such that $0 \in G(v)$, one can find a more simple estimate in $L^\infty(\Omega)$ for every solution θ to Problem $(\tilde{\mathrm{P}})$:

$$\theta \leq M = \max(v, \| h \|_{L^\infty(\Gamma_H)}).$$

Indeed, it is sufficient to consider $\zeta = [\theta - M]^+$ in (3.6) and to recall that the monotonicity of G implies $g \geq 0$ if $\theta > M$.

Remark 3. The results of this section can be easily extended to the case of a lateral boundary condition

$$-\frac{\partial \theta}{\partial n}(X) \in G(z, \theta(X)), \qquad \text{for } X = (x, y, z) \in \Gamma_1,$$

where, for each $z \in]0, H[$, $G(z, \cdot)$ denotes a maximal monotone graph satisfying (3.2), (3.3) and l in (3.14) being uniformly bounded in z.

An interesting case could be a lateral boundary submitted to N differents cooling zones, that is, when, for $i = 1, \ldots, N$,

$$G(z, \cdot) = G_i(\cdot), \qquad 0 = z_0 < \cdots < z_{i-1} < z < z_i < \cdots z_N = H.$$

4. Comparison results. If the cooling is given by a monotone function one can adapt the technique of [5] to prove

PROPOSITION 2. Let θ^ε (resp. $\hat{\theta}^\varepsilon$) a solution to Problem (P_ε) and corresponding to g and h (resp. $\hat{g}$ and $\hat{h}$), where g and $\hat{g}$ are monotone functions satisfying (3.8). Then if $\hat{h} \geq h$ and $\hat{g} \leq g$ it follows that $\hat{\theta}^\varepsilon \geq \theta^\varepsilon$.

Proof. Set $f_\delta(t) = [1 - \delta/t]^+$, $t \in R$ and $\delta > 0$.

From (2.7) and denoting $\eta = \theta^\varepsilon - \hat{\theta}^\varepsilon$, one has

$$\int_\Omega \nabla\eta \cdot \nabla\zeta = b \int_\Omega \{\eta + \lambda[\chi_\varepsilon(\theta^\varepsilon) - \chi_\varepsilon(\hat{\theta}^\varepsilon)]\}\zeta_z - \int_{\Gamma_1} [g(\theta^\varepsilon) - \hat{g}(\hat{\theta}^\varepsilon)]\zeta$$

for every $\zeta \in H^1(\Omega)$, $\zeta = 0$ on $\Gamma_0 \cup \Gamma_H$. In particular, for $\zeta = f_\delta(\eta)$, which is different from zero if $\theta^\varepsilon \geq \hat{\theta}^\varepsilon$ where $g(\theta^\varepsilon) \geq g(\hat{\theta}^\varepsilon) \geq \hat{g}(\hat{\theta}^\varepsilon)$, it follows

$$\left| \int_\Omega \nabla\eta \cdot \nabla f_\delta(\eta) \right| \leq bL_\varepsilon \int_\Omega |\eta| \cdot |[f_\delta(\eta)]_z|, \tag{4.1}$$

being L_ε the Lipschitz constant of $t \mapsto t + \lambda\chi_\varepsilon(t)$.

As in [5], (4.1) implies, for any $\delta > 0$,

$$\int_\Omega \log\left(1 + \frac{[\eta - \delta]^+}{\delta}\right)^2 \leq C(= \text{constant independent of } \delta)$$

from which it follows $\theta^\varepsilon - \hat{\theta}^\varepsilon = \eta \leq 0$.

Remark 4. This argument also proves the uniqueness of the solution of the Problem (P_ε) when g is monotone. Of course if θ(resp. $\hat{\theta}$) is a solution of (P) which is the limit of the subsequence $\theta^{\varepsilon'}$ (resp. $\hat{\theta}^{\varepsilon'}$) the above proposition implies that $\hat{\theta} \geq \theta$.

Next we shall prove comparison results with respect to the extraction velocity b.

PROPOSITION 3. Assume that there exists constants μ, M such that

$$0 < \mu \leq h(x, y) \leq M, \qquad \text{a.a. } (x, y) \in \Gamma_H. \tag{4.2}$$

and that the function g satisfies (3.8) with

$$\{t\colon g(t) = 0\} \subset [M, +\infty[, \tag{4.3}$$

or else that g verifies (2.2, 3, 4, 9). Then if $b \leq 1/H \log(1 + \mu/\lambda)$ a solution θ to Problem (P_1) is also a solution to Problem (P) with $\chi = 1$.

Proof. If g satisfies (3.8), then the Problem (P_1) has a unique solution (let $\chi_\varepsilon \equiv 0$ in (3.9)). Moreover by (4.3) one has $g(\theta) \leq 0$ (see Lemma 1).

Under assumptions (2.2, 3, 4, 9) the existence of θ may be shown essentially as in Proposition 1, being also $g(\theta) \leq 0$, by hypothesis.

Consider now the function $\theta_\mu(z) = \mu(e^{bz} - 1)(e^{bH} - 1)^{-1}$. Taking $\zeta = (\theta_\mu - \theta)^+$ in (1.10) and since $g(\theta) \leq 0$ in both cases, one easily finds that $\theta \geq \theta_\mu$. Therefore, it follows

$$\frac{\partial\theta}{\partial n} \leq \frac{\partial\theta_\mu}{\partial n} = -\mu b(e^{bH} - 1)^{-1} \text{ on } \Gamma_0.$$

Using the Green's formula with a smooth function ζ such that $\zeta \geq 0$ on Γ_0 and $\zeta = 0$ on Γ_H, one has

$$\int_\Omega (\nabla\theta \cdot \nabla\zeta + b\theta_z\zeta - \lambda b\zeta_z) + \int_{\Gamma_1} g(\theta)\zeta = \int_{\Gamma_0} \left(\frac{\partial\theta}{\partial n} + \lambda b\right)\zeta \leq 0$$

for $\lambda b \leq \mu b(e^{bH} - 1)^{-1}$. This means that, for all $bH \leq \log(1 + \mu/\lambda)$, $(\theta, 1)$ is also a solution to Problem (P).

This proposition suggests that, for small velocities b, the whole region Ω is occupied by solid metal, since if the Problem (P) admits only one solution θ, one has $\theta > 0$ in Ω for

$0 < b \leq 1/H \log(1 + \mu/\lambda)$. Conversely the next proposition suggests that for big velocities the free boundary exists, since we will show that the volume of the set $\{\theta > 0\}$ vanishes when $b \uparrow \infty$.

PROPOSITION 4. Under assumptions of the Theorem 1 or Theorem 3 and denoting by $|\Omega_+|$ the Lebesgue measure of the set $\Omega_+ = \{X \mid \theta(X) > 0\}$, one has

$$|\Omega_+| \leq \frac{C}{\lambda b}, \tag{4.4}$$

where C is a positive constant independent of λ and b. Moreover, for b big enough, one has $\chi \not\equiv 1$.

Proof. Let $\zeta = H - z$ in (1.8) and in (3.6). One has

$$-\int_\Omega \theta_z + b\int_\Omega \theta_z(H - z) + \lambda b \int_\Omega \chi + \int_{\Gamma_1} g(H - z) \leq 0, \tag{4.5}$$

where $g = g(\theta)$ and $g \in G(\theta)$, respectively. In the first case, g is a bounded function and from $0 \leq \theta \leq M$ (see Theorem 1 and Lemma 1), we may assume $-l_1 \leq g \leq 0$, with l_1 independent of b and λ. In the second one, by (3.17) and (3.14) we have $\| g \|_{L^\infty} \leq l$ and l is also independent of b and λ.

Denoting $L = \max(l, l_1)$ from (4.5) it follows that

$$\lambda b \int_\Omega \chi \leq \int_{\Gamma_H} h + L \int_{\Gamma_1} (H - z),$$

since one has

$$\int_\Omega \theta_z = \int_{\Gamma_H} h \quad \text{and} \quad \int_\Omega \theta_z(H - z) = \int_\Omega \theta \geq 0.$$

Recalling that $0 \leq \chi \leq 1$ and $\chi = 1$ in Ω_+, one has

$$|\Omega_+| \leq \int_\Omega \chi \leq |\Gamma|(M + LH^2/2)/\lambda b,$$

which completes the proof of the proposition.

Now we assume the existence of d, $0 < d < H$, such that

$$g(X, \rho, \theta) = 0 \qquad \text{for} \qquad 0 < z < d, \qquad \forall (X, \rho, \theta) \in \Gamma_1 \times R_+ \times R \tag{4.6}$$

or, for the monotone case (see Remark 3),

$$G(z, \cdot) \equiv 0 \qquad \text{for } 0 < z < d < H. \tag{4.7}$$

THEOREM 4. Let (θ, χ) (resp. (θ, χ, g)) a solution to Problem (P) (resp. $(\tilde{\mathrm{P}})$) under assumptions of Theorem 1 with (4.6) (resp. Theorem 3 with (4.7)). Then there exists δ, $0 < \delta < d$, such that

$$\theta(x, y, z) \leq \lambda b[z - \delta]^+, \qquad \forall (x, y, z) \in \bar{\Omega} \tag{4.8}$$

$$\theta = \chi = 0 \qquad \text{for } 0 < z < \delta, \tag{4.9}$$

for all $b > M/\lambda d$, where $M \geq \| \theta \|_{L^\infty}$ is a constant independent of b (see (2.10) and (3.15)). The proof of this theorem uses the following lemma.

LEMMA 3. Under assumptions of Theorem 4, one has

$$\int_{Z_\delta} \chi(\lambda b\chi - \theta_z) \leq \int_{Z_\delta} (b\theta + \lambda b\chi - \theta_z) \leq 0 \tag{4.10}$$

for $0 < \delta \leq d$ and $Z_\delta = \{(x, y, z) \in \Omega \mid 0 < z < \delta\}$.

Proof. Let $\zeta = [\delta - z]^+$ in (1.8) or in (3.6). One has

$$\int_{Z_\delta} [-\theta_z + b\theta_z(\delta - z) + \lambda b\chi] \leq 0,$$

because (4.6) or (4.7) imply $g[\delta - z]^+ = 0$. Since

$$\int_{Z_\delta} \theta_z(\delta - z) = \int_{Z_\delta} \theta \geq 0 \quad \text{and} \quad 0 \leq \chi \leq 1$$

it follows

$$\int_{Z_\delta} \chi(\lambda b\chi - \theta_z) \leq \int_{Z_\delta} (\lambda b\chi - \theta_z) \leq \int_{Z_\delta} (b\theta + \lambda b\chi - \theta_z) \leq 0.$$

Proof of Theorem 4. Consider $\mu = \mu(z) = \lambda b[z - \delta]^+$ with δ fixed such that $0 < \delta \leq d - M/\lambda b$. The function $\zeta = [\theta - \mu]^+$ vanishes on $z = 0$ and for $z \geq d$. Therefore $g[\theta - \mu]^+ = 0$ and from (1.8) or from (3.6), one has

$$\int_\Omega \nabla\theta \cdot \nabla[\theta - \mu]^+ + b\int_\Omega \theta_z[\theta - \mu]^+ - \lambda b\int_\Omega \chi[\theta - \mu]_z^+ \leq 0$$

or

$$\int_{Z_\delta} (|\nabla\theta|^2 - \lambda b\chi\theta_z) + \int_{(\Omega\setminus Z_\delta) \cap \{\theta > 0\}} \{\nabla\theta \cdot \nabla[\theta - \mu]^+ - \lambda b[\theta - \mu]_z^+\} + b\int_\Omega \theta_z[\theta - \mu]^+ \leq 0.$$

Adding the quantity

$$\lambda b\int_{Z_\delta} \chi(\lambda b\chi - \theta_z) - b\int_{\Omega\setminus Z_\delta} \lambda b[\theta - \mu]^+$$

which is non-positive by Lemma 3, one obtains

$$\int_{Z_\delta} \{\theta_x^2 + \theta_y^2 + (\theta_z - \lambda b\chi)^2\} + \int_{\Omega\setminus Z_\delta} |\nabla[\theta - \mu]^+|^2 + b\int_\Omega (\theta - \mu)_z[\theta - \mu]^+ \leq 0.$$

Since the last term is zero, it follows that $\theta \leq \mu$ in $\Omega\setminus Z_\delta = \{z \geq \delta\}$ and $\theta_x = \theta_y = 0$, $\theta_z = \lambda b\chi$ in $Z_\delta = \{z < \delta\}$. Since $\theta = 0$ for $z = 0$ and $z = \delta$, one has $\theta = 0$ for $z \leq \delta$ and consequently also $\chi = 0$ for $z \leq \delta$.

5. Regularity of the free boundary. The goal of Theorem 4 is to provide sufficient conditions in order to assume the global existence of a free boundary. In this case we shall prove that the free boundary is an analytic surface.

We begin with the following

PROPOSITION 5. A solution (θ, χ) (resp. (θ, χ, g)) to Problem (P) (resp. $(\tilde{\mathrm{P}})$) satisfies

$$-\Delta\theta + b\theta_z + \lambda b\chi_z = 0 \text{ in } \mathscr{D}'(\Omega), \tag{5.1}$$

$$\chi_z \geq 0 \text{ in } \Omega. \tag{5.2}$$

Proof. The equation (5.1) follows immediately by taking $\zeta \in \mathscr{D}(\Omega)$ in (1.8) or in (3.6)

Choosing as a test function in (1.8) or in (3.6) $\zeta = \min(\theta, \varepsilon\eta)$, where $\varepsilon > 0$ and $\eta \in \mathscr{D}(\Omega)$, $\eta \geq 0$ one has

$$I = \int_\Omega \nabla\theta \cdot \nabla \min(\theta, \varepsilon\eta) + b \int_\Omega \theta_z \min(\theta, \varepsilon\eta) - \lambda b \int_\Omega [\min(\theta, \varepsilon\eta)]_z \leq 0$$

since $\chi = 1$ in $\{\theta > 0\}$. Since $\min(\theta, \varepsilon\eta) = 0$ on $\partial\Omega$, the last integral is zero and it follows

$$\begin{aligned} I &= \int_{\{\theta \leq \varepsilon\eta\}} |\nabla\theta|^2 + \varepsilon \int_{\{\theta > \varepsilon\eta\}} \nabla\theta \cdot \nabla\eta + b \int_\Omega \{\varepsilon\eta\theta_z + \theta_z [\min(\theta, \varepsilon\eta) - \varepsilon\eta]\} \\ &\geq \varepsilon \int_{\{\theta > \varepsilon\eta\}} \nabla\theta \cdot \nabla\eta + \varepsilon b \int_\Omega \theta_z \eta - b \int_\Omega \theta_z [\varepsilon\eta - \theta]^+, \end{aligned}$$

from which one concludes

$$\int_\Omega \chi\{\theta > \varepsilon\eta\} \nabla\theta \cdot \nabla\eta + b \int_\Omega \theta_z \eta \leq b \int_\Omega \theta_z \left[\eta - \frac{\theta}{\varepsilon}\right]^+.$$

Passing to the limit $\varepsilon \searrow 0$, one obtains

$$\int_\Omega (\nabla\theta \cdot \nabla\eta + b\theta_z \eta) \leq 0, \qquad \forall \eta \in \mathscr{D}(\Omega): \eta \geq 0$$

and using (5.1), one deduces (5.2).

From (5.1) it follows that the function θ is locally Hölder continuous. Therefore the set

$$\Omega_+ \equiv \{X \in \Omega \mid \theta(X) > 0\} \tag{5.3}$$

is an open set. Since χ is montonous increasing in the z-coordinate one can introduce

$$\phi(x, y) = \inf\{z: \theta(x, y, z) > 0, (x, y, z) \in \Omega\} \tag{5.4}$$

where ϕ is an upper semi-continuous function, by the continuity of θ. Then we can state.

THEOREM 5. For any solution of Problem (P) or $(\tilde{\mathrm{P}})$ one has

$$\Omega_+ \equiv \{\theta > 0\} = \{X \in \Omega: z > \phi(x, y)\} \tag{5.5}$$

where ϕ is an upper semi-continuous function given by (5.4)

COROLLARY 1. Under conditions of Theorem 4, for all $b > M/\lambda d$, one has

$$H > \phi(x, y) \geq d - M/\lambda d > 0, \quad \text{for all } (x, y) \in \Gamma,$$

which, in particular, assures the existence of a free boundary.

Consider now the function

$$u(x, y, z) = \int_0^z \theta(x, y, t)\, dt, \quad \text{for} \quad (x, y, z) \in \bar{\Omega}, \tag{5.6}$$

which is a Baiocchi type transformation (see [3] for instance).

THEOREM 6. Let (θ, χ) (resp. $(0, \chi, g)$) be a solution to Problem (P) (resp. $(\tilde{\mathrm{P}})$ under the assumptions of Theorem 4. Then the function u defined by (5.6) satisfies the following variational inequality in Ω

$$u \geq 0, \qquad (-\Delta u + bu_z + \lambda b) \geq 0, \qquad u \cdot (-\Delta u + bu_z + \lambda b) = 0, \tag{5.7}$$

and χ is a characteristic function, being

$$\chi = \chi(\theta) = \chi(u) \qquad \text{a.e. in } \Omega \tag{5.8}$$

where $\chi(v)$ denotes the characteristic function of the set $\{v > 0\}$.

Proof. From definition (5.6) and recalling $\theta \geq 0$ it is obvious that $u \geq 0$. Since $\theta = u_z$ and θ satisfies (5.1) one has

$$(-\Delta u + bu_z + \lambda b\chi)_z = -\Delta\theta + b\theta_z + \lambda b\chi_z = 0$$

which, together with (4.9) and $0 \leq \chi \leq 1$, imply

$$0 = -\Delta u + bu_z + \lambda b\chi \leq -\Delta u + bu_z + \lambda b. \tag{5.9}$$

Recalling (5.5) it is clear that

$$\{\theta > 0\} = \{u > 0\} \tag{5.10}$$

from which one deduces $\chi = 1$ if $u > 0$, and the third condition of (5.7) follows by (5.9).

From the classical regularity to solutions of variational inequalities one has

$$u \in W^{2,\infty}_{\text{loc}}(\Omega) \tag{5.11}$$

(see [11], for instance) and (5.8) follows easily from (5.9) and (5.10).

Remark 5. If one considers a linear flux

$$g(X, \rho(X), \theta(X)) = \alpha(z)[\theta(X) - \rho(X)] \tag{5.12}$$

with $\rho \geq 0$ and $\alpha(z) = 0$ for $0 < z < d$ and $\alpha(z) = \alpha = \text{constant} > 0$ for $d < z < H$, then we have that u is the unique solution of the following variational inequality with mixed boundary conditions (see [6] and [15]):

$$u \in K = \{v \in H^1(\Omega) \,|\, v \geq 0 \text{ in } \Omega,\ v = 0 \text{ on } \Gamma_0\}$$

$$\int_\Omega \nabla u \cdot \nabla(v-u) + b\int_\Omega u_z(v-u) + \alpha\int_{\Gamma_1} u(v-u)$$
$$\geq \int_{\Gamma_H} h(v-u) - \lambda b\int_\Omega (v-u) + \alpha\int_{\Gamma_1} \tilde{\rho}(v-u), \qquad \forall v \in |K,$$

where $\tilde{\rho}(z) = \int_d^z \rho(t)\,dt$ for $z \geq d$.

In particular, this implies the uniqueness of the solution of Problem (P) for a linear cooling given by (5.12).

The transformation (5.6) and its consequence (5.8) allow us to include the study of the free boundary

$$\Phi = \Omega \cap \partial\Omega_+$$

in the known results of Caffarelli [7], Kinderlehrer and Nirenberg [10]. In order to apply these results we must show that Φ has not singular points. This may be done by using a technique due to Alt [2] for the dam problem.

LEMMA 4. Let $X_0 \in \Phi$ and $B_r(X_0) \subset \Omega$. Then there is a cone $\Lambda_r \subset \{X \in R^3 \,|\, z < 0\}$. Such that

$$\frac{\partial u}{\partial \eta}(X) = \nabla u(X) \cdot \eta \leq 0 \qquad \text{for } X \in B_{r/2}(X_0), \qquad \text{whenever } \eta \in \Lambda_r \cap S^2. \tag{5.13}$$

Proof. Recalling (5.11) and that $u_z = \theta \geq 0$ in Ω, the proof of this lemma is a simple adaptation of Lemma 6.9 of [11], page 255, and therefore we omit it.

THEOREM 7. Let (θ, χ) (resp. (θ, χ, g)) be a solution to Problem (P) (resp. $(\tilde{\mathrm{P}})$) under conditions of Theorem 4. Then the free boundary Φ is an analytic surface given by

$$\Phi\colon z = \phi(x, y) \quad \text{for} \quad (x, y) \in \Gamma,$$

and θ is also a classical solution of Problem (C).

Proof. By (5.13) the function ϕ defined by (5.4) is a Lipschitz function in Γ and we can apply Theorem 3 of [7] to conclude that (5.14) ϕ is C^1 and $u \in C^2(\Omega_+ \cup \Phi)$. Therefore from equation (5.1) and Green's formula one finds that condition (1.5) is verified in every point of the free boundary $z = \phi(x, y)$, for all $(x, y) \in \Gamma$, by Corollary 1.

To conclude that Φ is an analytic surface it is sufficient to apply Theorem 1 of [10], using (5.14) and recalling that the equation satisfied by u in Ω_+ has constant coefficients.

6. Unicity in the monotone case. In Remark 5 we have already stated the uniqueness of the solution of Problem (P) with a particular linear cooling.

Adapting to our problem the technique of Carrillo and Chipot [8] we shall prove an uniqueness result for the maximal monotone case assuming that χ is a characteristic function, that is, assuming

$$\chi = \chi(\theta), \tag{6.1}$$

to which we have already stated sufficient conditions in Theorems 4 and 6.

Denote by (θ_i, χ_i, g_i), with $\chi_i = \chi(\theta_i)$ and $g_i \in G(\theta_i)$, for $i = 1, 2$, two solutions of the Problem $(\tilde{\mathrm{P}})$ and set

$$\theta_0 = \min(\theta_1, \theta_2), \qquad \chi_0 = \min(\chi_1, \chi_2), \qquad \phi_0 = \sup(\phi_1, \phi_2).$$

LEMMA 5. Assuming (6.1), one has

$$\int_\Omega \{\nabla(\theta_i - \theta_0)\cdot\nabla\eta + b(\theta_i - \theta_0)_z\eta - \lambda b(\chi_i - \chi_0)\eta_z\}\,dx\,dy\,dz \leq \lambda b \int_{D_i} \eta(x, y, \phi_i(x, y))\,dx\,dy \tag{6.2}$$

for any $\eta \in H^1(\Omega) \cap C^0(\bar{\Omega})$, $\eta \geq 0$, where

$$D_i = \{(x, y) \in \Gamma \mid \phi_i(x, y) < \phi_0(x, y)\}, \qquad i = 1, 2.$$

Proof. Choosing the test functions $\pm\zeta = \pm\min(\theta_i - \theta_0, \varepsilon\eta)$, $\varepsilon > 0$, from (3.6) one obtains for $i \neq j$ $(i, j = 1, 2)$

$$\int_\Omega \{\nabla(\theta_i - \theta_j)\cdot\nabla\zeta + b(\theta_i - \theta_j)_z\zeta - \lambda b(\chi_i - \chi_j)\zeta_z\} + \int_{\Gamma_1}(g_i - g_j)\zeta = 0.$$

By the monotonicity of G, one has

$$\int_{\Gamma_1}(g_i - g_j)\min(\theta_i - \theta_0, \varepsilon\eta) \geq 0$$

since it is sufficient to integrate in $\{\theta_i > \theta_0\}$ where $\theta_j = \theta_0$.

Then it follows

$$\int_\Omega \{\nabla(\theta_i - \theta_0)\cdot\nabla\min(\theta_i - \theta_0, \varepsilon\eta) + b(\theta_i - \theta_0)_z\min(\theta_i - \theta_0, \varepsilon\eta) - \lambda b(\chi_i - \chi_0)[\min(\theta_i - \theta_0, \varepsilon\eta)]_z\} \leq 0$$

or, using $\min(u, v) = v - [v - u]^+$,

$$\int_{\{\theta_i - \theta_0 > \varepsilon\eta\}} \nabla(\theta_i - \theta_0) \cdot \nabla\eta + b \int_\Omega (\theta_i - \theta_0)_z \eta + \lambda b(\chi_i - \chi_0)\eta_z$$

$$\leq b \int_\Omega \left\{(\theta_i - \theta_0)_z \left[\eta - \frac{\theta_i - \theta_0}{\varepsilon}\right]^+ - \lambda(\chi_i - \chi_0)\left[\eta - \frac{\theta_i - \theta_0}{\varepsilon}\right]_z^+\right\}.$$

Since the χ_i are characteristic functions, integrating in z, one has

$$-\int_\Omega (\chi_i - \chi_0)\left[\eta - \frac{\theta_i - \theta_0}{\varepsilon}\right]_z^+ = -\int_{\{\phi_i < z < \phi_0\}} \left[\eta - \frac{\theta_i - \theta_0}{\varepsilon}\right]_z^+$$

$$\leq \int_{D_i} \left[\eta - \frac{\theta_i}{\varepsilon}\right]^+ (x, y, \phi_i) \leq \int_{D_i} \eta(x, y, \phi_i)$$

and (6.2) follows by passing to the limit $\varepsilon \searrow 0$ in

$$\int_{\{\theta_i - \theta_0 > \varepsilon\eta\}} \nabla(\theta_i - \theta_0) \cdot \nabla\eta + b \int_\Omega [(\theta_i - \theta_0)_z \eta - \lambda(\chi_i - \chi_0)\eta_z]$$

$$\leq b \int_\Omega (\theta_i - \theta_0)_z \left[\eta - \frac{\theta_i - \theta_0}{\varepsilon}\right]^+ + \lambda b \int_{D_i} \eta(x, y, \phi_i).$$

THEOREM 8. Assuming (6.1), the Problem ($\tilde{\mathrm{P}}$) has at most one solution.

Proof. For $\varepsilon > 0$, consider a smooth function α_ε, such that, $0 \leq \alpha_\varepsilon \leq 1$, and

$$\alpha_\varepsilon = 1 \text{ in } A_0 = \{\theta_0 > 0\} \cup \Gamma_1 \quad \text{and} \quad \alpha_\varepsilon(X) = 0 \text{ if } d(X, A_0) > \varepsilon.$$

Since $1 - \alpha_\varepsilon = 0$ on $\{\theta_0 > 0\}$, for all $\eta \in H^1(\Omega)$, one has

$$\int_\Omega \{\nabla\theta_0 \cdot \nabla(1 - \alpha_\varepsilon)\eta + b\theta_{0z}(1 - \alpha_\varepsilon)\eta - \lambda b\chi_0[(1 - \alpha_\varepsilon)\eta]_z\} = 0.$$

For $\eta \in H^1(\Omega) \cap C^0(\bar{\Omega})$, $\eta \geq 0$, $\zeta = (1 - \alpha_\varepsilon)\eta$ is a test function in (3.6), and it follows (since $1 - \alpha_\varepsilon = 0$ on Γ_1)

$$\int_\Omega \{\nabla(\theta_i - \theta_0) \cdot \nabla(1 - \alpha_\varepsilon)\eta + b(\theta_i - \theta_0)_z(1 - \alpha_\varepsilon)\eta$$

$$- \lambda b(\chi_i - \chi_0)[(1 - \alpha_\varepsilon)\eta]_z\} \leq 0 \qquad (i = 1, 2).$$

Using (6.2), we obtain

$$\int_\Omega \{\nabla(\theta_i - \theta_0) \cdot \nabla\eta + b(\theta_i - \theta_0)_z \eta - \lambda b(\chi_i - \chi_0)\eta_z\}$$

$$\leq \lim_{\varepsilon \downarrow 0} \lambda b \int_{D_i} (\alpha_\varepsilon \eta)(x, y, \phi(x, y)) = 0.$$

Choosing in this inequality $\eta = z$ and $\eta = H - z$, after a simple calculation one obtains

$$\int_\Omega (\theta_i - \theta_0) + \lambda \int_\Omega (\chi_i - \chi_0) = 0,$$

from where one deduces $\theta_i = \theta_0$ and $\chi_i = \chi_0$, for $i = 1, 2$, which proves the uniqueness of the solution.

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York (1975)

[2] H. Alt, *The fluid flow through porous media: regularity of the free surface*, Manus. Math **21**, 255–272 (1977)

[3] C. Baiocchi and A. Capelo, *Disequazioni variazionali e quasi variazional. Applicazioni a problemi di frontiera libera*, I, II, Quarderni dell'U.M.I., Pitágora Ed. Bologna (1978)

[4] H. Brézis, *Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert*, Math. Studies, **5**, North Holland (1973)

[5] H. Brézis, D. Kinderlehrer and G. Stampacchia, *Sur une nouvelle formulation du problème de l'écoulement à travers une digne*, C.R. Acad. Sc. Paris **287-A**, 711–714 (1978)

[6] T. Brière, *Application des méthodes variationnelles à la cristallisation d'un métal par passage dans une gaîne de reproidissement*, Ann. Fac. Sc. Toulouse **2**, 219–247 (1980)

[7] L. A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math. **139**, 155–184 (1978)

[8] J. Carrillo and M. Chipot, *On the dam problem*, J. Diff. Eqs. (to appear)—cf. also Note in C.R. Acad. Sc. Paris, **292-I**, 191–194 (1981)

[9] G. Duvaut, and J. L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris (1972)

[10] D. Kinderlehrer and L. Nirenberg, *Regularity in free boundary value problems*, Ann. Scuola Norm. Sup. Pisa Ser. IV, **4**, 373–391 (1977)

[11] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications.* Academic Press, New York (1980)

[12] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Dunod, Paris (1969)

[13] M. K. V. Murthy and G. Stampacchia, *A variational inequality with mixed boundary conditions*, Israel J. Math. **13**, 188–224 (1972)

[14] G. Stampacchia, *Problemi al contorno ellitici con datti discontinui dotati di soluzioni hölderiane*, Ann. di Mat. Pura Appl. **51**, 1–32 (1960)

[15] J. F. Rodrigues, *Sur un problème à frontière libre stationnaire traduisant la cristallisation d'un métal*, C.R. Acad. Sc. Paris **290-A**, 823–825 (1980)

[16] L. Rubinstein, *The Stefan problem*, A.M.S. Transl. Monograph 27, (1971)