Tôhoku Math. Journ.
22(1970), 191-199.

ON THE STONE-WEIERSTRASS THEOREM OF C^*-ALGEBRAS

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(Received August 28, 1969)

1. Introduction. Let A be the C^*-algebra of all complex valued continuous functions vanishing at infinity on a locally compact space. The Stone-Weierstrass theorem gives the conditions under which a C^*-subalgebra B coincides with A. A plausible non-commutative extension of the Stone-Weierstrass theorem is

Conjecture. Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$. Let $P(\mathfrak{A})$ be the set of all pure states of $\mathfrak{A}$ and let 0 be the identically zero function on $\mathfrak{A}$. Suppose that $\mathfrak{B}$ separates $P(\mathfrak{A})\cup(0)$, then $\mathfrak{A}=\mathfrak{B}$.

Kaplansky [9] proved a theorem equivalent to the conjecture for GCR C^*-algebras (equivalently, type I C^*-algebras [6], [13]). Glimm [5], Ringrose [10] and Akemann [1] gave some considerations related to this conjecture.

The purpose of this paper is to present another consideration to the conjecture. Unfortunately, we can not solve the problem completely; but the author feels that the results obtained here indicate strongly that the conjecture will be true for all separable C^*-algebras. Throughout the present paper, we shall deal with separable C^*-algebras only. The main tool to attack the problem is the reduction theory. As corollaries of our results, we shall show: (1) Let $\mathfrak{A}$ be a separable C^*-algebra and let $\mathfrak{B}$ be a uniformly hyperfinite C^*-subalgebra of $\mathfrak{A}$. Suppose that $\mathfrak{B}$ separates $P(\mathfrak{A})\cup(0)$, then $\mathfrak{A}=\mathfrak{B}$; (2) A new proof of Kaplansky's theorem in the separable case; (3) Let $\mathfrak{A}$ be a separable C^*-algebra and let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$. Suppose that there exists a *-representation $\{\pi,\mathfrak{H}\}$ of $\mathfrak{A}$ such that $\overline{\pi(\mathfrak{B})}\subsetneqq\overline{\pi(\mathfrak{A})}$ and the commutant of $\pi(\mathfrak{B})$ is hyperfinite, where $\overline{\pi(\cdot)}$ is the weak closure of $\pi(\cdot)$. Then, $\mathfrak{B}$ can not separate $P(\mathfrak{A})\cup(0)$; (4) Let $\mathfrak{A}$ be a separable C^*-algebra and let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$. Suppose that there exists a *-representation $\{\pi,\mathfrak{H}\}$ of $\mathfrak{A}$ such that $\overline{\pi(\mathfrak{A})}$ is a finite W^*-agebra and $\overline{\pi(\mathfrak{B})}\subsetneqq\overline{\pi(\mathfrak{A})}$, where $\overline{\pi(\cdot)}$ is the weak closure of $\pi(\cdot)$. Then, $\mathfrak{B}$ can not separate $P(\mathfrak{A})\cup(0)$.

2. Theorems. Let $\mathfrak{A}$ be a C^*-algebra and let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$. Let $P(\mathfrak{A})$ be the set of all pure states of $\mathfrak{A}$, and let 0 be the identically zero function on $\mathfrak{A}$. Throughout this section, we shall assume that $\mathfrak{B}$ separates $P(\mathfrak{A})\cup(0)$—namely, for any two different $\varphi_1,\varphi_2\in P(\mathfrak{A})\cup(0)$, there exists an element b such that $\varphi_1(b)\neq\varphi_2(b)$.

If $\mathfrak{A}$ has not the unit, we shall consider the C^*-algebra $\mathfrak{A}_1=\mathfrak{A}+\lambda 1$ and the subalgebra $\mathfrak{B}_1=\mathfrak{B}+\lambda 1$ obtained by adjoining the unit 1, where λ are complex numbers. Any pure state φ on $\mathfrak{A}$ can be uniquely extended to a pure state $\widetilde{\varphi}$ on $\mathfrak{A}_1$; therefore $P(\mathfrak{A}+\lambda 1)=\widetilde{P(\mathfrak{A})}+\lambda\varphi_0$, where φ_0 is the pure state of $\mathfrak{A}_1$ such that $\varphi_0(\mathfrak{A})=0$. Then, clearly $\mathfrak{B}_1$ separates $P(\mathfrak{A}_1)\cup(0)$; therefore it is enough to assume that $\mathfrak{A}$ has the unit 1.

LEMMA 1. *$\mathfrak{B}$ contains the unit* 1.

PROOF. Suppose that $1\notin\mathfrak{B}$. Then $\|b+1\|\geqq 1$ for $b\in\mathfrak{B}$—in fact, if $\|b+1\|<1$, $-b$ is invertible and $(-b)^{-1}\in\mathfrak{B}$; hence $1\in\mathfrak{B}$. Therefore, there exists a bounded linear functional f on $\mathfrak{A}$ such that $f(\mathfrak{B})=0$ and $\|f\|=f(1)=1$; hence f is a state (cf. [4], [11]). Let $\mathfrak{J}=\{x\,|\,f(x^*x)=0, x\in\mathfrak{A}\}$, then $\mathfrak{J}$ is a closed left ideal of $\mathfrak{A}$ and $\mathfrak{B}\subset\mathfrak{J}$. Let $\mathfrak{L}$ be a maximal left ideal of $\mathfrak{A}$ such that $\mathfrak{J}\subset\mathfrak{L}$, then there exists a pure state φ on $\mathfrak{A}$ such that $\mathfrak{L}=\{x\,|\,\varphi(x^*x)=0, x\in\mathfrak{A}\}$(cf. [4], [8]); this implies that $\mathfrak{B}$ can not separate φ and 0. This is a contradiction and completes the proof.

Henceforward, we shall assume that $\mathfrak{A}$ has the unit and so $\mathfrak{B}$ contains the unit. In this case, the separation of $P(\mathfrak{A})\cup(0)$ by $\mathfrak{B}$ is equivalent to the separation of $P(\mathfrak{A})$ by $\mathfrak{B}$.

DEFINITION 1. A W^*-algebra M is said to be atomic, if it is a direct sum of type I-factors.

DEFINITION 2. Let A be a C^*-algebra and let $\{\pi,\mathfrak{H}\}$ be a $*$-representation of A on a Hilbert space $\mathfrak{H}$. By $\overline{\pi(A)}$, we shall denote the weak closure of $\pi(A)$ on $\mathfrak{H}$. The representation $\{\pi,\mathfrak{H}\}$ is called to be atomic, if the W^*-algebra $\overline{\pi(A)}$ is atomic.

DEFINITION 3. Let φ be a state on a C^*-algebra A, $\{\pi_\varphi,\mathfrak{H}_\varphi\}$ the $*$-representation of A on a Hilbert space $\mathfrak{H}_\varphi$ constructed via φ. φ is called to be atomic, if the representation $\{\pi_\varphi,\mathfrak{H}_\varphi\}$ is atomic.

LEMMA 2. *Let φ_1,φ_2 be two states on $\mathfrak{A}$ such that the restriction $\varphi_1|\mathfrak{B}$, $\varphi_2|\mathfrak{B}$ on $\mathfrak{B}$ are atomic. Suppose that $\varphi_1=\varphi_2$ on $\mathfrak{B}$, then $\varphi_1=\varphi_2$ on $\mathfrak{A}$.*

PROOF. Put $\varphi=\dfrac{\varphi_1+\varphi_2}{2}$ and consider the $*$-representation $\{\pi_\varphi,\mathfrak{H}_\varphi\}$ of $\mathfrak{A}$. Let $\varphi(x)=\langle\pi_\varphi(x)\xi,\xi\rangle$ for $x\in\mathfrak{A}$, where $\langle\,,\rangle$ is the inner product of $\mathfrak{H}_\varphi$ and ξ is a vector in $\mathfrak{H}_\varphi$, and let e' be the projection of $\mathfrak{H}_\varphi$ onto the closed subspace $[\pi_\varphi(\mathfrak{B})\xi]$ generated by $\pi_\varphi(\mathfrak{B})\xi$; then the representation $b\rightarrow\pi_\varphi(b)e'(b\in\mathfrak{B})$ is atomic. Let z be the central envelope of e' in the commutant $\pi_\varphi(\mathfrak{B})'$ of $\pi_\varphi(\mathfrak{B})$, then the mapping $yz \to ye'$ of $\overline{\pi_\varphi(\mathfrak{B})}z$ onto $\overline{\pi_\varphi(\mathfrak{B})}e'$ is a *-isomorphism; hence $\overline{\pi_\varphi(\mathfrak{B})}$ contains a direct summand of an atomic W^*-algebra. Let p' be a minimal projection in $\pi_\varphi(\mathfrak{B})'$, then $b \to \pi_\varphi(b)p'(b \in \mathfrak{B})$ is irreducible. Take η $(\|\eta\|=1) \in p'\mathfrak{H}_\varphi$ and consider a state $\psi_0(x) = <\pi_\varphi(x)\eta, \eta>$ for $x \in \mathfrak{A}$. Then, $\psi_0|\mathfrak{B}$ is pure; we shall show that ψ_0 is pure on $\mathfrak{A}$. Let $\Gamma = \{\psi | \psi = \psi_0$ on $\mathfrak{B}$, ψ states on $\mathfrak{A}\}$, then Γ is a $\sigma(\mathfrak{A}^*, \mathfrak{A})$-compact convex set in $\mathfrak{A}^*$, where $\mathfrak{A}^*$ is the dual Banach space of $\mathfrak{A}$. Arbitrary extreme point in Γ is also extreme in the state space of $\mathfrak{A}$; hence it is pure. If Γ contains two points, there are two different pure states ψ_1, ψ_2 on $\mathfrak{A}$ such that $\psi_1 = \psi_2$ on $\mathfrak{B}$; hence Γ consists of only one point and it is pure.

Now suppose that $p'\mathfrak{H}_\varphi \subsetneqq [\pi_\varphi(\mathfrak{A})\eta]$, and let V be the orthocomplement of $p'\mathfrak{H}_\varphi$ in $[\pi_\varphi(\mathfrak{A})\eta]$. Let $\xi_1(\neq 0) \in p'\mathfrak{H}_\varphi$, $\xi_2(\neq 0) \in V$ and $\|\xi_1+\xi_2\| = 1$. Then, $g_1(x) = <\pi_\varphi(x)(\xi_1+\xi_2), (\xi_1+\xi_2)>$ and $g_2(x) = <\pi_\varphi(x)(\xi_1-\xi_2), (\xi_1-\xi_2)>$ for $x \in \mathfrak{A}$ are pure states of $\mathfrak{A}$ and $g_1 = g_2$ on $\mathfrak{B}$. Hence $g_1 = g_2$ on $\mathfrak{A}$. Since the restriction of $\pi_\varphi(\mathfrak{A})$ on $[\pi_\varphi(\mathfrak{A})\eta]$ is irreducible, $\xi_1+\xi_2 = \lambda(\xi_1-\xi_2)$ for some complex number $\lambda(|\lambda| = 1)$. This is a contradiction; hence $[\pi_\varphi(\mathfrak{A})\eta] = [\pi_\varphi(\mathfrak{B})\eta]$ and so $p' \in \pi_\varphi(\mathfrak{A})'$. Let c be the greatest central projection of $\pi_\varphi(\mathfrak{B})'$ such that $\pi_\varphi(\mathfrak{B})'c$ is atomic; then any non-zero projection of $\pi_\varphi(\mathfrak{B})'c$ is a sum of mutually orthogonal minimal projections; hence $c \in \pi_\varphi(\mathfrak{A})'$.

Since $\xi \in c\mathfrak{H}_\varphi$, $[\pi_\varphi(\mathfrak{A})\xi] \subset c\mathfrak{H}_\varphi$; hence $c\mathfrak{H}_\varphi = \mathfrak{H}_\varphi$ and so $c = 1_{\mathfrak{H}_\varphi}$, where $1_{\mathfrak{H}_\varphi}$ is the identity operator on $\mathfrak{H}_\varphi$; therefore $\pi_\varphi(\mathfrak{B})' \subset \pi_\varphi(\mathfrak{A})'$ and so $\overline{\pi_\varphi(\mathfrak{B})} = \overline{\pi_\varphi(\mathfrak{A})}$. Since $\varphi_1, \varphi_2 \leqq 2\varphi$, there exists vectors η_1, η_2 such that $\varphi_1(x) = <\pi_\varphi(x)\eta_1, \eta_1>$ and $\varphi_2(x) = <\pi_\varphi(x)\eta_2, \eta_2>$ for $x \in \mathfrak{A}$. For $a \in \mathfrak{A}$, there exists a direct set $\{\pi_\varphi(b_\alpha)\}$ $(b_\alpha \in \mathfrak{B})$ such that $\pi_\varphi(b_\alpha) \to \pi_\varphi(a)$(strongly); hence $\varphi_1(b_\alpha) \to \varphi_1(a)$ and $\varphi_2(b_\alpha) \to \varphi_2(a)$; $\varphi_1(b_\alpha) = \varphi_2(b_\alpha)$ implies $\varphi_1(a) = \varphi_2(a)$. This completes the proof.

LEMMA 3. *Let φ_1, φ_2 be two states on $\mathfrak{A}$ and suppose that one of them is atomic and $\varphi_1 = \varphi_2$ on $\mathfrak{B}$, then $\varphi_1 = \varphi_2$ on $\mathfrak{A}$.*

PROOF. Suppose that φ_1 is atomic. Consider the *-representation $\{\pi_{\varphi_1}, \mathfrak{H}_{\varphi_1}\}$ of $\mathfrak{A}$, then $\pi_{\varphi_1}(\mathfrak{A})$ is atomic; hence, there exists a family of mutually orthogonal minimal projections $(e_i' | i = 1, 2, \cdots)$ in $\pi_\varphi(\mathfrak{A})'$ such that $\sum_i e_i' = 1_{\mathfrak{H}_{\varphi_1}}$. Let $\varphi_1(x) = <\pi_{\varphi_1}(x)\xi, \xi>$, then $\varphi_1(x) = \sum_i <\pi_{\varphi_1}(x)e_i'\xi, e_i'\xi> = \sum \|e_i'\xi\|^2 <\pi_{\varphi_1}(x)\frac{e_i'\xi}{\|e_i'\xi\|}, \frac{e_i'\xi}{\|e_i'\xi\|}>$.

Since $<\pi_{\varphi_1}(x)\frac{e_i'\xi}{\|e_i'\xi\|}, \frac{e_i'\xi}{\|e_i'\xi\|}>$ is pure, its restriction on $\mathfrak{B}$ is also pure (cf. the proof of Lemma 2); hence $\varphi_1|\mathfrak{B}$ is atomic and so by Lemma 2, $\varphi_1 = \varphi_2$ on $\mathfrak{A}$. This completes the proof.

Now we shall explain some results of the reduction theory (cf. [3], [11], [12]). Let M be a type I W^*-algebra on a separable Hilbert space, M_* the predual of M. Then, $M=\sum_{i=1}^{\infty}\oplus M_i$, where M_i is a homogenuous type I_{n_i} W^*-algebra ($n_i \leqq \aleph_0$). Moreover, $M_i=B_i\otimes Z_i$, where B_i is a type I_{n_i}-factor, and Z_i is the center of M_i. Let B_{i*} be the predual of B_i, then we can consider the weak *-topology $\sigma(B_i, B_{i*})$ on B_i.

Then, we have the realization $B_i\otimes Z_i=L^\infty(B_i, \Omega_i, \mu_i)$, where (Ω_i, μ_i) is a measure space with a probability measure μ_i and $L^\infty(B_i, \Omega_i, \mu_i)$ is the W^*-algebra of all essentially bounded B_i-valued weakly*-measurable functions on Ω_i. For $a\in B_i\otimes Z_i$, the corresponding element of $L^\infty(B_i, \Omega_i, \mu_i)$ is denoted by $\int a(t)$, then $\|a\|=\text{ess. }\sup_{t\in\Omega_i}\|a(t)\|$ and $a_1+a_2=\int a_1(t)+a_2(t)$, $\lambda a_1=\int \lambda a_1(t)$, $a_1a_2=\int a_1(t)a_2(t)$ and $a_1{}^*=\int a_1(t)^*$ for $a_1, a_2\in B_i\otimes Z_i$ and λ are complex numbers.

Moreover the predual of $L^\infty(B_i, \Omega_i, \mu_i)=L^1(B_{i*}, \Omega_i, \mu_i)$, where $L^1(B_{i*}, \Omega_i, \mu_i)$ is the Banach space of all B_{i*}-valued Bochner integrable functions f on Ω_i with the norm $\|f\|=\int\|f(t)\|d\mu_i(t)$. Therefore, we have the realization $M_{i*}=L^1(B_{i*}, \Omega_i, \mu_i)$ For $g\in M_{i*}$, the corresponding element in $L^1(B_{i*}, \Omega_i, \mu_i)$ is denoted by $\int g(t)$. Then we have: $\|g\|=\int\|g(t)\|d\mu_i(t)$, $g_1+g_2=\int g_1(t)+g_2(t)$, $\lambda g_1=\int \lambda g_1(t)$, and if φ is a normal state on M_i, $\varphi(t)$ is a normal state on B_i for almost all t; moreover let $\mathcal{D}$ be a separable C^*-subalgebra of M_i, then we can choose a null set Q_i such that $d\to d(t)$ $(d\in\mathcal{D})$ is a *-homomorphism of $\mathcal{D}$ into B_i for all $t\in\Omega_i-Q_i$; moreover, if the W^*-subalgebra $(\mathcal{D}, Z_i)$ of M_i generated by $\mathcal{D}$ and Z_i coincides with M_i, the weak closure $\overline{\mathcal{D}(t)}=B_i$ for all $t\in\Omega_i-Q_i$, where $\mathcal{D}(t)=\{d(t)\,|\,d\in\mathcal{D}\}$ and $\overline{\mathcal{D}(t)}$ is the weak closure of $\mathcal{D}(t)$.

Since $M=\sum_{i=1}^{\infty}\oplus M_i$, by considering the direct sum $\left(\Omega=\bigcup_{i=1}^{\infty}\Omega_i,\ \mu=\sum_{i=1}^{\infty}\oplus\mu_i\right)$ of the measure spaces (Ω_i, μ_i), M can be realized as the W^*-algebra of vector valued functions $\int x(t)$ such that $x_i\in L^\infty(B_i, \Omega_i, \mu_i)$, $\|x\|=\sup\|x_i\|$, where x_i is the restriction of x on Ω_i. This realization will be denoted by $M=\sum_{i=1}^{\infty}\oplus L^\infty(B_i, \Omega_i, \mu_i)$. Now let $\mathcal{E}$ be a separable C^*-subalgebra of M such that the W^*-subalgebra of M generated by $\mathcal{E}$ and Z coincides with M, where Z is the center of M. Then $\mathcal{E}z_i$ and Z_i generate M_i, where z_i is the identity of M_i; hence there exists a null set Q in Ω such that $a\to a(t)$ $(a\in\mathcal{E})$ is a *-homomorphism and $\overline{\mathcal{E}(t)}=B_i$ for all $t\in\Omega_i-Q$ and all i.

Henceforward, the algebra $\mathfrak{A}$ will be assumed to be separable. Let $\{\pi, \mathfrak{H}\}$

be a *-representation of $\mathfrak{A}$ on a separable Hilbert space $\mathfrak{H}$. Put $\mathfrak{A}_0 = \pi(\mathfrak{A})$ and $\mathfrak{B}_0 = \pi(\mathfrak{B})$ and let $\mathfrak{A}_0'$ (resp. $\mathfrak{B}_0'$) be the commutant of $\mathfrak{A}_0$ (resp. $\mathfrak{B}_0$). Let C be a maximal abelian *-subalgebra of $\mathfrak{A}_0'$, then the W^*-algebra $(\mathfrak{A}_0, C)$ generated by $\mathfrak{A}_0$ and C is of type I and C is the center of $(\mathfrak{A}_0, C)$, because $(\mathfrak{A}_0, C)' = \mathfrak{A}_0' \cap C = C$.

By putting $(\mathfrak{A}_0, C) = M$, we can apply the reduction theory.

THEOREM 1. *Let T be a linear mapping of $\mathfrak{A}_0$ into $(\mathfrak{A}_0, C)$ such that (α) $\|T(x)\| \leqq \|x\|$ for $x \in \mathfrak{A}_0$; (β) $T(y) = y$ for $y \in \mathfrak{B}_0$. Then, $T(x) = x$ for $x \in \mathfrak{A}_0$.*

PROOF. Suppose that $T(x_0) \neq x_0$ for some $x_0 \in \mathfrak{A}_0$. Then, there exists a normal state ψ of $(\mathfrak{A}_0, C)$ such that $\psi(T(x_0)) \neq \psi(x_0)$. $(\mathfrak{A}_0, C) = \sum_{i=1} \oplus L^\infty(B_i, \Omega_i, \mu_i)$. Now let D be the C^*-subalgebra of $(\mathfrak{A}_0, C)$ generated by $\mathfrak{A}_0$ and $T(x_0)$, then D is separable.

By the previous considerations, we can assume that $x \to x(t)$ $(x \in D)$ is a *-homomorphism of D into B_i and $\overline{\mathfrak{A}_0(t)} = B_i$ for all $t \in \Omega_i - \mathfrak{N}$ with $\mu(\mathfrak{N}) = 0$, where $\mathfrak{A}_0(t) = \{x(t) \mid x \in \mathfrak{A}_0\}$.

Let $\psi = \int \psi(t)$, then $\psi(x_0) = \int \psi(t)(x_0(t))d\mu(t)$ and $\psi(T(x_0)) = \int \psi(t)(T(x_0)(t))d\mu(t)$. Since $\psi(x_0) \neq \psi(T(x_0))$, there exists a set $\mathfrak{M}$ with $\mu(\mathfrak{M}) > 0$ such that $\psi(t)(x_0(t)) \neq \psi(t)(T(x_0)(t))$ for all $t \in \mathfrak{M}$. Therefore, there exists a t_0 such that $\psi(t_0)$ is a positive linear functional on B_{i0} and $\psi(t_0)(x_0(t_0)) \neq \psi(t_0)(T(x_0)(t_0))$, $x \to x(t_0)$ $(x \in D)$ is a *-homomorphism of D into B_{i0} and $\overline{\mathfrak{A}_0(t_0)} = B_{i0}$. Now we shall define a linear functional ψ_1 on $\mathfrak{A}$ as follows: $\psi_1(a) = \psi(t_0)(\pi(a)(t_0))$ for $a \in \mathfrak{A}$. Then, ψ_1 is an atomic state on $\mathfrak{A}$. *Let* $x_0 = \pi(a_0)$ for some $a_0 \in \mathfrak{A}$; we shall define a linear functional ψ_2' on $\mathfrak{B} + \lambda a_0$ (λ complex numbers) as follows: $\psi_2'(b + \lambda a_0) = \psi(t_0)(\pi(b)(t_0) + \lambda T(x_0)(t_0))$ for $b \in \mathfrak{B}$. Then,

$$|\psi_2'(b + \lambda a_0)| \leqq \|\psi(t_0)\| \|\pi(b) + \lambda T(x_0)\| = \|\psi(t_0)\| \|T(\pi(b) + \lambda\pi(a_0))\|$$
$$\leqq \|\psi(t_0)\| \|\pi(b) + \lambda\pi(a_0)\| \leqq \|\psi(t_0)\| \|b + \lambda a_0\| .$$

Therefore, ψ_2' is well-defined and bounded. Let ψ_2 be a linear functional on $\mathfrak{A}$ such that $\|\psi_2\| = \|\psi_2'\|$ and $\psi_2 = \psi_2$ on $\mathfrak{B} + \lambda a_0$. Since $\psi_2(1) = \psi_2'(1) = \|\psi(t_0)\|$, ψ_2 is positive and clearly $\psi_1 = \psi_2$ on $\mathfrak{B}$. Therefore by Lemma 3, $\psi_1 = \psi_2$ on $\mathfrak{A}$; hence $\psi_1(a_0) = \psi(t_0)(\pi(a_0)(t_0)) = \psi(t_0)(x_0(t_0)) = \psi_2(a_0) = \psi(t_0)(T(x_0)(t_0))$. This is a contradiction and completes the proof.

Let $B(\mathfrak{H})$ be the W^*-algebra of all bounded operators on $\mathfrak{H}$. For any $w \in B(\mathfrak{H})$, let $K(w)$ be the weakly closed convex subset of $B(\mathfrak{H})$ generated by $\{u^*wu \mid u \in C_u\}$, where C_u is the set of all unitary elements in C. A family of weakly continuous linear mappings $\{w \to u^*wu \mid u \in C_u\}$ on $B(\mathfrak{H})$ is commutative; hence by the theorem of Kakutani-Markoff (cf. [2]), $K(w)$ contains at least one fixed point w_0—namely, $u^*w_0u = w_0$ for all $u \in C_u$; hence $w_0 \in C' = (\mathfrak{A}_0, C)$. Therefore, there exists a projection P with norm one of $B(\mathfrak{H})$ onto $(\mathfrak{A}_0, C)$ (cf. [14]).

Now we shall show

THEOREM 2. *For $x \in \mathfrak{A}_0$, let $\Gamma(x)$ be the weakly closed convex subset of $B(\mathfrak{H})$ generated by $\{u'^*xu' \mid u' \in \mathfrak{B}'_{0,u}\}$, where $\mathfrak{B}'_{0,u}$ is the set of all unitary elements of the commutant $\mathfrak{B}'_0$ of $\mathfrak{B}_0$. Then, $P(r) = x$ for all $r \in \Gamma(x)$.*

PROOF. Let $L(B(\mathfrak{H}))$ be the algebra of all bounded operators of $B(\mathfrak{H})$ into $B(\mathfrak{H})$. Then, $L(B(\mathfrak{H}))$ is the dual of $B(\mathfrak{H}) \otimes_\gamma B(\mathfrak{H})_*$, where γ is the greatest cross norm and $B(\mathfrak{H})_*$ is the predual of $B(\mathfrak{H})$ (cf. [7]). We shall consider the weak *-topology $\sigma(L(B(\mathfrak{H})), B(\mathfrak{H}) \otimes_\gamma B(\mathfrak{H})_*)$ on $L(B(\mathfrak{H}))$. Then, the unit sphere S of $L(B(\mathfrak{H}))$ is compact. The linear mapping $V_{u'} : w \to u'^*wu'$ ($w \in B(\mathfrak{H})$) belongs to S; let S_0 be the weakly *-closed convex subset of S generated by $\{V_{u'} \mid u' \in \mathfrak{B}'_{0,u}\}$, then for arbitrary $r \in \Gamma(x)$, there exists a $V \in S_0$ such that $V(x) = r$.

Now, consider a linear mapping $d \to P(V(d))$ ($d \in \mathfrak{A}_0$) of $\mathfrak{A}_0$ into $(\mathfrak{A}_0, C)$, then $P(V(y)) = P(y) = y$ for $y \in \mathfrak{B}_0$; hence by Theorem 1, $P(V(x)) = P(r) = x$. This completes the proof.

COROLLARY 1. *Let $\overline{\mathfrak{B}}_0$ be the weak closure of $\mathfrak{B}_0$, then $\|w-r\| = \|w-x\|$ for $w \in \overline{\mathfrak{B}}_0$ and $r \in \Gamma(x)$, where $x \in \mathfrak{A}_0$.*

PROOF. For $u' \in \mathfrak{B}'_{0u}$, $\|w - u'xu'^*\| = \|u'^*wu' - x\| = \|w - x\|$; therefore $\|w - \sum_{i=1}^n \lambda_i u_i'^* x u_i'\| \leqq \|w-x\|$, where $\lambda_i \geqq 0$ and $\sum_{i=1}^n \lambda_i = 1$, $u_i' \in \mathfrak{B}'_{0u}$; hence $\|w-r\| \leqq \|w-x\|$.

On the other hand, if $\|w_0 - r_0\| < \|w_0 - x\|$ for some $w_0 \in \overline{\mathfrak{B}}_0$ and $r_0 \in \Gamma(x)$, then $\|P(w_0 - r_0)\| = \|w_0 - P(r_0)\| \leqq \|w_0 - r_0\|$. But, $w_0 - P(r_0) = w_0 - x$. This is a contradiction and completes the proof.

COROLLARY 2. *$\|v-r\| \geqq \|v-x\|$ for $v \in (\mathfrak{A}_0, C)$ and $r \in \Gamma(x)$, where $x \in \mathfrak{A}_0$.*

The proof is quite similar with the second part of the proof of Corollary 1.

3. Applications. We shall show some applications of the results in the section 2.

DEFINITION 4. Let M be a W^*-algebra. M is called to be hyperfinite, if there exists an increasing sequence of type I_{n_i}-factors $\{M_i\}$ $(n_i < +\infty)$ containing the unit of M in M such that $\overline{\bigcup_{i=1}^{\infty} M_i} = M$, where $\overline{(\cdot)}$ is the weak closure of $(\cdot)$.

PROPOSITION 1. *Let $\mathfrak{A}$ be a separable C^*-algebra and $\mathfrak{B}$ a C^*-subalgebra of $\mathfrak{A}$. Suppose that there exists a $*$-representation $\{\pi, \mathfrak{H}\}$ of $\mathfrak{A}$ such that $\overline{\pi(\mathfrak{B})} \subsetneqq \overline{\pi(\mathfrak{A})}$ and the commutant $\pi(\mathfrak{B})'$ of $\pi(\mathfrak{B})$ is hyperfinite. Then, $\mathfrak{B}$ can* not separate $P(\mathfrak{A}) \cup (0)$.

PROOF. Suppose that $\mathfrak{B}$ separates $P(\mathfrak{A}) \cup (0)$. Put $\mathfrak{A}_0 = \pi(\mathfrak{A})$ and $\mathfrak{B}_0 = \pi(\mathfrak{B})$. By the result of Schwartz (cf. [14]), $\Gamma(x) \cap \overline{\mathfrak{B}}_0 \neq (\phi)$ for $x \in \mathfrak{A}_0$; hence by Corollary 1, $\inf_{w \in \overline{\mathfrak{B}}_0} \|x - w\| = 0$ and so $x \in \overline{\mathfrak{B}}_0$. This is a contradiction and completes the proof.

DEFINITION 5. Let A be a C^*-algebra. A is called to be uniformly hyperfinite, if there exists an increasing sequence of type I_{n_i}-factors $\{A_i\}$ $(n_i < +\infty)$ containing the unit of A in A such that the uniform closure of $\bigcup_{i=1}^{\infty} A_i = A$.

PROPOSITION 2. *Let $\mathfrak{A}$ be a separable C^*-algebra and let $\mathfrak{B}$ be a uniformly hyperfinite C^*-subalgebra of $\mathfrak{A}$. Suppose that $\mathfrak{B}$ separates $P(\mathfrak{A}) \cup (0)$, then $\mathfrak{A} = \mathfrak{B}$.*

PROOF. Suppose that $\mathfrak{B} \subsetneqq \mathfrak{A}$ and let f be a bounded selfadjoint linear functional on $\mathfrak{A}$ such that $f(\mathfrak{B}) = 0$ and $f \neq 0$. Let $f = f^+ - f^-$ be the orthogonal decomposition such that $f^+, f^- \geqq 0$, and $\|f^+\| + \|f^-\| = \|f\|$. Put $\varphi = f^+ + f^-$ and take the $*$-representation $\{\pi_\varphi, \mathfrak{H}_\varphi\}$ of $\mathfrak{A}$ as the $\{\pi, \mathfrak{H}\}$ in §2. Then, $\overline{\mathfrak{B}}_0 \subsetneqq \overline{\mathfrak{A}}_0$. Since $\mathfrak{B}_0$ is uniformly hyperfinite, there exists an increasing sequence of type I_{n_i}-factors (B_i) $(n_i < +\infty)$ in $\mathfrak{B}_0$ such that the uniform closure of $\bigcup_{i=1}^{\infty} B_i = \mathfrak{B}_0$. We can easily find a projection Q_i with norm 1 of $B(\mathfrak{H}_\varphi)$ onto B_i, because $B(\mathfrak{H}_\varphi) = B_i \otimes B_i'$. Let Q be an accumulate point of the set $\{Q_i \mid i = 1, 2, \cdots\}$ in $L(B(\mathfrak{H}_\varphi))$ with $\sigma(L(B(\mathfrak{H}_\varphi)), B(\mathfrak{H}_\varphi) \otimes_\gamma B(\mathfrak{H}_\varphi)_*)$, then clearly $Q(y) = y$ for $y \in \mathfrak{B}_0$; moreover $Q(\mathfrak{A}_0) \subset \overline{\left(\bigcup_{i=1}^{\infty} B_i\right)} = \overline{\mathfrak{B}}_0 \subset (\mathfrak{A}_0, C)$; hence by Theorem 1, $Q(x) = x$ for $x \in \mathfrak{A}_0$ and so $\mathfrak{A}_0 \subset \overline{\mathfrak{B}}_0$. This is a contradiction and completes the proof.

PROPOSITION 3. *Let $\mathfrak{A}$ be a separable C^*-algebra and let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$. Suppose that there exists a *-representation $\{\pi, \mathfrak{H}\}$ of $\mathfrak{A}$ such that $\overline{\pi(\mathfrak{A})}$ is a finite W^*-algebra and $\overline{\pi(\mathfrak{B})} \subsetneqq \overline{\pi(\mathfrak{A})}$. Then, $\mathfrak{B}$ can not separate $P(\mathfrak{A}) \cup (0)$.*

PROOF. Suppose that $\mathfrak{B}$ separates $P(\mathfrak{A}) \cup (0)$. By the result of Umegaki (cf. [15]), there exists a projection Q with norm 1 of $\overline{\pi(\mathfrak{A})}$ onto $\overline{\pi(\mathfrak{B})}$. On the other hand, by Theorem 1, $Q(\pi(a)) = \pi(a)$ for $a \in \mathfrak{A}$; hence $\overline{\pi(\mathfrak{A})} = \overline{\pi(\mathfrak{B})}$. This is a contradiction and completes the proof.

PROPOSITION 4 (Kaplansky [9]). *Let $\mathfrak{A}$ be a separable C^*-algebra and let $\mathfrak{B}$ be a type I C^*-subalgebra of $\mathfrak{A}$. Suppose that $\mathfrak{B}$ separates $P(\mathfrak{A}) \cup (0)$, then $\mathfrak{A} = \mathfrak{B}$.*

PROOF. Suppose that $\mathfrak{B} \subsetneqq \mathfrak{A}$. Take a *-representation $\{\pi, \mathfrak{H}\}$ of $\mathfrak{A}$ such that $\overline{\pi(\mathfrak{A}) \supsetneqq \pi(\mathfrak{B})}$. Since $\mathfrak{B}$ is a type I C^*-algebra, $\pi(\mathfrak{B})'$ is a type I W^*-algebra, By the theorem of Kakutani-Markoff, the structure theorem of type I W^*-algebras and the considerations of Schwartz (cf. [14]), we can easily see that $\Gamma(x) \cap \overline{\mathfrak{B}}_0 \neq (\phi)$ for $x \in \mathfrak{A}_0$; hence by Corollary 1, $x \in \overline{\mathfrak{B}}_0$. This is a contradiction and completes the proof.

REFERENCES

[1] C. AKEMANN, The Stone-Weierstrass problem for C^*-algebras, to appear.

[2] N. BOURBAKI, Espaces vectoriels topologiques, chap. I-II, Actualités Sci. Indust., Hermann, Paris, 1953.

[3] J. DIXMIER, Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann), Gauthier-Villars, Paris, 1957.

[4] J. DIXMIER, Les C^*-algèbres et leurs représentations, Gauthier-Villars, Paris, 1964.

[5] J. GLIMM, A Stone-Weierstrass theorem for C^*-algebras, Ann. of Math., 72(1960), 216-244.

[6] J. GLIMM, Type I C^*-algebras, Ann. of Math., 73(1961), 572-612.

[7] A. GROTHENDIECK, Product tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc., 16.

[8] R. KADISON, Irreducible operator algebras, Proc. Nat. Acad. Sci. U. S. A., 43(1957), 273-276.

[9] I. KAPLANSKY, The structure of certain operator algebras, Trans. Amer. Math. Soc., 70(1951), 219-255.

[10] J. RINGROSE, On subalgebras of a C^*-algebra, Pacific J. Math., 15(1965), 1377-1382.

[11] S. SAKAI, The theory of W^*-algebras, Lecture notes, Yale University, 1962.

[12] S. SAKAI, On the reduction theory of von Neumann, Bull. Amer. Math. Soc., 70(1964), 393-398.

[13] S. SAKAI, On a characterization of type I C^*-algebras, Bull. Amer. Math. Soc., 72(1966), 508-512.

[14] J. SCHWARTZ, Two finite, non-hyperfinite, non isomorphic factors, Comm. Pure Appl. Math., 16(1963), 19-26.
[15] H. UMEGAKI, Conditional expectation in an operator algebra, I, Tôhoku Math. J., 6(1954), 177-181.

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