ON THE STEADY-STATE PROPAGATION OF AN ANTI-PLANE SHEAR CRACK IN AN INFINITE GENERAL LINEARLY VISCOELASTIC BODY*

By

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Abstract. The steady-state propagation of a semi-infinite anti-plane shear crack is considered for a general infinite homogeneous and isotropic linearly viscoelastic body. Inertial terms are retained and the only restrictions placed on the shear modulus are that it be positive, continuous, decreasing and convex. For a given integrable distribution of shearing tractions travelling with the crack, a simple closed-form solution is obtained for the stress intensity factor and for the entire stress field ahead of and in the plane of the advancing crack. As was observed previously for the standard linear solid, the separate considerations of two distinct cases, defined by parameters c and c^*, arises naturally in the analysis. Specifically, c and c^* denote the elastic shear wave speeds corresponding to zero and infinite time, and the two cases are (1) $0 < v < c^*$ and (2) $c^* < v < c$, where v is the speed of propagation of the crack. For case (1) it is shown that the stress field is the same as in the corresponding elastic problem and is hence independent of v and all material properties, whereas, for case (2) the stress field depends on both v and material properties. This dependence is shown to be of a very elementary form even for a general viscoelastic shear modulus.

1. Introduction. Very few exact closed-form solutions have appeared in the literature to viscoelastic fracture problems for which the inertia terms in the equations of motion are retained. Willis [5] considered the steady-state problem of a semi-infinite anti-plane shear crack propagating through an infinite linearly viscoelastic body which was modelled as a standard linear solid. The only loading assumed on the body was an arbitrary integrable distribution of shearing tractions travelling with the crack. By use of the Wiener-Hopf technique, Willis obtained a simple expression for the stress intensity factor. In the course of the analysis, it became necessary to consider two separate cases defined by parameters c and c^*. In the notation of Willis, the shear modulus had the form $G(t) = (\mu/(1 + f))(1 + f\exp(-(1 + f)t/\tau))$ and c and c^* were defined by $c^2 = \mu/\rho$ and $(c^*)^2 = \mu/\rho(1 + f)$, where ρ is the mass density of the material. It should be noted that c and c^* are the elastic shear wave speeds corresponding to $G(0)$ and $G(\infty)$. If v denotes the speed of propagation of the crack, then the two cases distinguished by Willis are (1) $0 < v < c^*$ and (2) $c^* < v < c$. For case (1) it was observed that the stress intensity factor was the same as in the corresponding static

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elastic problem and hence independent of v and material properties, whereas, for case (2) inertial effects occur and the stress intensity factor depends on material properties and v.

This paper addresses the same problem but assumes a general viscoelastic shear modulus. Moreover, the resulting boundary-value problem is not solved by the Wiener-Hopf technique. The main obstacle to applying the Wiener-Hopf technique is the construction of the required factorization of one or more functions of a real variable as a product of two analytic functions whose domains of analyticity are the upper and lower half-planes respectively. Even for the standard linear solid, clever insight was required by Willis to effect the factorization, and separate factorizations were required for cases (1) and (2).

In the present work, the Wiener-Hopf problem is reformulated as a Riemann-Hilbert boundary-value problem. It is then possible to construct the solution for both cases simultaneously in terms of singular integrals and Fourier transforms. What is somewhat surprising is that the complicated-appearing integrals for the stress and displacement fields can be reduced to extremely simple forms, even for a general viscoelastic shear modulus. From these simplified expressions the differences between cases (1) and (2) become transparent. In particular, from the consideration of a sample loading, the characteristics of the shear modulus that determine the magnitude of the dynamic effects for case (2) become illuminated.

The reader should compare the present paper with the analysis by Atkinson and Popelar [3] of transient effects for an anti-plane shear crack in a viscoelastic layer. They discuss the problem of a crack, initially at rest, which is forced to propagate at a constant speed by the sudden action of a constant applied load or displacement on the upper and lower faces of the layer. For much of their analysis, only very general restrictions are imposed on the shear modulus (though more restrictive than here). By successive application of the Laplace and Fourier transforms, the problem is reduced to a Wiener-Hopf equation which is formally factored. Of course, for this problem it is not necessary to consider special cases. The formal factorization produces a very complicated expression for the Laplace transform of the stress intensity factor. Consequently, the Laplace inversion is performed only for the steady state limit $t \to \infty$. It is also appropriate to call the reader's attention to the recent papers by Atkinson [1] and Atkinson and Coleman [2] which shed additional light on the dynamic fracture of linear viscoelastic material.

The approach adopted here produces a formal solution in terms of Fourier transforms and singular integrals to the problem considered by Atkinson and Popelar as well as to many other physical scenarios. However, effecting a simplification of the formal solution to a tractable form may prove to be difficult in general. The technique is currently being adapted to an analysis of the steady-state propagation of a semi-infinite Mode I (plane strain) crack in a general infinite linearly viscoelastic body. The analysis even allows for non-constant Poisson's ratio. The results of that investigation will appear in a forthcoming paper.

2. Formulation of the problem. The boundary-value problem considered here is identical to that treated by Willis [5]. For the sake of completeness, this section begins with a brief review of his derivation of the resulting Wiener-Hopf equation. It is at that point that our analysis departs from his.

The governing field equations for the motion of a linearly viscoelastic solid are

$$\sigma_{ij,j} = \rho u_{i,tt}, \qquad \varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \qquad \sigma_{ij} = 2\mu * d\varepsilon_{ij} + \delta_{ij}\lambda * d\varepsilon_{kk},$$

where σ_{ij}, ε_{ij} and u_i denote the stress, strain and displacement fields respectively. The summation convention is employed, $f_{,i}$ denotes partial differentiation of the function f and $\mu * d\varepsilon$ denotes the Riemann-Stieltjes convolution

$$\mu * d\varepsilon = \int_{-\infty}^{t} \mu(t-\tau)\, d\varepsilon(\tau).$$

Since the deformation of the body is assumed to be anti-plane strain, the only equation of motion not identically satisfied is

$$\mu * d\Delta u_3 = \rho u_{3,tt}$$

where Δu_3 is the two-dimensional Laplacian, $\Delta = (\partial^2/\partial x_2^2) + (\partial^2/\partial x_2^2)$. A semi-infinite crack is assumed to propagate along the x_1-axis with speed v, driven by loads $\sigma_{23}(x_1, 0, t) = f(x_1 - vt)$ which follow it. Adoption of the Galilean variables $x = x_1 - vt$, $y = x_2$ yields the boundary value problem

$$\mu * d\Delta u_3 = \rho v^2 u_{3,xx}, \tag{2.1}$$

$$\sigma_{23}(x, 0) = \frac{\partial}{\partial y}(\mu * du_3) = f(x), \qquad x < 0, \tag{2.2}$$

$$u_3(x, 0) = 0, \qquad x > 0,$$

$$\sigma_{ij}(x, y) \to 0, \qquad x^2 + y^2 \to \infty.$$

Eqs. (2.1) and (2.2) are solved by an application of the Fourier transform defined by

$$\hat{f}(p, y) = \int_{-\infty}^{\infty} e^{ipx} f(x, y)\, dx = F_+(p, y) + F_-(p, y)$$

where

$$F_+ = \int_0^{\infty} e^{ipx} f(x, y)\, dx, \qquad F_- = \int_{-\infty}^{0} e^{ipx} f(x, y)\, dx.$$

Under suitable restrictions on $f(x, y)$, F_+ and F_- are analytic functions of p for $\mathrm{Im}(p) > 0$ and $\mathrm{Im}(p) < 0$ respectively. Transforming Eq. (2.1) and solving the resulting ordinary differential equation in y yields

$$\hat{u}_3(\mathrm{p}, y) = A(p) e^{-\gamma y}$$

where $\gamma^2 = p^2 + i\rho v p/\hat{\mu}(-vp)$. To insure that the stresses and displacement vanish as $y \to \infty$, it is necessary to select a square root of γ^2 with positive real part. This point will be discussed later. Application of the boundary conditions (2.2) yields the relation on $y = 0$

$$v\hat{\mu}(-vp)\gamma(p)\widehat{u_{3,1}} = \widehat{\sigma_{23}^+} + \widehat{\sigma_{23}^-} \tag{2.3}$$

where σ_{23}^+ and σ_{23}^- denote σ_{23} restricted to the positive and negative real axis respectively. It is Eq. (2.3) that Willis solves by the Wiener-Hopf technique for the standard linear solid. Here, Eq. (2.3) will be viewed as the Riemann-Hilbert problem

$$F^+(p) = \mathscr{G}(p) F^-(p) + g(p) \tag{2.4}$$

where

$$F^+(p) = \widehat{\sigma_{23}^+}, \qquad F^-(p) = \widehat{u_{3,1}^-} \quad \text{(recall that } u_{3,1}^+ \equiv 0), \tag{2.5}$$
$$g(p) = -\widehat{\sigma_{23}^-} = -\hat{f}, \qquad \mathscr{G}(p) = v\hat{\mu}(-vp)\gamma(p).$$

We remark that it is reasonable to assume a priori (and may be verified a posteriori) that $\sigma_{23}^+(x, 0)$ and $u_{3,1}^-(x, 0)$ are such that $F^+(x) = \widehat{\sigma_{23}^+}$ and $F^-(x) = \widehat{u_{3,1}^-}$ define functions analytic for $\text{Im}(z) \gtrless 0$ respectively and satisfy $\lim_{\text{Im}(z)\to\infty} F^{\pm}(z) = 0$. Moreover, the boundary limits

$$F^+(p) = \lim_{q\to 0^+} F^+(p + iq), \qquad F^-(p) = \lim_{q\to 0^-} F^-(p + iq)$$

both exist and satisfy (2.4).

The remainder of this section is devoted to the solution of (2.4). The reader who is unfamiliar with the theory of complex boundary-value problems of the Riemann-Hilbert type may wish to consult one of the many treatises on the subject in the literature, such as that by Gakov [4].

A formal solution to (2.4) may be readily constructed once the mapping properties of the coefficient $\mathscr{G}(p)$ have been determined. To this end it is convenient to rewrite $\mathscr{G}(p)$ as

$$\begin{aligned}\mathscr{G}(p) &= v\hat{\mu}(-vp)\gamma(p) = -i\ \text{sgn}(p)\bar{\mu}(-vp)\gamma_1(p)\\ &\equiv \text{sgn}(p)\mathscr{G}_1(p)\end{aligned} \tag{2.6}$$

where

$$\bar{\mu}(-vp) \equiv ipv\hat{\mu}(-vp) = \mu(0) + \int_0^\infty e^{-ivpt}\, d\mu(t) \tag{2.7}$$

and

$$\gamma_1(p) = [1 - \rho v^2/\bar{\mu}(-pv)]^{1/2}. \tag{2.8}$$

It is appropriate to specify now the hypotheses on the shear modulus, $\mu(t)$. Henceforth, it will be assumed that $\mu(t)$ is positive, continuously differentiable, non-increasing, convex and such that

$$\mu(\infty) \equiv \lim_{t\to\infty} \mu(t) > 0.$$

The limiting case $\mu(\infty) = 0$ will follow in an obvious way when the final results are presented.

It is easy to see that these hypotheses on $\mu(t)$ are sufficient (but certainly not necessary) to prove that

(i) $\bar{\mu}(0) = \mu(\infty) \leq \text{Re}(\bar{\mu}(-vp)) \leq \mu(0) = \bar{\mu}(\infty)$;

(ii) $\text{Im}(\bar{\mu}(-vp)) = -\text{Im}(\bar{\mu}(vp))$;

(iii) $\arg(\bar{\mu}(-vp))\begin{cases} \geq 0, & p > 0 \\ \leq 0, & p < 0 \end{cases}.$

From (i), (ii) and (iii) it follows that

(iv) $\text{Im}(\gamma_1^2(-p)) = -\text{Im}(\gamma_1^2(p))$, $\text{Re}(\gamma_1^2(-p)) = \text{Re}(\gamma_1^2(p))$,

(v) $\text{Im}(\gamma_1^2(p)) > 0$, $0 < p < \infty$,

(vi) $1 - (v/c^*)^2 = \gamma_1^2(0) \leq \text{Re}(\gamma_1^2(p)) \leq \gamma_1^2(\infty) = 1 - (v/c)^2$, where $c^* = \mu(\infty)/\rho$ and $c = \mu(0)/\rho$ are the elastic shear wave speeds corresponding to the value of $\mu(t)$ for infinite and zero time respectively. To take the square root of $\gamma_1^2(p)$, it is necessary to distinguish two cases: (1) $0 < v < c^*$, (2) $c^* < v < c$.

Since $\gamma_1(p)$ is required to have positive real part for all real p, the branch cut for the square root of $\gamma_1^2(p)$ is taken to be the negative axis. Hence for case (1), $\gamma_1(p)$ is Hölder-continuous for all real p and

(vii) $\mathrm{Im}(\gamma_1(p)) = -\mathrm{Im}(\gamma_1(-p))$, $\mathrm{Re}(\gamma_1(p)) = \mathrm{Re}(\gamma_1(-p))$,
(viii) $\mathrm{Im}(\gamma_1(p)) > 0,\ 0 < p < \infty$,
(ix) $(1 - (v/c^*)^2)^{1/2} = \gamma_1(0) \leq \gamma_1(\infty) = (1 - (v/c)^2)^{1/2}$,

whereas, for case (2), properties (vii) and (viii) still hold, but $\gamma_1(p)$ is now discontinuous for $p = 0$. In particular,

(x) $\gamma_1(\pm\infty) = (1 - (v/c)^2)^{1/2}$, $\gamma_1(0\pm) = \pm i((v/c^*)^2 - 1)^{1/2}$.

The image in the complex plane of the real p-axis under the transformation $\gamma_1(p)$ is illustrated in Fig. 1 for both cases (1) and (2).

It is clear from properties (i)–(x) for $\bar{\mu}(-vp)$ and $\gamma_1(p)$ that the image in the complex plane of the real p-axis under the transformation $\mathscr{G}_1(p)$ is as depicted in Fig. 2. We remark that in both cases (1) and (2), $\mathscr{G}_1(p)$ is Hölder-continuous for all p (except $p = 0$ for case (2)) on the extended real line, that is, equating the points $p = \pm\infty$. Consequently, $\log(\mathscr{G}_1(p))$ can be defined to be single-valued and Hölder-continuous for all p, including $p = \pm\infty$ (except as noted above).

The solution of (2.4) is now straightforward. We consider first the homogeneous problem of finding functions $X^\pm(z)$ analytic for $\mathrm{Im}(z) \gtrless 0$, respectively, and which satisfy the homogeneous boundary relation

$$X^+(p) = \mathscr{G}(p)X^-(p). \tag{2.10}$$

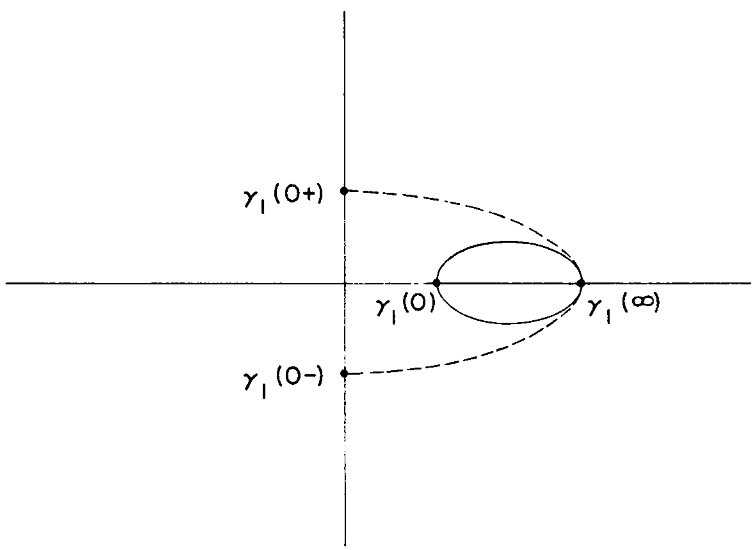

FIG. 1. Image in $\mathbb{C}$ of $\gamma_1(p)$, p real, for case (1) (———) and case (2) (------------).

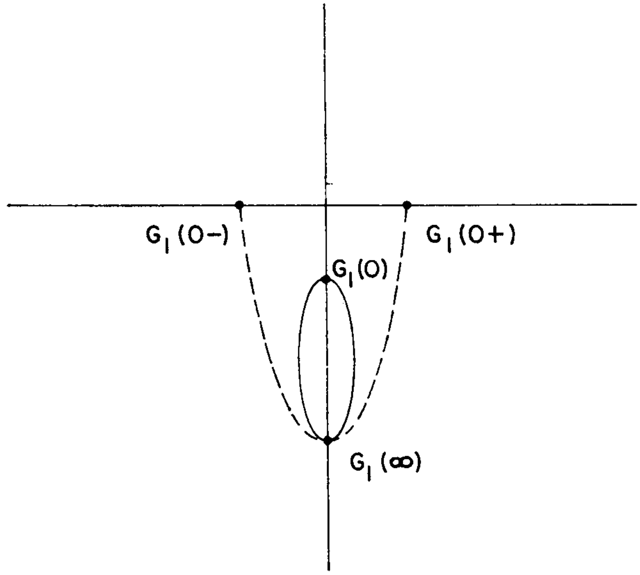

FIG. 2. Image in $\mathbb{C}$ of $\mathscr{G}_1(p)$, p, real, for case (1) (————) and case (2) (-----------).

Auxiliary functions $X_1^{\pm}(z)$ are defined by

$$X^{\pm}(z) = \omega^{\pm}(z)X_1^{\pm}(z)$$

where $\omega^{\pm}(z)$ denotes branches of $z^{1/2}$ whose branch cuts are the negative and positive imaginary axes, respectively. Then $\omega^{+}(z)$ is analytic for $\mathrm{Im}(z) > 0$, $\omega^{-}(z)$ is analytic for $\mathrm{Im}(z) < 0$ and for real p,

$$\omega^{+}(p)/\omega^{-}(p) = \mathrm{sgn}(p).$$

The functions $X_1^{\pm}(z)$ satisfy the boundary relation

$$X_1^{+}(p) = \mathscr{G}_1(p)X_1^{-}(p). \tag{2.11}$$

From the above discussion of $\mathscr{G}_1(p)$, it is clear that the canonical solution (see Gakov [4] for an explanation of the terminology) to (2.11) that is bounded at infinity is given by

$$X_1^{\pm}(z) = \exp(\Gamma_1^{\pm}(z)),$$

$$\Gamma_1^{\pm}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(\mathscr{G}_1(\tau))}{\tau - z}\, d\tau. \tag{2.12}$$

The integral in (2.12) is to be interpreted as a principal value for $\tau \to \pm\infty$. Application of the Plemelj formulas to (2.12) yields

$$\Gamma_1^{\pm}(p) = \pm\tfrac{1}{2}\log(\mathscr{G}_1(p)) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(\mathscr{G}_1(\tau))}{\tau - p}\, d\tau,$$

from which it follows that

$$X_1^{+}(\pm\infty) = e^{-\pi i/4}\,|\mathscr{G}_1(\infty)|^{1/2}, \qquad X_1^{-}(\pm\infty) = e^{\pi i/4}\,|\mathscr{G}_1(\infty)|^{-1/2}.$$

We remark that for case (1), $X_1^{\pm}(p)$ are both continuous on the extended real p-axis, whereas, for case (2), $X_1^{\pm}(p)$ have a discontinuity for $p = 0$. In particular, $X_1^{\pm}(p)$ is easily shown to have the form

$$X_1^{\pm}(p) = \exp(\pm\tfrac{1}{2}\log(\mathscr{G}_1(p)))\,|p|^{-1/2}X_2(p) \tag{2.13}$$

where $X_2(p)$ is continuous for $p \to 0$. Combining (2.10), (2.13) and the above observations, we may deduce that

$$X^{\pm}(p) = O(|p|^{1/2}), \qquad |p| \to \infty, \qquad \text{cases (1) and (2)},$$
$$X^{\pm}(p) = O(|p|^{1/2}), \qquad |p| \to 0, \qquad \text{case (1)},$$
$$\lim_{p\to 0} X^{\pm}(p) \neq 0, \qquad \text{case (2)}.$$

In view of these considerations, it is evident that the unique solution $F^{\pm}(z)$ to the Riemann-Hilbert problem (2.4) for both cases (1) and (2) which vanishes as $|z| \to \infty$ is given by

$$F^{\pm}(z) = X^{\pm}(z)\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g(\tau)/X^{+}(\tau)}{\tau - z}\,d\tau. \tag{2.14}$$

This section concludes with a derivation of the stress intensity factor. It will prove to be convenient to introduce notation for the inverse Fourier transform. The symbol $[F]^{\vee}$ denotes the transform

$$[F]^{\vee}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixp}F(p)\,dp.$$

From (2.5), and (2.14) and the Plemelj formulas it follows that

$$\widehat{\sigma_{23}^{+}}(p) = F^{+}(p) = \tfrac{1}{2}g(p) + X^{+}(p)\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g(\tau)/X^{+}(\tau)}{\tau - p}\,d\tau. \tag{2.15}$$

Fourier inverse of (2.15) and the fact that $g(p) = -\widehat{\sigma_{23}^{-}}$ show that for $x > 0$

$$\sigma_{23}^{+}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixp}X^{+}(p)\,dp\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g(\tau)/X^{+}(\tau)}{\tau - p}\,d\tau. \tag{2.16}$$

From the identity $1/(\tau - p) = -(1/p) + (\tau/p)/(\tau - p)$ and the assumption $\tau g(\tau) = O(1)$, $|\tau| \to \infty$ (this is merely a smoothness restriction on the applied tractions σ_{23}^{-}), it is clear that

$$\int_{-\infty}^{\infty} \frac{g(\tau)/X^{+}(\tau)}{\tau - p}\,d\tau = -\frac{1}{p}\int_{-\infty}^{\infty} g(\tau)/X^{+}(\tau)\,d\tau + G(p) \tag{2.17}$$

where $G(p) = O(|p|^{-3/2})$, $|p| \to \infty$. (In fact, $X^{+}(p)G(p) \in L^1(R)$.) Consequently, from (2.16), (2.17) it follows that for x near zero

$$\sigma_{23}^{+}(x) \sim -\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)/X^{+}(\tau)\,d\tau\,\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{X^{+}(p)}{p}\,e^{-ixp}\,dp. \tag{2.18}$$

Noting that

$$X^{+}(p) = \omega^{+}(p)X_1^{+}(p)$$
$$\sim |p|^{1/2}|\mathscr{G}_1(\infty)|^{1/2}e^{-i\pi/4}\begin{cases} 1, & p \to +\infty \\ i, & p \to -\infty \end{cases}$$

we may conclude that

$$-X^+(p) \sim |p|^{-1/2}\begin{cases} K^+ & p \to +\infty \\ K^- & p \to -\infty \end{cases} \tag{2.19}$$

where

$$K^+ = -|\mathscr{G}_1(\infty)|^{1/2} e^{-i\pi/4} \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(\tau)/X^+(\tau)\, d\tau, \qquad K^- = -iK^+.$$

From (2.18), (2.19) and the Abelian theorems for the Fourier transform, it follows that the dominant term in the asymptotic expansion of $\sigma_{23}^+(x)$ for x near zero is given by

$$\begin{aligned} \sigma_{23}^+(x) &\sim \frac{1}{2\pi}\left[K^+\int_0^{\infty} e^{-ipx} p^{-1/2}\, dp + K^- \int_{-\infty}^{0} e^{-ipx}|p|^{-1/2}\, dp\right] \\ &= x^{-1/2} K^+ \frac{1}{2\pi}\left[\left[\int_0^{\infty} e^{-ip} p^{-1/2}\, dp - i\int_0^{\infty} e^{ip} p^{-1/2}\, dp\right]\right. \\ &= x^{-1/2} \frac{|\mathscr{G}_1(\infty)|^{1/2}}{\sqrt{\pi}} \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tau)/X^+(\tau)\, d\tau. \end{aligned} \tag{2.20}$$

The expression for the stress intensity factor appearing in (2.20) is valid for both cases (1) and (2). In the next section, a simplified form for the stress intensity factor is given from which the fundamental difference in the stress field between cases (1) and (2) is illuminated. Moreover, an elementary expression is presented for the entire stress distribution in front of and in the plane of the advancing crack.

3. Stress analysis. In this section the stress intensity factor and entire in-plane stress field are determined for both cases (1) and (2). From (2.20) we have $\sigma_{23}^+(x) = Kx^{-1/2}$ where

$$\begin{aligned} K &= \frac{|\mathscr{G}_1(\infty)|^{1/2}}{\sqrt{\pi}} \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tau)/X^+(\tau)\, d\tau \\ &= |\mathscr{G}_1(\infty)|^{1/2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\widehat{-\sigma_{23}^-})(\check{h(x)})(\tau)\, d\tau \\ &= -|\mathscr{G}_1(\infty)|^{1/2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sigma_{23}^-(x) h(x)\, dx \end{aligned}$$

and where $h(x)$ is defined by

$$1/X^+(\tau) = \int_{-\infty}^{\infty} e^{-i\tau x} h(x)\, dx. \tag{3.1}$$

The program is to determine $X^+(z)$ for $\mathrm{Im}(z) > 0$, then compute $X^+(\tau)$ for real τ and finally invert (3.1). It is in the calculation of $X^+(z)$ that the two cases must be considered separately.

Case (1): From (2.6) and the stated hypotheses on $\mu(t)$, it is clear that $\mathscr{G}_1(p)$ has a natural extension $\mathscr{G}_1(z)$, $z = p + iq$ with $q < 0$. Moreover, for all $q < 0$,

(i) $\mathscr{G}_1(\pm\infty + iq) = \mathscr{G}_1(\infty)$,

(ii) $i\mathscr{G}_1(iq_1) < i\mathscr{G}_1(iq_2)$, $q_1 < q_2$,
(iii) $\lim_{q\to-\infty} \mathscr{G}_1(iq) = \mathscr{G}_1(\infty)$.

Fig. 3 depicts the mapping properties of $\mathscr{G}_1(z)$ for $\text{Im}(z) < 0$. We recall that $X_1^+(z) = \exp(\Gamma_1^+(z))$ and define $\mathscr{G}_2(z)$ by

$$\mathscr{G}_2(z) = \mathscr{G}_1(z)/\mathscr{G}_1(\infty). \tag{3.2}$$

Then for $q < 0$, $\lim_{|p|\to\infty} \mathscr{G}_2(p + iq) = 1$ and

$$\Gamma_1^+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(\mathscr{G}_1(\tau))}{\tau - z}\, d\tau$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(\mathscr{G}_2(\tau))}{\tau - z}\, d\tau = \tfrac{1}{2}\log(\mathscr{G}_1(\infty)).$$

Since $\log(\mathscr{G}_2(z))$ is analytic for $\text{Im}(z) < 0$ and $\log(\mathscr{G}_2(\infty)) = 0$, it follows by Cauchy's theorem that $\Gamma_1^+(z) = \frac{1}{2}\log(\mathscr{G}_1(\infty))$ and

$$X_1^+(z) = \exp(\Gamma_1^+(z)) = \exp(\tfrac{1}{2}\log(\mathscr{G}_1(\infty)))$$
$$= |\mathscr{G}_1(\infty)|^{1/2} e^{-i\pi/4}. \tag{3.3}$$

Taking the limit $z \to \tau$, $\text{Im}(z) > 0$ in (3.3) we conclude that

$$X_1^+(\tau) = |\mathscr{G}_1(\infty)|^{1/2} e^{-i\pi/4}. \tag{3.4}$$

Case (2): It will prove to be convenient to compute $\Gamma_1^-(\tau)$ and then appeal to the Plemelj formula

$$\Gamma_1^+(\tau) = \Gamma_1^-(\tau) + \log(\mathscr{G}_1(\tau)). \tag{3.5}$$

For this case we observe that $\mathscr{G}_1(p)$ has a natural extension $\mathscr{G}_1(z)$, $z = p + iq$, that is analytic for $q < 0$ except for a branch cut along the negative q-axis from $q = 0$ to $q = -q_0$ for some $q_0 > 0$. Moreover,

(iv) for $0 < q < q_0$, $\lim_{\tau\to0^+} \mathscr{G}_1(\tau - iq)$ is a decreasing real valued function of q with $\lim_{\tau\to0^+} \mathscr{G}_1(\tau - iq_0) = 0$;

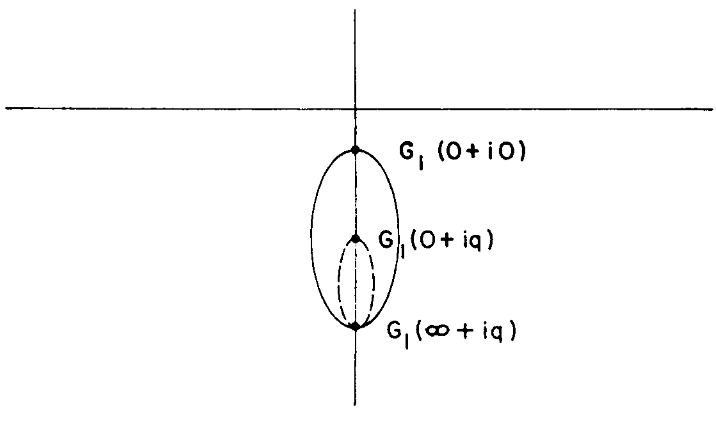

FIG. 3. Image in $\mathbb{C}$ of $\mathscr{G}_1(p + iq)$ for case (1) with $q = 0$ (————) and $q < 0$ (------------).

(v) for $0 < q < q_0$, $\lim_{\tau \to 0^-} \mathscr{G}_1(\tau - iq) = -\lim_{\tau \to 0^+} \mathscr{G}_1(\tau - iq)$;

(iv) for $q < -q_0$, $\mathscr{G}_1(z)$ is analytic and satisfies properties (i)–(iii) listed above for case (1).

The mapping properties of $\Gamma_1(z)$ for case (2) are depicted in Fig. 4.

Since we require $\lim_{q \to 0^-} \Gamma_1^-(p + iq)$, we consider $\Gamma_1^-(p + iq)$ for $0 > q > -q_0$. In addition, for convenience we assume $0 < p$. The case $p < 0$ follows similarly. $\Gamma_1^-(z)$ is defined by integration along the real axis. By properties (iv)–(vi) and Cauchy's theorem, that integral may be replaced by integrals along the segment on the negative imaginary axis from $q = 0$ to $q = -q_0$ and along the horizontal line $q = -q_0$ below which $\log(\mathscr{G}_1(p + iq))$ is analytic. Specifically, we have for $z = p + iq$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \log(\mathscr{G}_1(\tau)) \frac{d\tau}{\tau - z}$$

$$= \frac{1}{2\pi i} \left[\int_{-\infty}^{(0-)} + \int_{(0-)+i0}^{(0-)-iq_0} + \int_{(0-)-iq_0}^{-\infty - iq_0} \log(\mathscr{G}_1(\tau)) \frac{d}{\tau - z} \right] \tag{3.6}$$

$$+ \frac{1}{2\pi i} \left[\int_{(0-)-iq_0}^{(0-)+i0} + \int_{(0+)+i0}^{(0+)-iq_0} \log(\mathscr{G}_1(\tau)) \frac{d\tau}{\tau - z} \right] \tag{3.7}$$

$$+ \frac{1}{2\pi i} \left[\int_{-\infty - iq_0}^{(0+)-iq_0} + \int_{(0+)-iq_0}^{(0+)+i0} + \int_{0}^{\infty} \log(\mathscr{G}_1(\tau)) \frac{d\tau}{\tau - z} \right] \tag{3.8}$$

$$+ \frac{1}{2\pi i} \int_{-\infty - iq_0}^{\infty - iq_0} \log(\mathscr{G}_1(\tau)) \frac{d\tau}{\tau - z}. \tag{3.9}$$

Line (3.6) equals zero since $p > 0$. Line (3.8) equals $-\log(\mathscr{G}_1(z))$ since $p > 0$ and $0 > q > -q_0$. Line (3.9) equals $\frac{1}{2}\log(\mathscr{G}_1(\infty))$, as in case (1). To compute (3.7) we note that from property (v) it follows that for $0 < q < q_0$, $\log(\mathscr{G}_1((0+) - iq)) - \log(\mathscr{G}_1((0-) - iq)) = \pi i$. Consequently, line (3.7) equals

$$\frac{1}{2} \int_{i0}^{-iq_0} \frac{d\tau}{\tau - z} = \tfrac{1}{2}[\log(-iq_0 - z) - \log(-z)]. \tag{3.10}$$

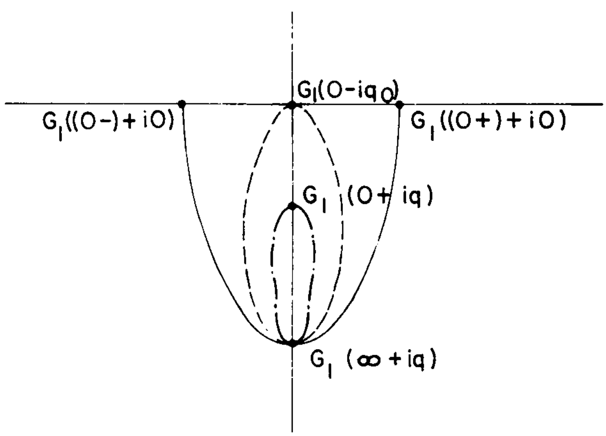

FIG. 4. Image in $\mathbb{C}$ of $\mathscr{G}_1(p + iq)$ for case (2) with $q = 0$ (————), $q = +q_0$ (---------) and $q < -q_0$ (—— · ——).

Combining (3.5)–(3.10) and letting $q \to 0-$ we obtain

$$\Gamma_1^+(\tau) = \tfrac{1}{2}\log(\mathscr{G}_1(\infty)) + \tfrac{1}{2}\log(-iq_0 - \tau) - \tfrac{1}{2}\log|\tau| + \begin{Bmatrix} \pi i/2, & \tau > 0 \\ 0, & \tau < 0 \end{Bmatrix}. \tag{3.11}$$

From (3.11) it follows that for case (2)

$$\begin{aligned} X_1^+(\tau) &= |\mathscr{G}_1(\infty)|^{1/2} e^{-i\pi/4} \exp(\tfrac{1}{2}\log(-iq_0 - \tau))|\tau|^{-1/2}\begin{Bmatrix} i, & \tau > 0 \\ 1, & \tau < 0 \end{Bmatrix} \\ &= i|\mathscr{G}_1(\infty)|^{1/2} e^{-i\pi/4} \exp(\tfrac{1}{2}\log(-iq_0 - \tau))/\omega^+(\tau). \end{aligned} \tag{3.12}$$

Finally, combining the fact that $X^+(\tau) = \omega^+(\tau)X_1^+(\tau)$ with (3.3) and (3.12), we arrive at

$$\begin{aligned} X^+(\tau) &= \omega^+(\tau)|\mathscr{G}_1(\infty)|^{1/2} e^{-i\pi/4}, \qquad \text{case (1)} \\ &= i|\mathscr{G}_1(\infty)|^{1/2} e^{-i\pi/4} \exp(\tfrac{1}{2}\log(-iq_0 - \tau)), \qquad \text{case (2)}. \end{aligned} \tag{3.13}$$

The computation of $h(x)$ is completed by recalling the well-known integrals

$$\int_{-\infty}^{0} e^{-ix\tau}|x|^{-1/2}\,dx = \Gamma(1/2)e^{+i\pi/4}/\omega^+(\tau), \tag{3.14}$$

$$\int_{-\infty}^{0} e^{-ix\tau}e^{qx}|x|^{-1/2}\,dx = \Gamma(1/2)e^{-i\pi/4}(-iq - \tau)^{-1/2}, \qquad q > 0. \tag{3.15}$$

Lines (3.13)–(3.15) together with (3.1) yield

$$h(x) = |\mathscr{G}_1(\infty)|^{-1/2}\frac{H(-x)}{\sqrt{\pi}}|x|^{-1/2}\begin{Bmatrix} 1, & \text{case (1)} \\ e^{xq_0}, & \text{case (2)} \end{Bmatrix} \tag{3.16}$$

where $H(x)$ denotes the Heaviside step function. The desired form for the stress intensity factor follows directly from (3.16) as

$$\begin{aligned} K &= -\frac{1}{\pi}\int_{-\infty}^{0} \sigma_{23}^-(x)|x|^{-1/2}\,dx, \qquad \text{case (1)} \\ &= -\frac{1}{\pi}\int_{-\infty}^{0} \sigma_{23}^-(x)|x|^{-1/2}e^{xq_0}\,dx, \qquad \text{case (2)}. \end{aligned} \tag{3.17}$$

It should be noted in (3.17) that even for general viscoelastic material, when dynamic effects occur for case (2), the stress intensity factor is modified only by a simple exponential damping factor, as was observed by Willis for the standard linear solid. Since the dependence of K on material properties and the speed of propagation occurs only through q_0, it is worthwhile to investigate the relationship among v, $\mu(t)$ and q_0. It is easy to see that q_0 is the unique value of q for which $\mathscr{G}_1(-iq) = 0$ which, from (2.6), must also be the unique value of q for which $\gamma_1(-iq) = 0$. It will prove to be convenient to define a nondimensional shear modulus $\mu^*(t)$ by

$$\mu^*(t) = \mu(t)/\mu(0).$$

From (2.8), the requirement $\gamma_1(-iq_0) = 0$ is seen to be equivalent to

$$vq_0\int_0^{\infty} \mu^*(t)e^{-q_0vt}\,dt = (v/c)^2. \tag{3.18}$$

Eq. (3.18) is easily solved numerically for q_0.

The problem of determining $\sigma_{23}^{+}(x, 0)$ for all $x > 0$ will now be addressed. Use will be made of the following well-known identity relating the Fourier and Hilbert transforms:

$$H(f) = [i\ \mathrm{sgn}(\tau) f^{\vee}]^{\wedge} \tag{3.19}$$

where

$$H(f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)\, dt}{t - \tau}. \tag{3.20}$$

Recall that

$$g(\tau)/X^{+}(\tau) = -2\pi \widehat{\sigma_{23}^{-}} \widecheck{(h(x))} = -\widehat{\sigma_{23}^{-}}[h(-x)]^{\vee}(\tau). \tag{3.21}$$

From (2.16) we have

$$\sigma_{23}^{+}(x) = [L(p)]^{\vee} = -\frac{d}{dx}[L(p)/ip]^{\vee} \tag{3.22}$$

where

$$L(p) = X^{+}(p) \frac{1}{2i} H[g(\tau)/X^{+}(\tau)]. \tag{3.23}$$

Lines (3.19), (3.21) and (3.23) together yield

$$L(p) = -\tfrac{1}{2} X^{+}(p) [\mathrm{sgn}(\tau)(\sigma_{23}^{-}(x) * h(-x))(\tau)\hat{]}. \tag{3.24}$$

Observe further that

$$X^{+}(p)/ip = -\frac{|\mathscr{G}_1(\infty)|^{1/2}}{\Gamma(1/2)} \widehat{l^{+}(x)} \tag{3.25}$$

and

$$1/X^{+}(p) = \frac{|\mathscr{G}_1(\infty)|^{-1/2}}{\Gamma(1/2)} \widehat{l^{+}(x)} = \widehat{h(-x)} \tag{3.26}$$

where

$$l^{+}(x) = H(z)|x|^{-1/2}.$$

Defining $k(x)$ by

$$k(x) = \sigma_{23}^{-}(x) * l^{+}(x)$$

and making use of (3.22), (3.24)–(3.26) we arrive at

$$\sigma_{23}^{+}(x) = -\frac{1}{2\pi} \frac{d}{dx} [l^{+}(x) * (k(x)\mathrm{sgn}(x))] \tag{3.27}$$

as the desired form for $\sigma_{23}^{+}(x)$.

If $\sigma_{23}^{-}(x)$ is smooth for $x \le 0$, then $k(x)$ is smooth for all real x and

$$k'(x) = l^{+} * \sigma_{23}^{-\prime}(x).$$

Moreover, we may integrate by parts the convolution in (3.27) and perform the differentiation to simplify (3.27) further to

$$\sigma_{23}^{+}(x) = -x^{-1/2}\frac{1}{\pi}\int_{-\infty}^{0}\sigma_{23}^{-}(x)|x|^{-1/2}\,dx - \int_{-\infty}^{x}(x-t)^{-1/2}k'(t)\mathrm{sgn}(t)\,dt. \qquad (3.28)$$

Line (3.28) exhibits again the stress intensity factor for case (1).

For case (2) a slightly different calculation is required. Since

$$X^{+}(p) = i|\mathscr{G}_1(\infty)|^{1/2}e^{-i\pi/4}(-iq_0 - p)^{1/2},$$

we write

$$\sigma_{23}^{+}(x) = [L(p)]^{\vee} = -i[(q_0 - ip)L(p)/(-iq_0 - p]^{\vee} \qquad (3.29)$$

where $L(p)$ is given by (3.23). Corresponding to (3.25) and (3.26) there are

$$\begin{aligned} 1/X^{+}(p) &= -i|\mathscr{G}_1(\infty)|^{-1/2}e^{i\pi/4}(-iq_0 - p)^{-1/2} \\ &= \frac{|\mathscr{G}_1(\infty)|^{-1/2}}{\Gamma(1/2)}\,l^{+}(x), \end{aligned} \qquad (3.30)$$

$$X^{+}(p)/(-iq_0 - p) = \frac{i|\mathscr{G}_1(\infty)|^{1/2}}{\Gamma(1/2)}\,l^{+}(x). \qquad (3.31)$$

where

$$l^{+}(x) = H(x)\exp(-q_0 x)x^{-1/2}). \qquad (3.32)$$

From (3.29) it is easily seen that

$$\sigma_{23}^{+}(x) = G'(x) + q_0 G(x) \qquad (3.33)$$

in which

$$G(x) = -i[L(p)/(-iq_0 - p)]^{\vee}. \qquad (3.34)$$

Combining (3.30)–(3.33) we obtain

$$G(x) = -\frac{1}{2\pi}\,l^{+}(x) * (k(x)\mathrm{sgn}(x)) \qquad (3.35)$$

where

$$k(x) = \sigma_{23}(x) * l^{+}(x).$$

Lines (3.32), (3.33) and (3.35) together provide the desired result for case (2) analogous to (3.27) for case (1). The analogue to (3.28) for case (2) is easy to derive but somewhat messy and will be omitted.

We remark that for the entire stress distribution ahead of the advancing crack, not merely for the stress intensity factor, the dynamic effect occurring for case (2) is solely through the exponential damping factor $\exp(-q_0 x)$, where q_0 is a function of v and $\mu(t)$ through (3.18). It should also be noted that

$$\lim_{v \to c*+} q_0(v, \mu) = 0$$

and

$$\lim_{v \to c-} q_0(v, \mu) = \infty.$$

Hence as $v \to c^*+$, the stress distribution converges continuously to the corresponding elastic distribution, while as $v \to c-$, the stress intensity factor tends to zero.

In principle it is easy to construct an expression for the displacement of the crack faces. It is obtained by convolving $\sigma_{23}(x)$ with the creep compliance and then convolving that result with the inverse Fourier transform of $\gamma_1(p)$. However, there obviously is no simple form for this multiple convolution which depends in a fundamental and complicated way upon $\mu(t)$ and v. Consequently, there will be no attempt here to analyze $u_3^-(x, 0)$ further.

We remark also that the techniques employed in the analysis presented in Secs. 2 and 3 may be applied to many other dynamic fracture problems for general viscoelastic material. In particular, the transient problem for a layer considered by Atkinson and Popelar is amenable to such an analysis. Of perhaps a greater interest is the opening mode problem, both transient and steady-state, for general material including those with non-constant Poisson's ratio. An analysis of this latter problem is currently being completed and will be the subject of a forthcoming paper.

It should be emphasized that the method employed here will, in general, produce only formal solutions to dynamic fracture problems more complicated than the one considered here. The factor $X^+(p)$ will be expressed in terms of Fourier transforms and singular integrals. For many problems, effecting a simplification of the formal solution to a tractable form may prove to be quite difficult.

The next section contains an example which illustrates the results of Sec. 3. In particular, the consideration of a sample loading illuminates the characteristics of the shear modulus $\mu(t)$ that determine the magnitude of the dynamic effects appearing in case (2).

4. An example. Some insight into the properties of the shear modulus $\mu(t)$ that affect the magnitude of the dynamic effect for case (2) can be gained from the consideration of the special case of a shear crack driven by tractions of constant magnitude on a finite interval traveling with the crack tip. Specifically, we assume that

$$\sigma_{23}^-(x, 0) = -P(H(-x) - H(-a - x)) \tag{4.1}$$

where a and P are positive constants. To study the viscoelastic effect of case (2), it is appropriate to define a nondimensional stress intensity factor k by $k = K_2/K_1$. Here K_1 and K_2 are the stress intensity factors for cases (1) and (2), respectively. From (3.17), (4.1) and an obvious change of variables, we have

$$k = \int_0^1 \exp(-q_0 a x^2)\, dx.$$

The magnitude of the viscoelastic effect is determined by the nondimensional parameter $q_0 a$, where q_0 is the unique solution to (3.18). Recall that in (3.18), $\mu^*(t)$ is the nondimensional shear modulus defined by $\mu^*(t) = \mu(t)/\mu(0)$. Hence, $\mu^*(t)$ is positive, non-increasing and convex. It will prove to be useful to introduce two parameters, β and δ, by

$$\beta = \lim_{t \to \infty} \mu^*(t), \qquad -\delta = \frac{d}{dt}\mu^*(0).$$

Eq. (3.18) may be rewritten as

$$\int_0^\infty \mu^*(t/vq_0)e^{-t}\,dt = s^2 \tag{4.2}$$

with $s = v/c$. The non-dimensional crack velocity satisfies $\beta < s^2 < 1$, and for $s^2 = 1$, $q_0 = \infty$ while for $s^2 = \beta$, $q_0 = 0$.

For given s and $\mu(t)$, Eq. (4.2) is easily solved numerically for vq_0. However, some general observations can be made about the solution to (4.2) when s^2 is near β or 1, that is, for crack speeds near c^* and c. Consider the function $\phi(\lambda)$ given by

$$\phi(\lambda) = \int_0^\infty \mu^*(t\lambda)e^{-t}\,dt.$$

For s^2 near 1, an estimate of the solution to $\phi(\lambda) = s^2$ may be obtained from the asymptotic expansion of $\phi(\lambda)$ for λ near zero, whereas for s^2 near β, what is required is an asymptotic expansion for $\lambda \to \infty$.

As $\lambda \to 0$, the desired expansion follows from the fact that $\phi'(0) = (d/dt)\mu^*(0) = -\delta$. The asymptotic expansion for $\lambda \to \infty$ will be considered below.

To illustrate the dependence of q_0 on β and δ, we define $\mu_2(t)$ by

$$\mu_2(t) = (\mu^*(t(1-\beta)/\delta) - \beta)/(1-\beta).$$

Then $\mu_2(0) = 1$, $\mu_2(\infty) = 0$, $\mu_2'(0) = -1$ and $\mu_2(t) = O(\mu^*(t))$, $t \to \infty$. Eq. (4.2) becomes

$$\int_0^\infty \mu_2(t\alpha/s(1-\beta))e^{-t}\,dt = (s^2 - \beta)/(1-\beta), \tag{4.3}$$

with α defined by

$$aq_0 = \tau\delta/\alpha \tag{4.4}$$

and where τ is a characteristic time parameter $\tau = a/c$. If $\phi_2(\lambda)$ is now defined by

$$\phi_2(\lambda) = \int_0^\infty \mu_2(t\lambda)e^{-t}\,dt$$

then $\phi_2(0) = 1$, $\phi_2(\infty) = 0$, $\phi_2'(0) = -1$ and

$$\phi(\lambda) \sim \beta + \phi_2(\lambda) \quad \text{as} \quad \lambda \to \infty.$$

Consequently, for $s \to 1$ we have

$$\alpha = s(1-\beta)\phi_2^{-1}((s^2-\beta)/(1-\beta)) \sim s(1-\beta)(1-s^2).$$

We remark that in (4.4), the factor $\tau\delta$ is a nondimensional time parameter which is the product of the minimum time required for a shearing disturbance to travel the length of the interval on which the loading is applied and the instantaneous rate of stress relaxation at zero time corresponding to the nondimensional modulus $\mu^*(t)$.

The asymptotic form for α when $s^2 - \beta \ll 1$ depends on the rate of decay of $\mu_2(t)$ as $t \to \infty$. We distinguish two cases:

(a) $\displaystyle\int_0^\infty \mu_2(t)\,dt < \infty$,

(b) $\displaystyle\int_0^\infty \mu_2(t)\,dt = \infty$.

For case (a), it is clear that

$$\phi_2(\lambda) \sim \frac{1}{\lambda} \int_0^\infty \mu_2(t)\, dt \quad \text{as} \quad \lambda \to \infty.$$

From this it follows that

$$\alpha \sim s(1-\beta)^2 \|\mu_2\|_1/(s^2-\beta) \quad \text{as} \quad s^2 \to \beta, \tag{4.5}$$

where $\|\mu_2\|_1 = \int_0^\infty \mu_2(t)\, dt$. It should be noted that from (4.5) it is apparent that for $s^2 - \beta \ll 1$, the size of α, and hence the size of aq_0, is essentially independent of the rate of decay of $\mu(t)$ as $t \to \infty$, provided that $\|\mu_2\|_1 < \infty$. In particular, the magnitude of the viscoelastic effect in this case should be the same for exponential decay (as with the standard linear solid) and power-law decay with exponent greater than unity. Numerical calculations have shown this to indeed be the case.

For case (b) we observe that while there always exists a constant c_1 such that $c_1\mu_2(\lambda) \leq \phi_2(\lambda)$ as $\lambda \to \infty$, there need not exist a constant c_2 with $\phi_2(\lambda) \leq c_2\,\mu_2(\lambda)$. However, given the additional restriction

$$\frac{1}{N} \int_0^N \mu(t)\, dt = O(\mu(N)) \qquad \text{as} \qquad N \to \infty, \tag{4.6}$$

then it can be shown that

$$\phi_2(\lambda) = O(\mu_2(\lambda)) \qquad \text{as} \qquad \lambda \to \infty. \tag{4.7}$$

Note that (4.6) is not a growth condition (given that $\|\mu_2\|_1 = \infty$). In particular, (4.6) holds for all power-law material with exponent less than or equal to unity. Indeed, logarithmic decay or even $\mu(t) \equiv \mu_0$ is permitted. Rather, (4.6) is a regularity condition which, from a practical point of view, is no real restriction.

The paper concludes with the observation from (4.7) that the slower the rate of decay of $\mu(t)$ as $t \to \infty$, then the smaller the magnitude of the viscoelastic effect for case (2) for given s, β, δ and τ with $s^2 - \beta \ll 1$.

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