DE GRUYTER DOI 10.1515/jmbm-2013-0006 — J Mech Behav Mater 2012; 21(5-6): 181–186

Konstantinos A. Lazopoulos * and Anastasios K. Lazopoulos

On the strain gradient bending deformations

Abstract: Bending deformations are reviewed in the context of strain gradient linear elasticity, considering the complete set of strain gradient components. It is well understood that conventional bending deformations depend on the collective uniaxial extension of axial fibers resulting in the dependence on the curvature of the neutral geometry of various (linear or surface) structures. Nevertheless, the deformation of each fiber depends not only on the local curvature of the neutral geometry but also on the distance of the fiber from the neutral axis. Hence, the strain gradient tensor of the conventional bending strain should include not only components along the neutral axis but also those on the transverse direction. The problems of bending and buckling, along with geometrically non-linear and post-critical behavior, are reviewed in the context of strain gradient elasticity considering not only conventional bending strain but also the complete components of the strain gradient.

Keywords: bending; buckling; gradient elasticity.

***Corresponding author: Konstantinos A. Lazopoulos,** School of Mathematical Sciences (SEMFE), National Technical University of Athens, 5 Heroes of Polytechnion Avenue, Athens 15573, Greece, e-mail: kolazop@mail.ntua.gr
Anastasios K. Lazopoulos: Mathematical Sciences Department, Hellenic Army Academy, Vari 16673, Greece

1 Introduction

Because size effects have been observed in thin films, micro-electromechanical systems, and nano-electromechanical systems, those microstructures are studied, invoking strain gradient elasticity theories. Further, the thin beam theory has found many applications in the areas of micromechanics and nanomechanics. Altan and Aifantis [1] and Ru and Aifantis [2] have correlated the thin beam theory with strain gradient elasticity theories. The theory of gradient elasticity has been applied to many mechanics problems lifting various singularities in fracture problems [1] and around concentrated forces like the Flamant problem [3]. Further applications in plasticity and dislocation dynamics may be found in ref. [4]. Many researchers [5–7] have presented a Bernoulli thin beam theory that is a straightforward extension of the conventional thin elastic beam theory. Let us point out that Park and Gao [6] and Yang et al. [7] invoked a questionable principle in mechanics of the balance of the moment of rotational momentum. The authors have also presented a series of articles concerning thin structures based on Bernoulli's thin beam theory as well as on the Timoshenko beam assumption [8, 9].

In the above studies, the authors considered not only the gradient strain component along the axis of the beam but also on the transverse direction. New terms appeared in the governing equilibrium of the beam that do not depend on the bending but on the shear stiffness of the beam. These terms highly influence the bending behavior of thin strain gradient elastic beams. In fact, the fiber character of the Bernoulli conventional thin elastic beam breaks down as the couple elasticity effects combine the non-symmetric stress tensor, namely the shearing effects, with the bending moments. To understand the essence of the present ideas, let us consider the simple bending of a beam. It is well-known that bending deformation is the deformation of various fibers axially distributed along the beam (see Figure 1).

Hence, the conventional bending deformation is defined by

$$\varepsilon_{xx}=-y\frac{\partial^2 w}{\partial x^2}$$

Likewise, the strain gradient tensor components are defined by

$$\nabla\varepsilon_{xx}=\left(\varepsilon_{xxx}=-y\frac{\partial^3 w}{\partial x^3},\quad \varepsilon_{yxx}=-\frac{\partial^2 w}{\partial x^2}\right)$$

Considering the equilibrium of a representative element at some point of the cross section (see Figure 1), strain gradient linear elasticity yields the equilibrium diagram with balance stresses [10], $\sigma_{21}=-\frac{2}{3}\frac{\partial\mu_{211}}{\partial x}$, with $\mu_{211}=\frac{\partial W}{\partial\varepsilon_{211}}\neq 0$ (W denotes the strain energy density function), whereas $\sigma_{12}=0$. Nevertheless, there exist a distributed moment vector (e.g., ref. [10] and references quoted therein), $\mu_{13}=-\frac{2}{3}\mu_{211}$. Therefore, the distributed moment μ_{13} contributes into the equilibrium of the bending moment apart from the contribution of the axial stresses due to the

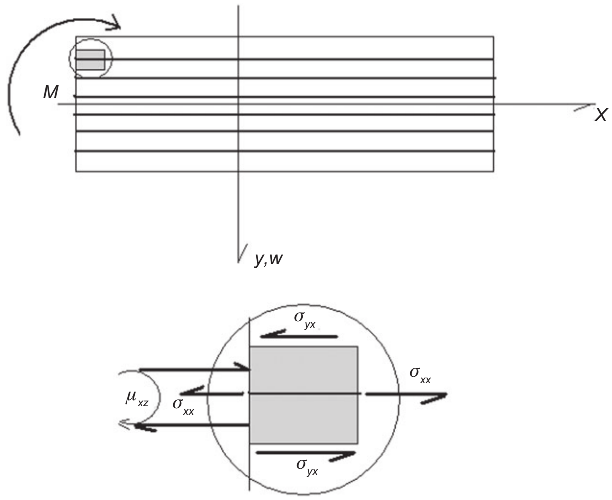

Figure 1 Representative element with the non-symmetric stresses and the distributed moments contributing to the equilibrium of the bending moment.

deformation of the fibers. Consequently, additional terms show up in the analysis of various bending problems, owing to the non-symmetric stress tensor, which is a substantial difference between conventional and strain gradient bending. Further, the absence of those terms in some strain gradient bending theories is considered a serious handicap, as these terms involve only the area instead of the moment of inertia of the bending cross-section. This suggests the primary importance of these terms in very thin bending structures. The influence of the complete set of strain gradient tensor components in bending has been proved by Spencer and Soldatos [11] in the context of continuum mechanics.

The problems that are outlined in the present work under the umbrella of strain gradient elastic bending, considering the complete set of the components of the strain gradient tensor, have also been discussed elsewhere [8, 9, 12–15].

2 Revised bending model of a strain gradient elastic beam

Following Aifantis [4], a linear theory of elasticity with microstructure, equipped with extra gradient terms and corresponding constitutive coefficients, apart from the Lame constants, is adopted. The intrinsic bulk length, g, is the additional constitutive parameter. Indeed, the strain energy density function for the case considered herein is expressed by

$$W=\frac{1}{2}\lambda\varepsilon_{mm}\varepsilon_{nn}+G\varepsilon_{mn}\varepsilon_{nm}+g^2\left(\frac{1}{2}\lambda\varepsilon_{kmm}\varepsilon_{knn}+G\varepsilon_{kmn}\varepsilon_{knm}\right) \tag{1}$$

where ε_{ij} denotes the infinitesimal strain and ε_{ijk} the infinitesimal strain gradient, with

$$\varepsilon_{ij}=\varepsilon_{ji}=\frac{1}{2}\left(\partial_i u_j+\partial_j u_i\right), \qquad \varepsilon_{ijk}=\varepsilon_{ikj}=\partial_i\varepsilon_{kj} \tag{2}$$

and $\mu_i=\mu_i(x_k)$ being the infinitesimal displacement field.

The usual stresses are defined by the relations

$$\tau_{ij}=\frac{\partial W}{\partial\varepsilon_{ij}}=\lambda\varepsilon_{kk}\delta_{ij}+2G\varepsilon_{ij} \tag{3}$$

and the hyperstresses by

$$\mu_{ijk}=\frac{\partial W}{\partial\varepsilon_{ijk}}=g^2\left(\lambda\varepsilon_{inn}\delta_{jk}+2G\varepsilon_{ijk}\right) \tag{4}$$

For the present study, we consider a beam shown in Figure 2. The x-axis is the axis of the beam, whereas the y-axis is the deflection axis.

The elastic line lies on the x-y plane. Considering the Bernoulli-Euler principle, the infinitesimal strain of the beam is defined by

$$\varepsilon_{xx}=-y\frac{\partial^2 w}{\partial x^2} \tag{5}$$

with w denoting the displacement of the elastic line.

For the formulation of the present problem, we need the stress

$$\tau_{xx}=\frac{\partial W}{\partial\varepsilon_{xx}}=E\varepsilon_{xx} \tag{6}$$

where E is the elastic Young's modulus, as well as the hyperstresses

$$\mu_{xxx}=g^2E\varepsilon_{xxx} \tag{7}$$

$$\mu_{yxx}=g^2E\varepsilon_{yxx} \tag{8}$$

Figure 2 Beam axis.

with the corresponding hyperstrains being

$$\varepsilon_{xxx}=-y\frac{\partial^3 w}{\partial x^3} \quad \text{and} \quad \varepsilon_{yxx}=-\frac{\partial^2 w}{\partial x^2} \tag{9}$$

Following the analysis of Lazopoulos and Lazopoulos [8], the equilibrium equation of the strain gradient beam is given by

$$E\left(I+g^2A\right)\frac{\partial^4 w}{\partial x^4}-g^2EI\frac{\partial^6 w}{\partial x^6}+q=0 \tag{10}$$

where I is the moment of inertia, A is the cross-sectional area, and q denotes the distributed load.

Furthermore, the corresponding boundary conditions are

$$V=E\left(I+g^2A\right)\frac{\partial^3 w}{\partial x^3}-g^2EI\frac{\partial^5 w}{\partial x^5} \text{ or } \delta w=0 \text{ at } x=0,L \tag{11}$$

$$M=E\left(I+g^2A\right)\frac{\partial^2 w}{\partial x^2}-g^2EI\frac{\partial^4 w}{\partial x^4} \text{ or } \delta w,_x=0 \text{ at } x=0,L \tag{12}$$

$$m=EIg^2\frac{\partial^3 w}{\partial x^3} \text{ or } \delta w,_{xx}=0 \text{ at } x=0,L \tag{13}$$

where V denotes the shear force, M the bending moment, and m the hypermoment. It should be pointed out that the equilibrium Eq. (10) departs from Aifantis's special theory of gradient elasticity (GRADELA model), as the considered stress tensor in the present case is not symmetric. Owing to that property of the stress tensor, beam bending ceases to be considered as the result of the uniaxial stretching of various beam fibers, but it is a result of a two-dimensional strain gradient elastic configuration, where the bending moment contributes not only to the extension of the set of the ideal fibers, but also interacts with the distributed moments on the cross-section of the beam, due to the asymmetry of the stress tensor. This makes the present bending strain gradient theory quite different not only from the conventional one but also from other strain gradient theories that do not account for the contribution of the asymmetry of the stress tensor. Let us point out that the terms of the equilibrium in Eqs. (10–13), including the cross-sectional area, correspond to the contribution of the asymmetry of the stress tensor. These terms become of primary importance when the beam is thin, as the moment of inertia I is proportional to h^3, while the cross-sectional area A is proportional to the thickness h.

In relation to the discussion of the beam buckling problem of a strain gradient elastic beam, it turns out the governing equation along with the boundary conditions are of the form [9]

$$E\left(I+g^2A\right)\frac{\partial^4 w}{\partial x^4}-g^2EI\frac{\partial^6 w}{\partial x^6}+Pw''=0 \tag{14}$$

for the governing differential equation with the boundary conditions (P denotes the point axial load applied on the right end of the beam)

$$V=\left[-Pw'-E\left(I+g^2A\right)\frac{\partial^3 w}{\partial x^3}+g^2EI\frac{\partial^5 w}{\partial x^5}\right] \text{ or } \delta w=0 \;\; at \;\; x=0,L \tag{15}$$

$$M=E\left(I+g^2A\right)\frac{\partial^2 w}{\partial x^2}-g^2EI\frac{\partial^4 w}{\partial x^4} \text{ or } \delta w'=0 \;\; at \;\; x=0,L \tag{16}$$

$$m=EIg^2\frac{\partial^3 w}{\partial x^3} \text{ or } \delta w''=0 \text{ at } x=0,L \tag{17}$$

It is also evident here the contribution of the asymmetry of the stress tensor as shown by the inclusion of the cross-sectional area A into the various terms.

3 Non-linear beam bending of strain gradient elastic beam

It is obvious that for thin beams, large deformations may be expected. Therefore, the constitutive relations will remain the same, while the effects of large deformations will be introduced through the non-linear equation for the curvature of the elastic curve. Indeed, for inextensible elastic curves, the curvature is given by

$$k=-\frac{w''(x)}{\sqrt{1-w'(x)^2}} \tag{18}$$

where $w(x)$ denotes the y-displacement of the inextensible beam elastic line.

As the displacement is small, the curvature k may be approximated by

$$k=-w''(x)\left(1+\frac{1}{2}w'^2\right)+o\left(w''w'^2\right) \tag{19}$$

In this case, the axial strain is approximated by

$$\varepsilon_{xx}\approx yk=-yw''(x)\left(1+\frac{1}{2}w'^2\right) \tag{20}$$

whereas

$$\varepsilon_{xxx}=-yw'w''^2-y\left(1+\frac{1}{2}w'^2\right)w''' \tag{21}$$

$$\varepsilon_{yxx}=k=-w''\left(1+\frac{1}{2}w'^2\right). \tag{22}$$

By following an analysis similar to that of Lazopoulos et al. [14], and by further adopting Aifantis' special theory of gradient elasticity [1], i.e., by assuming that surface effects are entirely accounted for by the bulk internal length g, the governing equation is given by

$$\begin{aligned}EI_g w^{IV}-g^2EIw^{VI}=&-\tilde{p}(t)\delta(x-t)+\frac{1}{2}\{w''\{-4EI_g w''^2\\&+24EIg^2w'''^2+2g^2E\,Iw''w'''+1.5Eg^2w^{IV}\}\\&+2w'\{EI_g\,w''^2+EIg^2w'''\left(w'''+8w^{IV}\right)\\&+6w''\left(-EI_g w'''+g^2EIw^{V}\right)\}\\&-2w'^2\left(EI_g w^{IV}-g^2EIw^{VI}\right)\}\end{aligned} \tag{23}$$

with $I_g=I+g^2A$ and the boundary conditions

$$\begin{aligned}V=&EI_gw'''-EIg^2w^V-EI_gw'w''^2-EIg^2w''^2w'''-EIg^2w'w'''^2\\&+2EI_gw'w''^2+EI_gw'^2w'''+2EIg^2w''^2w'''+2EIg^2w'w'''^2\\&+2EIg^2w'w''w^{IV}-EIg^2\left(5w''^2w'''+w'\left(2w'''^2+2w''w^{IV}\right)\right)\\&-EIg^2\left(w'''\left(2w''^2+2w'w'''\right)+4w'w''w^{IV}+w'^2w^V\right)\\&\text{or } \delta w=0 \text{ at } x=0,L\end{aligned} \tag{24}$$

$$\begin{aligned}M=&EI_gw''+EI_gw'^2w''+2EIg^2w'w''w'''-EIg^2w^{IV}\\&-4EIg^2w'w''w'''-EIg^2w'^2w^{IV} \text{ or } \delta w_x=0 \text{ at } x=0,L\end{aligned} \tag{25}$$

$$\begin{aligned}m=&EIg^2w'''+EIg^2w'w''^2+EIg^2w'^2w'''\\&\text{or } \delta w_{xx}=0 \text{ at } x=0,L\end{aligned} \tag{26}$$

One may use the successive approximations procedure for evaluating an approximate solution of the non-linear problem. Indeed, if we are looking for a solution of the type

$$w=\varepsilon w_1+\varepsilon^3w_3+o(\varepsilon^3) \tag{27}$$

for the loading

$$\tilde{p}(t)=p_1(t)+\varepsilon^2p_3(t) \tag{28}$$

and

$$M=M_1+\varepsilon^2M_3+o(\varepsilon^2),\ V=V_1+\varepsilon^2V_3+o(\varepsilon^2),\ m=m_1+\varepsilon^2m_3+o(\varepsilon^2)$$

then the solution w_1 for the first-order problem is defined as

$$EI_gw_1^{IV}-g^2EIw_1^{VI}=-p_1(t)\delta(x-t) \tag{29}$$

with the boundary conditions

$$V_1=EI_gw_1'''-EIg^2w_1^V \text{ or } \delta w_1=0 \text{ at } x=0,L \tag{30}$$

$$M_1=EI_gw_1''-EIg^2w_1^{IV} \text{ or } \delta w_{1,x}=0 \text{ at } x=0,L \tag{31}$$

$$m_1=EIg^2w_1''' \text{ or } \delta w_{1,xx}=0 \text{ at } x=0,L \tag{32}$$

As the symmetry of the problem should allow for two symmetrical solutions reflected on the beam axis, the second-order problem is not of relevance. In fact, consideration of the second-order problem breaks the symmetry of the problem.

Proceeding with the definition of the third-order problem, we obtain the following governing equation:

$$\begin{aligned}EI_gw_3^{IV}-g^2EIw_3^{VI}=&-p_3(t)\delta(x-t)+\frac{1}{2}\{\{-4EI_gw_1''w_1''^2+24EIg^2w_1'''^2\\&+2g^2EIw_1''w_1'''+1.5Eg^2w_1^{IV}\}+2w_1'\{EI_g\,w_1''^2+g^2EIw_1'''\\&\left(w_1'''+8w_1^{IV}\right)+w_1''\left(-6EI_gw_1'''+6g^2EIw_1^V\right)\}\\&-2Ew_1'^2\left(I_gw_1^{IV}-g^2Iw_1^{VI}\right)\}\end{aligned} \tag{33}$$

with the boundary conditions

$$\begin{aligned}V_3=&EI_gw_3'''-EIg^2w_3^V-EI_gw_1'w_1''^2-EIl_xw_1'w_1''w_1'''-EIg^2w_1''^2w_1'''\\&-EIg^2w_1'w_1'''^2+2EI_gw_1'w_1''^2+EI_gw_1'^2w_1'''+2EIg^2w_1''^2w_1'''\\&+2EIg^2w_1'w_1'''^2+2EIg^2w_1'w_1''w_1^{IV}-EIg^2\left(5w_1''^2w_1'''+w_1'\right.\\&\left.\left(2w_1'''^2+2w_1''w_1^{IV}\right)\right)-EIg^2\left(w_1'''\left(2w_1''^2+2w_1'w_1'''\right)+4w_1'w_1''w_1^{IV}\right.\\&\left.+w_1'^2w_1^V\right) \text{ or } \delta w_3=0 \text{ at } x=0,L\end{aligned} \tag{34}$$

$$\begin{aligned}M_3=&EI_gw_3''+EI_gw_1'^2w_1''-2g^2EIw_1'w_1''w_1'''-g^2EIw_3^{IV}\\&-EIg^2w_1'^2w_1^{IV} \text{ or } \delta w_{3,x}=0 \text{ at } x=0,L\end{aligned} \tag{35}$$

$$m_3=g^2EI\left(w_3'''+w_1'w_1''^2+w_1'^2w_1'''\right) \text{ or } \delta w_{3,xx}=0 \text{ at } x=0,L \tag{36}$$

4 Post-critical buckling problem of a strain gradient elastic beam

In the present section, the buckling problem of a strain gradient elastic beam along with its post-critical behavior is discussed. First, we list the governing equation along with the corresponding boundary conditions, and then perform a bifurcation analysis. Following Lazopoulos et al. [14], the governing equilibrium equation for the present non-linear buckling problem is expressed by

$$\begin{aligned} EI_g w^{IV} - g^2 EIw^{VI} + Pw'' = &+\frac{1}{2}\Big\{-4EI_g w'' w''^2 + 24EIg^2 w'''^2 \\ &+2EIg^2 w'' w''' + 1.5Eg^2 w^{IV}\Big\} + 2w'\Big\{EI_g w''^2 \\ &+EIg^2 w'''\left(w''' + 8w^{IV}\right) + 6w''\left(-EI_g w''' + EIg^2 w^V\right)\Big\} \\ &-2Ew'^2\left(I_g w^{IV} - g^2 Iw^{VI}\right) \end{aligned} \tag{37}$$

with the boundary conditions

$$\begin{aligned} V = &Pw' + EI_g w''' - EIg^2 w^V - EI_g w' w''^2 - EIg^2 w''^2 w''' \\ &-EIg^2 w' w'''^2 + 2EI_g w' w''^2 + EI_g w'^2 w''' \\ &+2EIg^2 w''^2 w''' + 2EIg^2 w' w'''^2 + 2EIg^2 w' w'' w^{IV} \\ &-EIg^2\left(5w''^2 w''' + w'\left(2w'''^2 + 2w'' w^{IV}\right)\right) \\ &-EIg^2\left(w'''\left(2w''^2 + 2w' w'''\right) + 4w' w'' w^{IV} + w'^2 w^V\right) \\ &\text{or } \delta w = 0 \text{ at } x = 0, L \end{aligned} \tag{38}$$

$$\begin{aligned} M = &EI_g w'' + EI_g w'^2 w'' - 2EIg^2 w' w'' w'' - EIg^2 w^{IV} \\ &-EIg^2 w'^2 w^{IV} \quad \text{or} \quad \delta w_x = 0 \quad \text{at } x = 0, L \end{aligned} \tag{39}$$

$$m = EIg^2\left(w''' + w' w''^2 + w'^2 w'''\right) \text{ or } \delta w_{xx} = 0 \text{ at } x = 0, L \tag{40}$$

Eqs. (38) and (39) are the counterparts of the classic boundary conditions, while Eq. (40) is a non-classic one. It is evident that the present equations first introduce the geometrical non-linearity through the non-linear curvature expression, and second, involve terms that correspond to the double stresses μ_{yxx}.

Looking for solutions of the type

$$w = \varepsilon w_1 + \varepsilon^3 w_3 + o(\varepsilon^3)$$

for the loading

$$P = P_o(1 + \varepsilon^2\lambda)$$

and

$$M = \varepsilon M_1 + \varepsilon^3 M_3 + o(\varepsilon^4),\ m = \varepsilon m_1 + \varepsilon^3 m_3 + o(\varepsilon^4),$$
$$V = \varepsilon V + \varepsilon^3 V + o(\varepsilon^4)$$

we apply the bifurcation theory to consider the multiplicity of rotations of the non-linear problem given by Eqs. (37)–(40), by utilizing the linear homogeneous problem, i.e., the governing differential equation

$$E\left(I + g^2 A\right)\frac{\partial^4 w}{\partial x^4} - g^2 EI\frac{\partial^6 w}{\partial x^6} + Pw'' = 0 \tag{41}$$

with the boundary conditions

$$V = \left[-Pw' + E\left(I + g^2 A\right)\frac{\partial^3 w}{\partial x^3} - g^2 EI\frac{\partial^5 w}{\partial x^5}\right] \text{ or } \delta w = 0 \text{ at } x = 0, L \tag{42}$$

$$M = E\left(I + g^2 A\right)\frac{\partial^2 w}{\partial x^2} - g^2 EI\frac{\partial^4 w}{\partial x^4} \text{ or } \delta w' = 0 \text{ at } x = 0, L \tag{43}$$

$$m = EIg^2\frac{\partial^3 w}{\partial x^3} \text{ or } \delta w'' = 0 \text{ at } x = 0, L \tag{44}$$

When the boundary value problem given by Eqs. (41)–(44) possesses only the trivial solution, the non-linear boundary value problem, Eqs. (37)–(41), admits only a unique solution. Nevertheless, whenever the linear problem, or Eqs. (41)–(44), has a non-trivial solution, for a specific value P_o,

$$w_1 = \phi(x) \tag{45}$$

the non-linear problem of Eqs. (37)–(41) admits a solution of the type

$$w = \varepsilon\xi\phi(x) + o(\xi^2) \tag{46}$$

e.g., ref. [16]. It is evident that the deflection of the beam can be described when the small parameter ξ is defined.

Furthermore, the third-order equilibrium equation is given by

$$\begin{aligned} E\left(I + g^2 A\right)\frac{\partial^4 w_3}{\partial x^4} - g^2 EI\frac{\partial^6 w_3}{\partial x^6} + P_o w_3'' = &-P_o w_1''\lambda + \frac{1}{2}\Big\{\Big\{-4EI_g w_1'' w_1''^2 \\ &+24EIg^2 w''' + 2g^2 EIw_1'' w_1''' + 1.5Eg^2 w_1^{IV}\Big\} + 2w_1'\Big\{EI_g w_1''^2 \\ &+g^2 EIw_1'''\left(w_1''' + 8w_1^{IV}\right) + w_1''\left(-6EI_g w_1''' + 6g^2 EIw_1^V\right)\Big\} \\ &-2Ew_1'^2\left(I_g w_1^{IV} - g^2 Iw_1^{IV}\right)\Big\} \end{aligned} \tag{47}$$

with the boundary conditions

$$\begin{aligned} V_3 = &EI_g w_3''' - EIg^2 w_3^V - EI_g w_1' w_1''^2 - EIl_x w_1' w_1'' w_1''' \\ &-EIg^2 w_1''^2 w_1''' - EIg^2 w_1' w_1'''^2 + 2EI_g w_1' w_1''^2 + EI_g w_1'^2 w_1''' \\ &+2EIg^2 w_1''^2 w_1''' + 2EIg^2 w_1' w_1'''^2 + 2EIg^2 w_1' w_1'' w_1^{IV} \\ &-EIg^2\left(5w_1''^2 w_1''' + w_1'\left(2w_1'''^2 + 2w_1'' w_1^{IV}\right)\right) \\ &-EIg^2\left(w_1'''\left(2w_1''^2 + 2w_1' w_1'''\right) + 4w_1' w_1'' w_1^{IV} + w_1'^2 w_1^V\right) \\ &\text{or } \delta w_3 = 0 \text{ at } x = 0, L \end{aligned} \tag{48}$$

$$M_3 = EI_g w_3'' + EI_g w_1'^2 w_1'' - 2g^2 EI w_1' w_1'' w_1''' - g^2 EI w_3^{IV} - EIg^2 w_1'^2 w_1^{IV} \text{ or } \delta w_{3,x} = 0 \text{ at } x = 0, L \quad (49)$$

$$m_3 = g^2 EI\left(w_3''' + w_1' w_1''^2 + w_1'^2 w_1'''\right) \text{ or } \delta w_{3,xx} = 0 \text{ at } x = 0, L \quad (50)$$

It is well known that the small parameter ξ may be determined by invoking Fredholm's alternative theorem [16] or by directly defining the minimum of the potential energy V. Fredholm's alternative theorem pertaining to the existence of the solution w_3 of Eq. (47), with boundary conditions given by Eqs. (48)–(50), requires that

$$\int_0^L R_3(\xi\phi(x), \lambda)\phi(x)dx + \left[V_3\phi(x)\right]_0^L - \left[M_3\phi'(x)\right]_0^L - \left[m_3\phi''(x)\right]_0^L = 0 \quad (51)$$

so that Eq. (51) is reduced to an algebraic equation of the type

$$\xi^3 - \gamma\lambda\xi = 0 \quad (52)$$

with the solutions for ξ of Eq. (52) defining w, through Eq. (46) and γ being a constant.

Specifically, the deflection of the beam defined by Eq. (46) is expressed by the equation

$$w = c_2 + c_1 x + c_3 \sin(r_1 x) + c_4 \cos(r_1 x) + c_5 \sinh(r_2 x) + c_6 \cosh(r_2 x) \quad (53)$$

where c_i's are constants defined by the boundary conditions and

$$r_1 = \sqrt{\frac{-a + \sqrt{a^2 + 4g^2 p}}{2g^2}} \quad r_2 = \sqrt{\frac{a + \sqrt{a^2 + 4g^2 p}}{2g^2}} \quad (54)$$

with

$$a = (I + g^2 A)/I, \; p = \sqrt{\frac{P - P_o}{EI}} \quad (55)$$

References

[1] Altan BS, Aifantis EC. *J. Mech. Behav. Mater.* 1997, 8, 231–282 [see also: Aifantis EC. *Int. J. Eng. Sci.* 3].

[2] Ru CQ, Aifantis EC. *Acta Mech.* 1993, 101, 59–68.

[3] Lazar M, Maugin GA. *Mech. Res. Commun*. 2006, 33, 674–680.

[4] Aifantis EC. *Mech. Mater.* 2003, 35, 259–280 [see also: Aifantis, EC. *Int. J. Fract.* 1999, 95, 299–314].

[5] Papargyri-Beskou S, Tsepoura KG, Polyzos D, Beskos DE. *Int. J. Solids Struct.* 2003, 40, 385–400.

[6] Park SK, Gao XL. *J. Micromech. Microeng.* 2006, 16, 2355–2359.

[7] Yang F, Chong ACM, Lam DCC, Tong P. *Int. J. Solids Struct.* 2002, 39, 2731–2743.

[8] Lazopoulos KA, Lazopoulos AK. *Eur. J. Mech./A Solids* 2010, 29, 837–843.

[9] Lazopoulos KA, Lazopoulos AK. *ZAMM* 2011, 91, 875–882.

[10] Vardoulakis I. In *Degradations and Instabilities in Geomaterials*, a CIMS/DIGA-sponsored course, Chapter 4, Darve F, Vardoulakis I, Eds. Springer: Wien, 2004.

[11] Spencer AJM, Soldatos KP. *Int. J. Non-Linear Mech.* 2007, 42, 355–368.

[12] Lazopoulos AK. *Int. J. Mech. Sci.* 2012, 58, 27–33.

[13] Lazopoulos KA, Lazopoulos AK. ZAMM 2009, 90, 174–184.

[14] Lazopoulos KA, Lazopoulos AK, Palassopoulos GV. *Appl. Maths* in press [see also: Lazopoulos KA, Alnefaie KA, Abnu-Hamdeh NH, Aifantis EC. In: *Nanoplates and Nanoshells, Advances in Materials and Mechanics (AMM)*, Sun B, Ed. HEP/Springer (in press)].

[15] Lazopoulos KA, Lazopoulos AK. *Eur. J. Mech./A Solids* 2010, 30, 286–292.

[16] Vainberg M, Trenogin V. *Theory of Branching of Solution of Nonlinear Equations*. Noordhoff: Leyden, The Netherlands, 1974.