On the Stokes conjecture for the wave of extreme form

by

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1. Introduction

1.1. The Stokes conjecture

In this paper we settle a question of the regularity, at one exceptional point, of the free boundary in a problem governed by the Laplace equation and a non-linear boundary condition.

The physical problem concerns gravity waves of permanent form on the free surface of an ideal liquid (that is, of a liquid having constant density, no viscosity and no surface tension). We suppose throughout that the motion is two-dimensional, irrotational and in a vertical plane. Of the various cases to be introduced in section 2, we consider here only the simplest: that of periodic waves on liquid of infinite depth. If we take axes moving with the wave (axes fixed relative to a crest) as in Figure 1, the problem becomes one of steady motion; the fluid domain is

$$\Omega = \{ (x,y): -\infty < x < \infty,\ -\infty < y < Y(x) \},$$

where the free surface $\Gamma=\{(x, Y(x)):\ x \in \mathbf{R}\}$ is unknown a priori, and Y is to have period λ. Moreover, we assume Γ to have a single crest (Y to have a single maximum) per wavelength, and Γ to be symmetrical about that crest. One seeks a stream function Ψ that (a) is harmonic ($\Delta\Psi=0$) in Ω, (b) satisfies $\Psi(x+\lambda, y)=\Psi(x,y)$, (c) is such that the fluid velocity $(\Psi_y, -\Psi_x)\to(c,0)$ as $y\to-\infty$, (d) satisfies the free-surface conditions

$$\Psi = 0 \quad \text{and} \quad \frac{1}{2}|\nabla\Psi|^2+gy = \text{constant on } \Gamma. \tag{1.1}$$

Here the wavelength λ and gravitational acceleration g are given positive constants, and the wave velocity $(-c,0)$, relative to the fluid at infinite depth, is to be found after

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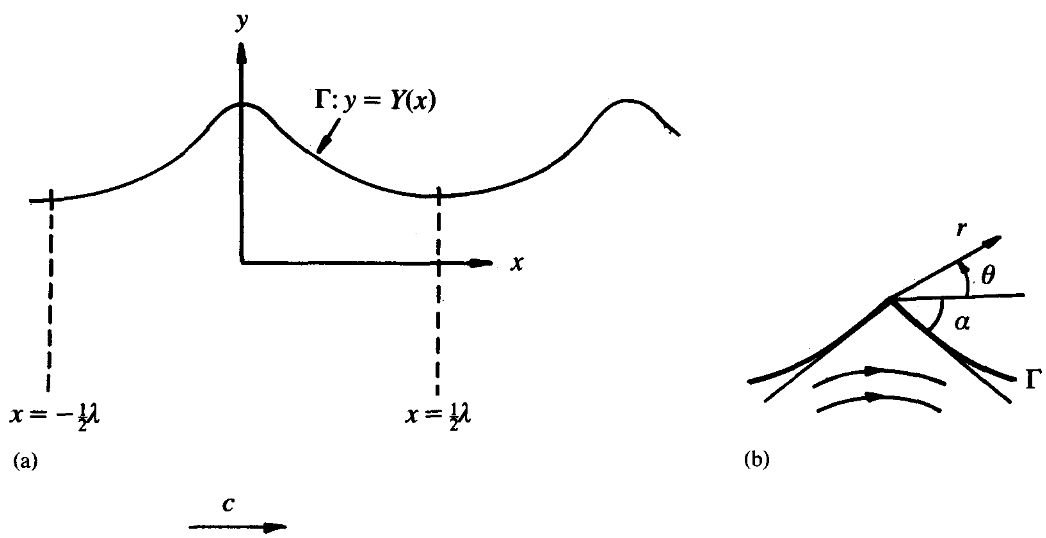

Figure 1. Some notation for (a) periodic waves on water of infinite depth, (b) the extreme wave contemplated by Stokes.

specification of some measure of the height or amplitude of the wave profile. The height H of the wave may be defined as

$$H = \max_{x \in \mathbf{R}} Y(x) - \min_{x \in \mathbf{R}} Y(x).$$

(Certain other definitions are equally legitimate.)

In the manner of his time, Stokes [9] assumed that, for fixed λ and g, there exists a family of solutions parametrized by H; he calculated various formal approximations to them for small values of H/λ. He argued [10] that the *wave of greatest height*, as it has come to be called, is distinguished by sharp crests of included angle $2\pi/3$. (At these crests, the fluid velocity, relative to our axes, must then be zero.) He also conjectured, although with less conviction and with no mathematical argument whatever, that between the sharp crests the wave profile is strictly convex: $Y''(x)>0$. (This summary Stokes's remarks may suggest a greater precision than he intended. Stokes did not define wave height, nor did he explicitly specify fixed wavelength. The words 'greatest height', 'steepest form' and 'limiting form' were all used in his paper to describe the extreme wave. However, there is no ambiguity in his description of the sharp crest and the convexity of the profile.)

Stokes gave essentially the following argument (which has served, since then, as an exercise for generations of students) to show that, if the slope $Y'(x)$ has a simple discontinuity at $x=0$, then the included angle must be $2\pi/3$. Suppose that

$$Y'(x) \to \pm \tan\alpha \quad \text{as} \quad x \uparrow 0 \quad \text{or} \quad x \downarrow 0, \text{ respectively;}$$

let $x+i\{y-Y(0)\} = r\,e^{i\theta}$, and suppose further that

$$\Psi(x,y) = Ar^b \cos b\left(\theta+\frac{\pi}{2}\right)+o(r^b) \quad \text{as } r\to 0,$$

where the constants α, A and b are to be determined, and the gradient of the o-term is assumed to be $o(r^{b-1})$. Then substitution into (1.1) yields $b=3/2$ and $\alpha=\pi/6$.

In the present paper, we shall prove that *the waves of extreme form whose existence has recently been established do indeed have sharp crests of included angle* $2\pi/3$. Before beginning the proof, however, we must relate this question to rigorous results for steady gravity waves of large height.

1.2. Recent analysis of the problem

The first existence proof for waves whose slope need not be small is due to Krasovskii [6], who applied positive-operator methods to a non-linear integral equation, but his analysis is restricted to wave angles strictly less than $\pi/6$; that is, to waves for which $\sup_x |Y'(x)|<1/\sqrt{3}$. Changing Krasovskii's approach slightly but significantly, Keady and Norbury [5] studied the integral equation due to Nekrasov; for periodic waves and infinite depth this is

$$\varphi(s) = \frac{1}{3}\int_0^\pi K(s,t)\frac{\sin\varphi(t)}{\nu+\int_0^t \sin\varphi}\,dt, \quad 0<s\leqslant\pi, \tag{1.2}$$

where

$$K(s,t) = \frac{1}{\pi}\log\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}|s-t|} = \frac{1}{\pi}\log\frac{\tan\frac{1}{2}s+\tan\frac{1}{2}t}{|\tan\frac{1}{2}s-\tan\frac{1}{2}t|} \tag{1.3}$$

for $(s,t)\in[0,\pi]\times[0,\pi]$ with $s\neq t$. We abbreviate $\int_0^t \sin\varphi(u)\,du$ to $\int_0^t \sin\varphi$ throughout this paper. Equation (1.2) results from (a) differentiation of (1.1) with respect to the velocity potential Φ, and some manipulation, (b) mapping one period of the flow domain ($-\frac{1}{2}\lambda\leqslant x<\frac{1}{2}\lambda$, $-\infty<y<Y(x)$ in the physical plane, or $-\frac{1}{2}c\lambda\leqslant\Phi<\frac{1}{2}c\lambda$, $-\infty<\Psi<0$ in the plane of the complex potential $\Phi+i\Psi$) conformally onto the punctured unit disk $\mathscr{D}\setminus\{0\}$, where $\mathscr{D}=\{\zeta\in\mathbf{C}: |\zeta|<1\}$ and $\zeta=\varrho e^{is}$. The unit circle $\partial\mathscr{D}$ represents one wavelength of Γ, with $s=\pi$ corresponding to the trough at $x=-\frac{1}{2}\lambda$ and $s=0$ to the crest at $x=0$; the point $\zeta=0$ corresponds to $y=-\infty$, and the negative real axis within $\mathscr{D}$ to $x=-\frac{1}{2}\lambda, y<Y(-\frac{1}{2}\lambda)$. The function $\varphi(s)=\tan^{-1} Y'(x)$ is the *local wave angle*, and $\nu=2\pi q_0^3/3g\lambda c$, where q_0 is the fluid velocity (relative to our axes) at the crest $(0, Y(0))$. One prescribes λ and g; then from a non-trivial solution (ν, φ) of (1.2) the value of c can be calculated, and the value of q_0 follows.

The linear integral operator with kernel K solves the Neumann problem for functions $v(\varrho, s)$ harmonic in the unit disk $\mathscr{D}$, vanishing on the real axis in $\mathscr{D}$, and sufficiently smooth on $\overline{\mathscr{D}}$ (sufficient smoothness is specified precisely in the Appendix); for such functions,

$$v(1,s)=\int_0^\pi K(s,t)\frac{\partial v}{\partial \varrho}(1,t)\,dt, \quad 0<s\leqslant\pi. \tag{1.4}$$

The parameter $\nu\uparrow 1/3$ as the wave height $H\to 0$, while $\nu=0$ for the *wave of extreme form*, which is defined to be such that the fluid velocity q_0 at the crest is zero.

Keady and Norbury [5] proved that, for any $\nu\in(0,1/3)$, equation (1.2) has a smooth solution φ_ν such that $\varphi_\nu(0)=0$, $\varphi_\nu(\pi)=0$ and $0\leqslant\varphi_\nu(s)<\frac{1}{2}\pi$ on $[0,\pi]$. A surprising property of these solutions was suggested by the numerical work of Longuet-Higgins and Fox [7], and proved by McLeod [8]: if ν is sufficiently small (but positive), then there exist small positive values s, of order ν, such that $\varphi_\nu(s)>\pi/6$.

The existence of a wave of extreme form ($\nu=0$) was proved by Toland [11], and then, in a more elementary manner, by McLeod [8]; in totally different ways, both these authors extracted convergent subsequences from sequences $\{\varphi_{\nu(n)}\}$ of solutions for which $\nu(n)\to 0$ as $n\to\infty$. We know from [2], [3], [8] and [11] that the limiting solution φ is real-analytic on $[\delta,\pi]$ for any $\delta>0$, that $\varphi(\pi)=0$, that

$$0<\varphi(s)<\frac{\pi}{3} \quad \text{for } s\in(0,\ \pi), \quad \liminf_{s\to 0}\varphi(s)>0, \tag{1.5}$$

and that $\varphi'(s)=O(s^{-1})$ as $s\to 0$. The upper bound $\pi/3$ in (1.5) is a recent result [3] and seems essential for our estimates in section 3. Both Toland and McLeod showed that, if a limiting value $\varphi(0+)=\lim_{s\to 0}\varphi(s)$ exists, then it must be $\pi/6$, but neither author was able to prove existence of $\varphi(0+)$. Thus the question of the celebrated Stokes angle remained open.

The difficulty of the case $\nu=0$ is clear from the integral equation (1.2). If $\nu=0$, the second factor of the integrand is singular at $t=0$, in contrast to the case $\nu>0$. Indeed, if we denote the right-hand side of (1.2) by $(T_\nu \varphi)(s)$, then the non-linear operator T_0 lacks certain properties needed for the application of fixed-point theorems and enjoyed by T_ν for $\nu>0$. These adverse features of T_0 are presented in the Appendix, and perhaps account for the unorthodox nature of our proofs in section 3, and for our failure to prove that Y is convex (or, equivalently, that φ is non-increasing on $[0,\pi]$) for the wave of extreme form.

1.3. Plan of the present proof

Our analysis involves the approximate integral equation

$$\theta(x)=\frac{1}{3}\int_0^\infty k(x,y)\frac{\sin\theta(y)}{\int_0^y \sin\theta}dy, \quad 0<x<\infty, \tag{1.6}$$

where

$$k(x,y)=\frac{1}{\pi}\log\frac{x+y}{|x-y|}. \tag{1.7}$$

This has a solution $\theta(x)=\pi/6$, because

$$\frac{1}{3}\int_0^\infty k(x,y)\frac{1}{y}dy=\frac{1}{3\pi}\int_0^\infty \log\frac{1+u}{|1-u|}\frac{1}{u}du=\frac{\pi}{6}. \tag{1.8}$$

(The integral may be evaluated by noting that the contributions of $0<u<1$ and $u>1$ are equal, and by expanding the logarithm in series for $u<1$. Alternatively, noting that the kernel $k(y,\eta)$ plays a role like that of $K(s,t)$ in (1.4) when the unit disk is replaced by the left half-plane, one can consider the function

$$v(x,y;\varepsilon)=-\operatorname{Im}\log\{(z-\varepsilon)e^{-i\pi}\}, \quad z=x+iy, \quad x\leqslant 0, \quad \varepsilon>0,$$

infer that

$$v(0,y;\varepsilon)=\int_0^\infty k(y,\eta)\frac{\partial v}{\partial x}(0,\eta;\varepsilon)\,d\eta, \quad 0<y<\infty,$$

and show that this formula remains valid as $\varepsilon\rightarrow 0$.)

We begin with the estimates (1.5) and the following observation (see the Acknowledgement below): if $\pi/6$ is the *only* solution θ of (1.6) such that

$$\inf_{x\in(0,\infty)} \theta(x)>0, \quad \sup_{x\in(0,\infty)} \theta(x)\leqslant \pi/3, \tag{1.9}$$

then $\varphi(s)\to\pi/6$ as $s\to 0$ for any solution φ of (1.2), with $\nu=0$, that satisfies (1.5). By a *solution of* (1.6) we mean a pointwise solution in the space $C_b(0,\infty)$ of functions continuous and bounded on $(0,\infty)$; by a *solution of* (1.2), with $\nu=0$, we mean a pointwise solution in $C_b(0,\pi]$. For any function $f\in C_b(0,\infty)$, the abbreviations

$$\inf f(x) = \inf_{x\in(0,\infty)} f(x), \quad \sup f(y) = \sup_{y\in(0,\infty)} f(y), \tag{1.10}$$

will be convenient.

To prove that $\pi/6$ is the only solution of (1.6) and (1.9), we use a method that is perhaps elaborate but is wholly elementary. For any solution θ, we derive from three variants of the integral equation, and from consideration of the sine function on $[0,\pi/3]$, five inequalities involving the four functionals

$$\sup\frac{1}{x}\left|\int_0^x\left(\theta-\frac{\pi}{6}\right)\right|, \quad \sup\left\{\theta(x)-\frac{\pi}{6}\right\}, \quad \inf\frac{1}{x}\int_0^x \sin\theta, \quad \sup\frac{1}{x}\int_0^x \sin\theta.$$

Let $p=p(\theta)$ denote the first of these. The numerically small coefficients c_2 and c_3, in Lemmas 3.2 and 3.3 below, offer the hope that the five inequalities may be combined to one of the form

$$p\leqslant h(p), \quad 0\leqslant p\leqslant \pi/6,$$

where

$$h(0)=0 \quad \text{and} \quad h(p)<p \quad \text{if } 0<p\leqslant\pi/6,$$

which implies that $p=0$. This turns out to be the case.

1.4. Remaining questions

Although this paper settles the matter of the sharp crest and its included angle, some interesting questions remain open.

(a) We say nothing of the *rate* at which $\varphi(s)\to\pi/6$ as $s\to 0$. This question is taken up in [1], where an exponent $\gamma \in (0, 1)$, in the estimate $\varphi(s)-\pi/6=O(s^\gamma)$, is established that is probably the best possible.

(b) With some natural definition of wave height, is the wave of extreme form that of greatest height? Numerical evidence and heuristic approximations [4], [7] suggest that it is, when the height H is defined as in section 1.1.

(c) We have not proved convexity of the wave profile.

1.5. Acknowledgement

We are heavily indebted to J. B. McLeod for an emphatic statement of Theorem 2.1 (for $\lambda<\infty$) to one of us, during a conversation in January 1980. Although we were aware already of the usefulness of the approximate kernel k, it was McLeod's remark that ultimately led us to concentrate attention on the approximate integral equation.

2. Reduction of the problem

We consider flows with wavelength $\lambda \in (0, \infty]$ in liquid of mean depth $h \in (0, \infty]$, the lower boundary in the physical plane being straight and horizontal when $h<\infty$. We write $\lambda=\infty$ for the *solitary wave*, and in that case $h<\infty$. It is known from [2], [3], [8] and [11] that waves of extreme form exist for all these values of λ and h. In section 1 we have considered the case $\lambda/h=0$; all essential statements there about the wave of extreme form remain true in the general situation.

The integral equation of Nekrasov type, for the wave of extreme form, is now [3]

$$\varphi(s) = \frac{1}{3}\int_0^\pi K(s,t)\frac{f_\lambda(t)\sin\varphi(t)}{\displaystyle\int_0^t f_\lambda \sin\varphi}\,dt, \quad 0<s<\pi, \tag{2.1}$$

where φ is still the local wave angle, K is as in (1.3) and (1.4),

$$f_\lambda(t) = \frac{1}{2}\left(\cos^2\frac{t}{2}+\beta_\lambda \sin^2\frac{t}{2}\right)^{-\frac{1}{2}},$$

and β_λ is a continuous, monotone function of $\lambda/h \in [0, \infty)$ such that $\beta_\lambda \uparrow 1$ as $\lambda/h \to 0$, while $\beta_\lambda \downarrow 0$ as $\lambda/h \to \infty$. We define $\beta_\infty=0$ for the case $\lambda=\infty$, $h<\infty$ of the solitary wave.

Recall that

$$k(x,y) = \frac{1}{\pi}\log\frac{x+y}{|x-y|} \quad (0<x<\infty,\ 0<y<\infty,\ x\neq y).$$

The word *solution* continues to have the meaning assigned to it after (1.9).

THEOREM 2.1. *If $\pi/6$ is the only solution of*

$$\theta(x)=\frac{1}{3}\int_0^\infty k(x,y)\frac{\sin\theta(y)}{\int_0^y \sin\theta}\,dy,\quad 0<x<\infty, \tag{2.2}$$

satisfying

$$\inf_{x\in(0,\infty)}\theta(x)>0,\qquad \sup_{x\in(0,\infty)}\theta(x)\leqslant\pi/3, \tag{2.3}$$

then any solution φ of (2.1) *satisfying*

$$\liminf_{s\to 0}\varphi(s)>0,\quad 0<\varphi(s)\leqslant\frac{\pi}{3}\quad \text{for}\quad s\in(0,\pi) \tag{2.4}$$

has the property: $\varphi(s)\to\pi/6$ as $s\to 0$. (Solutions φ of (2.1) *and* (2.4) *are known to exist for all $\lambda/h\in[0,\infty]$.)*

Proof. (i) Given any solution φ of (2.1) and (2.4), we shall construct a corresponding solution θ of (2.2) and (2.3). First, we cast (2.1) into a more convenient form. Under the transformation $\xi=\tan\frac{1}{2}s$, $\eta=\tan\frac{1}{2}t$ and $\psi(\xi)=\varphi(2\tan^{-1}\xi)=\varphi(s)$, equation (2.1) becomes

$$\psi(\xi)=\frac{1}{3}\int_0^\infty k(\xi,\eta)\frac{g_\lambda(\eta)\sin\psi(\eta)}{\int_0^\eta g_\lambda\sin\psi}\,d\eta,\quad 0<\xi<\infty,$$

where

$$g_\lambda(\eta)=(1+\eta^2)^{-\frac{1}{2}}(1+\beta_\lambda\eta^2)^{-\frac{1}{2}}.$$

For $\xi\to 0$, we may truncate the integral. By (2.4), there exists a constant $c_0>0$ such that $\sin\varphi(t)\geqslant 2c_0$ on $(0,\pi/2]$; since $f_\lambda(t)\geqslant\frac{1}{2}$, we have

$$\int_0^t f_\lambda\sin\varphi\geqslant\begin{cases} c_0 t & \text{for } 0\leqslant t\leqslant\pi/2,\\ \frac{1}{2}c_0 t & \text{for } 0\leqslant t<\pi,\end{cases}$$

whence

$$\int_0^\eta g_\lambda\sin\psi\geqslant c_0\tan^{-1}\eta,\quad 0\leqslant\eta<\infty.$$

Also, $k(\xi,\eta)\leqslant(8/3\pi)\,(\xi/\eta)$ when $\xi/\eta\leqslant\frac{1}{2}$. Accordingly, if $\xi\leqslant\frac{1}{2}$,

$$0\leqslant\int_1^\infty k(\xi,\eta)\frac{g_\lambda(\eta)\sin\psi(\eta)}{\displaystyle\int_0^\eta g_\lambda\sin\psi}\,d\eta\leqslant\text{constant}\cdot\int_1^\infty\frac{\xi}{\eta}(1+\eta^2)^{-\frac{1}{2}}\,d\eta=\text{constant}\cdot\xi,$$

so that

$$\psi(\xi)=\frac{1}{3}\int_0^1 k(\xi,\eta)\frac{g_\lambda(\eta)\sin\psi(\eta)}{\displaystyle\int_0^\eta g_\lambda\sin\psi}\,d\eta+O(\xi), \tag{2.5}$$

where now, since $\eta\in[0,1]$,

$$\frac{1}{2}\leqslant g_\lambda(\eta)\leqslant 1,\quad \int_0^\eta g_\lambda\sin\psi\geqslant\frac{c_0\pi}{4}\eta. \tag{2.6}$$

(ii) Let $\{\alpha_n\}$ be a decreasing sequence in (0,1) such that $\alpha_n\to 0$, and define $\theta_n\in C_b(0,1/\alpha_n]$ by

$$\theta_n(y)=\psi(\alpha_n y)\quad (0<\alpha_n y\leqslant 1). \tag{2.7}$$

Setting $\xi=\alpha_n x$ and $\eta=\alpha_n y$ in (2.5), we suppose that

$$x\in[a,b]\subset(0,\infty),$$

with a and b fixed, and that α_n is so small that $2\alpha_n b\leqslant 1$; then

$$\theta_n(x)=\frac{1}{3}\int_0^{1/\alpha_n}k(x,y)\frac{g_\lambda(\alpha_n y)\sin\theta_n(y)}{\displaystyle\int_0^y g_\lambda(\alpha_n z)\sin\theta_n(z)\,dz}\,dy+O(\alpha_n).$$

We now show that g_λ may be replaced by 1 in this equation. Let $M\in[2b,\,1/\alpha_n]$, so that $M\geqslant 2x$; then, by (2.6),

$$0\leqslant\int_M^{1/\alpha_n}k(x,y)\frac{g_\lambda(\alpha_n y)\sin\theta_n(y)}{\displaystyle\int_0^y g_\lambda(\alpha_n z)\sin\theta_n(z)\,dz}\,dy\leqslant\text{constant}\cdot\int_M^\infty\frac{x}{y}\frac{1}{y}\,dy\leqslant\frac{\text{constant}}{M},$$

and similarly if g_λ is replaced by 1 for $M\leqslant y\leqslant 1/\alpha_n$; moreover,

$$\left| \int_0^M k(x,y) \left\{ \frac{g_\lambda(\alpha_n y) \sin\theta_n(y)}{\int_0^y g_\lambda(\alpha_n z) \sin\theta_n(z)\,dz} - \frac{\sin\theta_n(y)}{\int_0^y \sin\theta_n} \right\} dy \right|$$

$$\leqslant \text{constant} \cdot \int_0^\infty k(x,y) \frac{\alpha_n^2 M^2}{y}\,dy = \text{constant} \cdot \alpha_n^2 M^2.$$

Choosing $M = \alpha_n^{-2/3}$, we obtain

$$\theta_n(x) = \frac{1}{3} \int_0^{1/\alpha_n} k(x,y) \frac{\sin\theta_n(y)}{\int_0^y \sin\theta_n}\,dy + O(\alpha_n^{2/3}), \tag{2.8}$$

uniformly for $x \in [a, b]$.

The sequence of functions defined by the integral in (2.8) is bounded and equicontinuous on $[a, b]$, by standard results for integral operators like that in question (note that $k(x, y) = O(y)$ as $y \to 0$, uniformly for $x \in [a, b]$); hence the sequence $\{\theta_n\}$ is relatively compact in $C[a, b]$. Now let $\{[a_m, b_m]\}$ be an expanding sequence of intervals whose union is $(0, \infty)$. We extract successive subsequences $\{\theta_{m,j}\}_{j=1}^\infty$ convergent in $C[a_m, b_m]$, and contained in $\{\theta_{m-1,j}\}_{j=1}^\infty$ and hence in $\{\theta_n\}$, and diagonalize; there results a sequence that converges uniformly on any compact subset of $(0, \infty)$ to a function $\theta \in C_b(0, \infty)$, and θ satisfies (2.2) and (2.3). Then $\theta(x) = \pi/6$ for all $x \in (0, \infty)$, by the hypothesis of the theorem.

(iii) Finally, suppose that $\varphi(s) \not\to \pi/6$ as $s \to 0$. Then there exists a sequence $\{\alpha_n\}$ as in (ii) such that $\psi(\alpha_n) \to \gamma \neq \pi/6$, and, by (2.7), $\theta_n(1) \to \gamma$. But this contradicts the result of (ii).

3. Uniqueness for (2.2) and (2.3)

3.1. Primary inequalities

Throughout this section, θ denotes any pointwise solution of (2.2) that is in $C_b(0, \infty)$ and satisfies (2.3); abbreviating as in (1.10), we define

$$p = \sup \frac{1}{x} \left| \int_0^x \left(\theta - \frac{\pi}{6}\right) \right|, \quad m = \sup \left\{ \theta(x) - \frac{\pi}{6} \right\}, \tag{3.1}$$

and note that $0 \leq p \leq \pi/6$, $m \leq \pi/6$. Our first two estimates involve the integrated kernel

$$q(x, y) = \int_0^x k(z, y)\, dz = \frac{1}{\pi}\left\{x \log \frac{x+y}{|x-y|} + y \log \frac{|x^2-y^2|}{y^2}\right\}, \tag{3.2 a}$$

and

$$q_y(x, y) = \frac{1}{\pi} \log \frac{|x^2-y^2|}{y^2}. \tag{3.2 b}$$

LEMMA 3.1. *Equation* (2.2) *implies that*

$$\int_0^x \theta = \frac{1}{3} \int_0^\infty q(x, y) \frac{\sin \theta(y)}{\int_0^y \sin \theta} dy, \tag{3.3}$$

which yields the estimate

$$\inf \frac{1}{x} \int_0^x \sin \theta \geq c_1, \quad \textit{where } c_1 = \sqrt{3}\left(\frac{1}{8} - \frac{\log 2}{\pi^2}\right) > 0.0948. \tag{3.4}$$

Proof. Obviously (3.3) results from integration of (2.2). Then, since $q(x, y) \geq 0$ and $q(x, y) = O(y^{-1})$ as $y \to \infty$ for fixed $x \in (0, \infty)$,

$$\begin{aligned}
\int_0^x \theta &\geq \frac{1}{3} \int_x^\infty q(x, y) \frac{\sin \theta(y)}{\int_0^y \sin \theta} dy \\
&= \frac{1}{3} \int_x^\infty q(x, y) \left\{\frac{d}{dy} \log\left(\frac{1}{y} \int_0^y \sin \theta\right) + \frac{1}{y}\right\} dy \\
&= -\frac{1}{3} q(x, x) \log\left(\frac{1}{x} \int_0^x \sin \theta\right) - \frac{1}{3} \int_x^\infty q_y(x, y) \log\left(\frac{1}{y} \int_0^y \sin \theta\right) dy \\
&\qquad + \frac{1}{3} \int_x^\infty q(x, y) \frac{1}{y} dy.
\end{aligned}$$

Referring to (3.2 a), we find that $q(x,x) = c_0 x$, with $c_0 = (\log 4)/\pi$, and we evaluate the last integral; from (3.2 b) we observe that $q_y(x,y) < 0$ for $y > x$. Defining

$$J = \inf \frac{1}{y} \int_0^y \sin \theta,$$

we then obtain

$$\int_0^x \theta \geqslant -\frac{1}{3} c_0 x \log\left(\frac{1}{x}\int_0^x \sin\theta\right) + \frac{1}{3} c_0 x \log J + \frac{x}{3}\left(\frac{\pi}{4} - \frac{2\log 2}{\pi}\right).$$

Since $(\sin t)/t$ is a decreasing function on $[0, \pi/3]$, we also have

$$\frac{\pi/3}{\sin(\pi/3)} \int_0^x \sin\theta \geqslant \int_0^x \theta,$$

and there results, upon division by $x/3$,

$$\frac{2\pi}{\sqrt{3}} \frac{1}{x} \int_0^x \sin\theta + c_0 \log\left(\frac{1}{x}\int_0^x \sin\theta\right) \geqslant c_0 \log J + \left(\frac{\pi}{4} - \frac{2\log 2}{\pi}\right).$$

The left-hand member of this inequality is an increasing function of $(1/x)\int_0^x \sin\theta$; taking its infimum, we obtain (3.4).

LEMMA 3.2. *Equation* (2.2) *implies that*

$$\int_0^x \left(\theta - \frac{\pi}{6}\right) = -\frac{1}{3} \int_0^\infty q_y(x, y) \log\left(\frac{C}{y}\int_0^y \sin\theta\right) dy, \tag{3.5}$$

for any constant $C > 0$. *With* p *as in* (3.1), *this yields the estimate*

$$\left.\begin{aligned} p \leqslant c_2 \log\left[\sup \frac{1}{y}\int_0^y \sin\theta \Big/ \inf \frac{1}{y}\int_0^y \sin\theta\right], \\ \textit{where } c_2 = \frac{2}{3\pi}\log(1+\sqrt{2}) < 0.1871. \end{aligned}\right\} \tag{3.6}$$

Proof. To derive (3.5), we combine (2.2) and (1.8), and integrate with respect to x, to obtain

$$\int_0^x \left(\theta - \frac{\pi}{6}\right) = \frac{1}{3} \int_0^\infty q(x, y) \frac{d}{dy} \log\left(\frac{C}{y}\int_0^y \sin\theta\right) dy. \tag{3.7}$$

Note from (3.2a) that, for fixed $x \in (0, \infty)$, we have $q(x, y) = O(y \log y)$ as $y \to 0$, and $q(x, y) = O(y^{-1})$ as $y \to \infty$. Thus integration by parts with respect to y gives (3.5). Now define

$$J = \inf \frac{1}{y}\int_0^y \sin\theta, \quad K = \sup \frac{1}{y}\int_0^y \sin\theta, \tag{3.8}$$

and choose $C=1/J$ in (3.5), so that the logarithm there is non-negative. Equation (3.2 b) shows that

$$q_y(x, y)>0 \quad \text{for } 0<y<x/\sqrt{2},$$

$$q_y(x, y)<0 \quad \text{for } x/\sqrt{2}<y<\infty, \quad y\neq x.$$

Accordingly, (3.5) implies that

$$\begin{aligned}\int_0^x \left(\theta-\frac{\pi}{6}\right) &\geqslant -\frac{1}{3}\int_0^{x/\sqrt{2}} q_y(x, y)\log\frac{K}{J}\,dy \\ &= -\frac{1}{3}q\left(x, \frac{x}{\sqrt{2}}\right)\log\frac{K}{J} \\ &= -c_2 x\log\frac{K}{J},\end{aligned}$$

while integration from $y=x/\sqrt{2}$ onwards yields

$$\int_0^x \left(\theta-\frac{\pi}{6}\right) \leqslant c_2 x\log\frac{K}{J}.$$

LEMMA 3.3. *Equation* (2.2) *implies that*

$$\theta(x)-\frac{\pi}{6}=\frac{1}{3\pi}\int_0^\infty \frac{1}{u}\log\left\{\frac{|1-u|}{1+u}\frac{\displaystyle\int_0^{x(1+u)}\sin\theta}{\displaystyle\int_0^{x|1-u|}\sin\theta}\right\}du. \tag{3.9}$$

With m as in (3.1), *this yields the estimate*

$$m\leqslant c_3\left\{\frac{\sin(\pi/6+m)}{\inf\displaystyle\frac{1}{y}\int_0^y\sin\theta}-1\right\}, \quad \textit{where } c_3=\frac{4\log 2}{3\pi}<0.2943. \tag{3.10}$$

Proof. Again we combine (2.2) and (1.8), and define

$$F(y)=\log\left(\frac{1}{y}\int_0^y\sin\theta\right),$$

so that $F\in C^1(0,\infty)$ and $|F(y)|\leqslant$ constant. For fixed $x\in(0,\infty)$ as elsewhere, we have

$$\begin{aligned}\theta(x)-\frac{\pi}{6}&=\frac{1}{3\pi}\int_0^\infty \log\frac{x+y}{|x-y|}F'(y)\,dy\\&=-\frac{1}{3\pi}\lim_{\delta\to 0}\left\{\int_0^{x-\delta}+\int_{x+\delta}^\infty\left(\frac{1}{y+x}-\frac{1}{y-x}\right)F(y)\,dy\right\}\\&=-\frac{1}{3\pi}\int_0^\infty\frac{F(y)}{y+x}dy+\frac{1}{3\pi}\lim_{\delta\to 0}\left\{\int_0^{x-\delta}+\int_{x+\delta}^\infty\frac{F(y)}{y-x}dy\right\}.\end{aligned}$$

Set $y=z-x$ in the first integral, $y=x-z$ in the second, and $y=x+z$ in the third; then the first two may be combined to give

$$\theta(x)-\frac{\pi}{6}=\frac{1}{3\pi}\lim_{\delta\to 0}\int_\delta^\infty\frac{F(x+z)-F(|x-z|)}{z}dz,$$

and $F(x+z)-F(|x-z|)\sim 2F'(x)z$ as $z\to 0$. Accordingly,

$$\theta(x)-\frac{\pi}{6}=\frac{1}{3\pi}\int_0^\infty\frac{1}{z}\log\left\{\frac{|x-z|}{x+z}\frac{\int_0^{x+z}\sin\theta}{\int_0^{|x-z|}\sin\theta}\right\}dz,$$

and we let $z=xu$ to obtain (3.9).

Now let J be as in (3.8), and let $l(u)$ denote the logarithm in (3.9). For $u\leqslant 1$,

$$l(u)=\log\left\{\frac{1-u}{1+u}\left(1+\frac{\int_{x(1-u)}^{x(1+u)}\sin\theta}{\int_0^{x(1-u)}\sin\theta}\right)\right\},$$

where

$$\int_{x(1-u)}^{x(1+u)}\sin\theta\leqslant 2xu\sin(\pi/6+m),\qquad \int_0^{x(1-u)}\sin\theta\geqslant x(1-u)J.$$

Hence

$$\begin{aligned}l(u)&\leqslant\log\left\{\frac{1-u}{1+u}\left(1+\frac{2u\sin(\pi/6+m)}{(1-u)J}\right)\right\}\\&=\log\left\{1+\frac{2u\sin(\pi/6+m)-2uJ}{(1+u)J}\right\}\end{aligned}$$

$$\leq \frac{2u}{1+u}\left\{\frac{\sin(\pi/6+m)}{J}-1\right\} \quad (u \leq 1).$$

Similarly, for $u \geq 1$,

$$l(u) \leq \log\left\{1+\frac{2\sin(\pi/6+m)-2J}{(1+u)J}\right\} \leq \frac{2}{1+u}\left\{\frac{\sin(\pi/6+m)}{J}-1\right\}.$$

Using these bounds for $l(u)$ in (3.9), we obtain (3.10).

Two more inequalities. By Jensen's inequality and the concavity of the sine function on $[0, \pi/3]$,

$$\sup \frac{1}{y}\int_0^y \sin\theta \leq \sup \sin\left(\frac{1}{y}\int_0^y \theta\right) \leq \sin\left(\frac{\pi}{6}+p\right). \tag{3.11}$$

Also, the elementary inequality

$$\frac{\sin\theta(z)}{\theta(z)} \geq \frac{\sin(\pi/6+m)}{\pi/6+m}$$

implies that

$$\inf \frac{1}{y}\int_0^y \sin\theta \geq \frac{\sin(\pi/6+m)}{\pi/6+m} \inf \frac{1}{y}\int_0^y \theta \geq \frac{\sin(\pi/6+m)}{\pi/6+m}\left(\frac{\pi}{6}-p\right). \tag{3.12}$$

3.2. Manipulation

LEMMA 3.4. $p<0.086.$

Proof. (i) First, we use Lemma 3.1 and (3.11) in Lemma 3.2 to obtain

$$p \leq c_2 \log \frac{\sin(\pi/6+p)}{c_1} = f(p), \text{ say.} \tag{3.13}$$

Here $p \in [0, \pi/6]$, and f is a strictly positive, increasing, concave function on $[0, \pi/6]$. Moreover, $f(0.398) < 0.398$, so that $f(p) < p$ if $p \in [0.398, \pi/6]$. But then (3.13) shows that $p < 0.398$.

(ii) Since $m \leq \pi/6$, it follows from (3.12) that

$$\inf \frac{1}{y}\int_0^y \sin\theta \geq \frac{\sin(\pi/3)}{\pi/3}\left(\frac{\pi}{6}-p\right).$$

Now using this and (3.11) in Lemma 3.2, we obtain

$$p \leqslant c_2 \log \frac{2\pi \sin(\pi/6+p)}{3\sqrt{3}\,(\pi/6-p)} = g(p), \text{ say.} \tag{3.14}$$

Then g is strictly positive and increasing on $[0, 0.398]$; it has one point of inflexion there, and is concave to the left of that point and convex to the right of it. Moreover, $g(0.086)<0.086$ and $g(0.398)<0.398$, so that $g(p)<p$ if $p \in [0.086, 0.398]$. But then (3.14) shows that $p<0.086$.

LEMMA 3.5. $$m \leqslant Qp,$$
where

$$Q = \frac{c_3}{\pi/6 - c_3 - 0.086} < 2.053.$$

Proof. Using (3.12) in Lemma 3.3, we obtain

$$m \leqslant c_3 \left\{ \frac{\pi/6+m}{\pi/6-p} - 1 \right\},$$

or, equivalently,

$$m \leqslant \frac{c_3 p}{\pi/6 - c_3 - p},$$

and we apply Lemma 3.4 to the denominator of this bound, recalling from Lemma 3.3 that $c_3 < 0.2943$.

THEOREM 3.6. $p=0$. *In other words, $\pi/6$ is the only solution of* (2.2) *and* (2.3).

Proof. Using (3.11) and (3.12) in Lemma 3.2, we obtain

$$p \leqslant c_2 \log \frac{(\pi/6+m)\sin(\pi/6+p)}{(\pi/6-p)\sin(\pi/6+m)}. \tag{3.15}$$

Now $t/\sin t$ is an increasing function on $[0, \pi/3]$; hence Lemma 3.5 implies that

$$\frac{\pi/6+m}{\sin(\pi/6+m)} \leqslant \frac{\pi/6+Qp}{\sin(\pi/6+Qp)} = F(p), \text{ say.}$$

Let $p_0=0.086$; by Lemma 3.4, it is sufficient to consider $p \in [0, p_0]$. Now F is convex on $[0, p_0]$; hence

$$F(p) \leqslant F(0) + \{F(p_0) - F(0)\} \frac{p}{p_0} = \frac{\pi}{3} + Rp,$$

where

$$R = \frac{1}{p_0}\left\{\frac{\pi/6+Qp_0}{\sin(\pi/6+Qp_0)} - \frac{\pi}{3}\right\} < 0.459.$$

Then, by (3.15),

$$p \leqslant c_2 \log\frac{(\pi/3+Rp)\sin(\pi/6+p)}{\pi/6-p} = G(p), \text{ say.} \tag{3.16}$$

Here $G(0)=0$, and for $p\in[0, p_0]$

$$\begin{aligned} G'(p) &= c_2\left\{\frac{R}{\pi/3+Rp} + \cot(\pi/6+p) + \frac{1}{\pi/6-p}\right\} \\ &\leqslant c_2\left\{\frac{3R}{\pi} + \cot\frac{\pi}{6} + \frac{1}{\pi/6-p_0}\right\} < 0.84. \end{aligned}$$

Therefore $G(p)<p$ if $p\in(0, p_0]$, and (3.16) now shows that $p=0$. Consequently

$$\int_0^x \left(\theta - \frac{\pi}{6}\right) = 0 \quad \text{for all } x\in(0,\infty),$$

and, by differentiation, $\theta(x) = \pi/6$.

Appendix. Adverse properties of the integral operator for $\nu = 0$

Let T denote the operator called T_0 in section 1.2, and defined by

$$(T\psi)(s) = \frac{1}{3}\int_0^\pi K(s,t)\frac{\sin\psi(t)}{\displaystyle\int_0^t \sin\psi}\,dt, \quad 0<s\leqslant\pi. \tag{A.1}$$

Here we present three properties of T which have frustrated our attempts to prove by means of fixed-point theorems that, for a wave of extreme form (a) $\varphi(s)\to\pi/6$ as $s\to 0$, (b) the profile function Y is convex (equivalently, that φ is non-increasing on $[0,\pi]$). We begin with a method of evaluating $T\psi$ explicitly for certain functions ψ.

A.1. Explicit evaluation of $T\psi$

As before, we write $\mathcal{D}=\{\zeta\in\mathbf{C}: |\zeta|<1\}$ and $\zeta=\varrho e^{is}$.

LEMMA A.1. *Assume that*

(a) *w is holomorphic on* $\overline{\mathscr{D}}\setminus\{1\}$,

(b) *there exist positive constants M and δ such that*

$$|w'(\zeta)| \leq M|\zeta-1|^{-2+\delta} \text{ for } \zeta\in\overline{\mathscr{D}}\setminus\{1\},$$

(c) $w(\zeta)$ *is real for real* $\zeta\in[-1,1)$.

Then, with the notation $w(e^{is})=a(s)+ib(s)$ *(where a and b are real),*

$$b(s) = -\int_0^\pi K(s,t)\, a'(t)\, dt, \quad 0<s\leq\pi.$$

Proof. Define

$$w_n(\zeta) = w\left(\frac{n}{n+1}\zeta\right) \text{ for } n=1,2,\ldots,$$

and let $w_n(e^{is})=a_n(s)+ib_n(s)$. Then w_n is holomorphic on $\overline{\mathscr{D}}$ and real on $[-1,1]$. The characterization (1.4) of $K(s,t)$, and the Cauchy-Riemann equation $\partial v_n/\partial\varrho=-\varrho^{-1}\,\partial u_n/\partial s$ (where $w_n=u_n+iv_n$), imply that

$$b_n(s) = -\int_0^\pi K(s,t)\, a_n'(t)\, dt, \quad 0\leq s\leq\pi.$$

Now $b_n(s)\to b(s)$ and $a_n'(s)\to a'(s)$ as $n\to\infty$ with $s\in(0,\pi]$; we claim that the lemma then follows from the Lebesgue dominated convergence theorem. Without loss of generality, let $\delta\in(0,2]$; then

$$\begin{aligned} K(s,t)\,|a_n'(t)| &\leq K(s,t)\, M\frac{n}{n+1}\left|\frac{n}{n+1}e^{it}-1\right|^{-2+\delta} \\ &\leq K(s,t)\, M\left(2\sin\frac{t}{2}\right)^{-2+\delta} = F(s,t), \text{ say.} \end{aligned}$$

Here $K(s,t)=O(t)$ as $t\to 0$, for fixed $s\in(0,\pi]$, and $K(s,t)$ has merely a logarithmic singularity at $t=s$, so that $F(s,\cdot)\in L_1(0,\pi)$ for $s\in(0,\pi]$.

LEMMA A.2. $$\int_0^\pi K(s,t)\cot\frac{t}{2}\,dt = \pi-s, \quad 0<s\leq\pi.$$

Proof. The function $w_0(\zeta)=-\log(1-\zeta)$ satisfies the hypotheses of Lemma A.1 if we choose $\arg(1-\zeta)\in(-\pi/2,\pi/2)$ for $\zeta\in\mathscr{D}$, and then, for $s\in(0,\pi]$,

$$a_0(s) = -\log\left(2\sin\frac{s}{2}\right), \quad -a_0'(s) = \frac{1}{2}\cot\frac{s}{2}, \quad b_0(s) = \frac{1}{2}(\pi-s).$$

LEMMA A.3. *Let w be as in Lemma* A.1, *with $\delta \geq 1$ in condition* (b), *and also restricted as follows*: (d) $s \exp\{-a(s)\} \to 0$ *as* $s \to 0$, (e) *the functions $a'(s)\sin(s/2)$ and $a(s)$ are so small that the equation*

$$\sin \psi(s) = \left\{\frac{1}{2}\cos\frac{s}{2} - a'(s)\sin\frac{s}{2}\right\} e^{-a(s)}, \quad 0 < s \leq \pi, \tag{A.2}$$

defines $\psi(s) \in [0, \pi/2)$ for $s \in (0, \pi]$. (Note that $a'(s) \to 0$ as $s \to \pi$.) Then

$$(T\psi)(s) = \frac{1}{6}(\pi - s) + \frac{1}{3} b(s), \quad 0 < s \leq \pi. \tag{A.3}$$

Proof. From (d) and (e) we have

$$\int_0^t \sin\psi = \sin\frac{t}{2} e^{-a(t)}, \quad \frac{\sin\psi(t)}{\int_0^t \sin\psi} = \frac{1}{2}\cot\frac{t}{2} - a'(t).$$

Hence, using Lemmas A.2 and A.1 to evaluate the integral in (A.1), we obtain (A.3).

A.2. Non-compactness

This section is included to excuse our failure to exploit the wide variety of fixed-point theorems now available for non-linear compact operators. Let

$$A = \{\psi \in C[0, \pi] : \psi(0) = \pi/6, \ \psi(\pi) = 0, \quad \psi(s) \geq 0\}.$$

An attractive proposition is to prove that T has a fixed point in A, and thus to obtain simultaneously the existence and regularity of an extreme wave.

A small difficulty is that T does not map A into itself (because $\sin \psi(t) < 0$ when $\pi < \psi(t) < 2\pi$), but this is easily overcome. Defining a truncation operator J by

$$J\psi(t) = \begin{cases} \psi(t) \text{ if } 0 \leq \psi(t) \leq \pi, \\ \pi \text{ if } \psi(t) > \pi, \end{cases}$$

we may replace the operator equation $\varphi = T\varphi$ by

$$\varphi = T \circ J\varphi, \quad \varphi \in A, \tag{A.4}$$

because a priori bounds can be obtained to show that every solution of (A.4) is also a solution of $\varphi = T\varphi$. The real difficulty, which obstructs further progress along these lines, is shown by the following theorem.

THEOREM A.4. *The operator $T\circ J$ maps A into itself, and $T\circ J: A\to A$ is continuous but not compact (the norm being that of $C[0,\pi]$).*

Proof. We omit the proof that $T\circ J$ maps A into itself continuously. (Given (1.8) or Lemma A.2, and step (i) in the proof of Theorem 2.1, it is not difficult to prove that $(T\circ J\psi)(s)\to\pi/6$ as $s\to 0$, when $\psi\in A$, and the rest is routine.) To prove the non-compactness, we use Lemma A.3 and choose

$$w(\zeta)=\frac{c}{\alpha}(1-\zeta)^{\alpha},\quad \alpha\in(0,1],$$

where the constant $c>0$ is so small that condition (e) holds ($c=1/2\pi$ will serve). Then

$$\sin\psi_\alpha(s)=\left\{\frac{1}{2}\cos\frac{s}{2}-\frac{c}{2}\left(2\sin\frac{s}{2}\right)^{\alpha}\sin\frac{(1+\alpha)(\pi-s)}{2}\right\}\exp\left\{-\frac{c}{\alpha}\left(2\sin\frac{s}{2}\right)^{\alpha}\cos\frac{\alpha(\pi-s)}{2}\right\}.$$

Note that the first factor is bounded below by $\frac{1}{2}\cos\frac{1}{2}s-c(\pi-s)(\sin\frac{1}{2}s)^{\alpha}$ and above by $\frac{1}{2}\cos\frac{1}{2}s$; the exponential does not exceed 1, and equals 1 for $s=0$. Thus $\psi_\alpha\in A$ for each $\alpha\in(0,1]$, and the set $\{\psi_\alpha : 0<\alpha\leq 1\}$ is bounded in $C[0,\pi]$. Now, by (A.3),

$$(T\psi_\alpha)(s)=\frac{1}{6}(\pi-s)-\frac{c}{3\alpha}\left(2\sin\frac{s}{2}\right)^{\alpha}\sin\frac{\alpha(\pi-s)}{2},$$

so that

$$(T\psi_\alpha)(0)=\frac{\pi}{6}\quad\text{for all } \alpha\in(0,1],$$

and

$$(T\psi_\alpha)(s)\to\frac{1-c}{6}(\pi-s)\quad\text{as } \alpha\to 0\quad\text{with } s\in(0,\pi].$$

Hence the sequence $\{T\psi_{1/n}\}$ (in which $\alpha=1/n$ and $n=1,2,\ldots$) contains no subsequence convergent in $C[0,\pi]$.

A.3. *T* preserves certain oscillations of unbounded variation

Previous estimates (summarized in section 1.2) of extreme waves did not rule out certain functions φ that have oscillations of unbounded variation as $s\to 0$. In particular, functions behaving near $s=0$ like $\pi/6+\frac{1}{4}\sin(\log s)$ were compatible with those estimates.

The following example shows that T does not have a smoothing effect on such functions, but maps them into functions of the same general kind. We choose

$$w(\zeta) = c \cos \{\log(1-\zeta)\}$$

in Lemma A.3 to obtain

EXAMPLE A.5. *If*

$$\sin \psi(s) = \left\{ \frac{1}{2} \cos \frac{s}{2} - a'(s) \sin \frac{s}{2} \right\} e^{-a(s)}, \quad 0 < s \leqslant \pi,$$

where

$$a(s) = c \cos \left\{ \log \left(2 \sin \frac{s}{2} \right) \right\} \cosh \frac{\pi - s}{2},$$

and c is a small positive constant, then

$$(T\psi)(s) = \frac{1}{6}(\pi - s) + \frac{c}{3} \sin \left\{ \log \left(2 \sin \frac{s}{2} \right) \right\} \sinh \frac{\pi - s}{2}.$$

A.4. *T* fails to preserve monotonicity

Define the closed, convex set $B \subset A$ by

$$B = \{\psi \in A : \psi \text{ is non-increasing on } [0, \pi]\}.$$

Any attempt to establish the convexity of the wave profile by finding a fixed point of T in B is thwarted by the failure of T to map B into itself. The following example shows this failure.

EXAMPLE A.6. *Define*

$$w(\zeta) = c(1-\zeta)^{\alpha} \left[\frac{k}{\alpha} + 2 \cos \{\log(1-\zeta)\} \right], \quad \zeta \in \overline{\mathscr{D}},$$

where $k > 2^{3/2} \cosh(\pi/2)$, *while* $c, \alpha \in (0, \frac{1}{2})$ *and are sufficiently small. Then the hypotheses of Lemma* A.3 *are satisfied, and* $\psi'(s) < 0$ *on* $(0, \pi]$. *However,* $T\psi$ *is not monotone; in fact,*

$$\liminf_{s \to 0} s^{1-\alpha} \frac{d}{ds}(T\psi)(s) < 0 \quad \text{and} \quad \limsup_{s \to 0} s^{1-\alpha} \frac{d}{ds}(T\psi)(s) > 0.$$

The demonstration is by direct calculation. However, this example has a weakness: the oscillations (of form $s^{\alpha} \sin(\log s)$ with α small) that occur in $\psi(s)$ and $T\psi(s)$ as $s \to 0$ cannot occur in a wave of extreme form; they are ruled out by the estimate $\varphi(s) - \pi/6 = O(s^{\gamma})$ that we quoted from [1] in section 1.4.

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