ON THE STOCHASTIC APPROXIMATION METHOD OF ROBBINS AND MONRO[1]

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1. Summary. In their interesting and pioneering paper Robbins and Monro [1] give a method for "solving stochastically" the equation in x: $M(x) = \alpha$, where $M(x)$ is the (unknown) expected value at level x of the response to a certain experiment. They raise the question whether their results, which are contained in their Theorems 1 and 2, are valid under a condition (their condition (4′), our condition (1) below) which is statistically plausible and is weaker than the condition which they require to prove their results. In the present paper this question is answered in the affirmative. They also ask whether their conditions (33), (34), and (35) (our conditions (25), (26) and (27) below) can be replaced by their condition (5″) (our condition (28) below). A counterexample shows that this is impossible. However, it is possible to weaken conditions (25), (26) and (27) by replacing them by condition (3) (abc) below. Thus our results generalize those of [1]. The statistical significance of these results is described in [1].

2. Statement of the problem. Let $H(y \mid x)$ be a family of distribution functions which correspond to real values of the parameter x. Write

$$M(x) = \int_{-\infty}^{\infty} y \, dH(y \mid x).$$

We postulate that

$$(1) \qquad | M(x) | \leqq C < \infty, \qquad \int_{-\infty}^{\infty} (y - M(x))^2 \, dH(y \mid x) \leqq \sigma^2 < \infty,$$

and that either

$$(2) \qquad \begin{aligned} M(x) &\leqq \alpha - \delta, \qquad &\text{for } x < \theta, \\ M(x) &\geqq \alpha + \delta, \qquad &\text{for } x > \theta, \end{aligned}$$

for some $\delta > 0$, or else

$$(3a) \qquad \begin{aligned} &M(x) < \alpha \text{ for } x < \theta, \\ &M(\theta) = \alpha, \\ &M(x) > \alpha \text{ for } x > \theta, \end{aligned}$$

and, for some positive δ,

(3b) $M(x)$ is strictly increasing if $| x - \theta | < \delta$

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and

$$\inf_{|x-\theta| \geqq \delta} | M(x) - \alpha | > 0. \tag{3c}$$

Let $\{a_n\}$ be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n = \infty, \tag{4}$$

$$\sum_{n=1}^{\infty} a_n^2 < \infty. \tag{5}$$

Let x_1 be an arbitrary number. The Robbins-Monro convergence scheme is defined recursively for all n by

$$x_{n+1} = x_n + a_n(\alpha - y_n),$$

where y_n is a chance variable with distribution function $H(y \mid x_n)$. We shall prove the following

THEOREM. *If* (1), (4), (5), *and either* (2) *or* (3) (abc) *hold, then* x_n *converges stochastically to* θ.

3. Proof of the Theorem. Let

$$b_n = E(x_n - \theta)^2,$$

$$d_n = E[(x_n - \theta)(M(x_n) - \alpha)],$$

$$e_n = E\left[\int_{-\infty}^{\infty} (y - \alpha)^2 \, dH(y \mid x_n)\right].$$

Then, from (1),

$$0 \leqq e_n \leqq \sigma^2 + (C + |\alpha|)^2 = h \quad \text{(say)}. \tag{6}$$

An examination of the proof of [1] shows that, since (6) is valid, the following results of [1] hold under our assumptions:

$$b_n \to \text{a limit, say } b, \text{ as } n \to \infty; \tag{7}$$

$$d_n \geqq 0; \tag{8}$$

$$\sum_{n=1}^{\infty} a_n d_n < \infty; \tag{9}$$

$$b_{n+1} = b_1 + \sum_1^n a_j^2 e_j - 2 \sum_1^n a_j d_j. \tag{10}$$

From (4) and (9) we obtain

$$\liminf d_n = 0. \tag{11}$$

Let $n_1 < n_2 < \cdots$ be an infinite sequence of positive integers such that

$$\lim d_{n_j} = 0. \tag{12}$$

We assert that x_{n_j} converges stochastically to θ. If this were not so, there would exist an infinite subsequence $t_1 < t_2 < \cdots$ of the sequence of n_j, and positive numbers ϵ and η such that, for all j,

$$P\{| \hat{x}_{t_j} - \theta | > \eta\} > \epsilon. \tag{13}$$

But then for all t_j we would have

$$\begin{aligned} d_{t_j} &= E[(x_{t_j} - \theta)(M(x_{t_j}) - \alpha)] \\ &= E[|x_{t_j} - \theta \mid | M(x_{t_j}) - \alpha |] \\ &\geqq \epsilon\eta \inf_{| x-\theta | \geqq \eta} | M(x) - \alpha|. \end{aligned} \tag{14}$$

From either (2) or (3) (abc) it follows that the last member of (14) is positive. This contradicts (12) and proves the assertion.

Let ϵ and η be arbitrary positive numbers. Our theorem is proved if we can demonstrate the existence of an integer $N(\eta, \epsilon)$ such that, if $n > N(\eta, \epsilon)$,

$$P\{| x_n - \theta |>\eta\} \leqq \epsilon. \tag{15}$$

Let s be a positive number such that

$$\frac{s^2 + s}{\eta^2} < \frac{\epsilon}{2}. \tag{16}$$

Since x_{n_j} converges stochastically to θ there exists an integer N_0 such that

$$P\{| x_{N_0} - \theta | \geqq s\} < \frac{\epsilon}{2}, \tag{17}$$

and

$$\sum_{n=N_0}^{\infty} a_n^2 < \frac{s}{2h}. \tag{18}$$

Define, for $n > N_0$,

$$b'_n(z) = E\{(x_n - \theta)^2 \mid x_{N_0} = z\}.$$

From (6) and (10) we obtain

$$b'_n(z) \leqq (z - \theta)^2 + h \sum_{j=N_0}^{\infty} a_j^2 \leqq (z - \theta)^2 + \frac{s}{2}. \tag{19}$$

Consequently, when $n > N_0$,

$$P\left\{ | x_n - \theta | \geqq \eta \,\middle|\, | x_{N_0} - \theta | < s \right\} < \frac{s^2 + s}{\eta^2} < \frac{\epsilon}{2} \tag{20}$$

by (19) and (16). From (17) and (20) we conclude that (15) holds with $N(\eta, \epsilon) = N_0$. Thus our proof is complete.

4. Two counterexamples.

4.1. We show that (1) ((4′) in [1]) cannot be completely removed. Suppose that $\theta = \alpha = 0$, and define $H(y \mid x)$ for $x < 1$ in any way whatever provided only that

$$M(x) = x, \qquad -1 \leqq x < 1,$$
$$M(x) = -1, \qquad x < -1.$$

For all x such that

$$\sum_{n=1}^{k} \frac{1}{n} \leqq x < \sum_{n=1}^{k+1} \frac{1}{n} \tag{21}$$

define $H(y \mid x)$ as follows:

$$H(y \mid x) = 0, \qquad y \leqq -1, \tag{22}$$

$$H(y \mid x) = 1 - \frac{1}{k^2}, \qquad -1 < y \leqq 2k^2 - 1, \tag{23}$$

$$H(y \mid x) = 1, \qquad y > 2k^2 - 1. \tag{24}$$

We have $M(x) \equiv 1$ for $x \geqq 1$. Thus $M(x)$ satisfies

(25) M(x) is nondecreasing,

$$M(\theta) = \alpha, \tag{26}$$

$$\left.\frac{dM(x)}{dx}\right|_{x=\theta} > 0. \tag{27}$$

(These are the conditions (33), (34), and (35) of [1].)

Now let $a_n = 1/n$ and $x_1 > 2$. We have

$$P\{x_{n+1} > x_n \mid y_1 = y_2 = \cdots = y_n = -1\} \geqq 1 - \frac{1}{n^2}$$

for all $n > 1$, and

$$P\{x_2 > x_1\} \geqq 3/4.$$

Hence there is a positive probability that $x_n \to \infty$.

4.2. We show that the condition

$$\begin{aligned} M(x) &< \alpha, \qquad x < \theta, \\ M(x) &> \alpha, \qquad x > \theta, \end{aligned} \tag{28}$$

(which is condition (5″) of [1]), and the condition (1) (or even the stronger condition (4) of [1], i.e., that there exist a positive constant C' such that

$$\int_{-C'}^{C'} dH(y \mid x) = 1 \tag{29}$$

identically in x) are not sufficient for the theorem to hold. Let $a_n = 1/(4n)$, $\theta = \alpha = 0$. For all x we define

$$H(y \mid x) = 0, \qquad y \leqq M(x), \tag{30}$$

$$H(y \mid x) = 1, \qquad y > M(x). \tag{31}$$

We define $M(0) = 0$, $M(1) = 1$. Let $x_1 = 1$. Then $x_2 = \frac{3}{4}$. If $x_n = y_n$ for $n \geqq 2$ we define $M(y_n) = (4n)^{-1}$. Since $\sum_{n=2}^{\infty} 1/(16n^2) < \frac{1}{2}$, we have

$$P\{x_n > \tfrac{1}{4}\} = 1$$

for all n. We can define $M(x)$ at points x not included in our construction above in any manner compatible with (28).

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REFERENCE

[1] HERBERT ROBBINS AND SUTTON MONRO, "A stochastic approximation method," *Annals of Math. Stat.*, Vol. 22 (1951), pp. 400–407.