

On the Strong-Mixing Property of Skew Product of Binary Transformation on 2-Dimensional Torus by Irrational Rotation

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1. Presentation of Theorem.

Let $(\mathbb{T}, \mathcal{F}, \mathbf{P})$ be 1-dimensional Lebesgue probability space, i.e., $\mathbb{T} = 1$ -dimensional torus $\cong [0, 1)$, $\mathcal{F} = \mathcal{B}(\mathbb{T})$, $\mathbf{P}(dx) = 1$ -dimensional Lebesgue measure on \mathbb{T} .

Let T be binary transformation on 2-dimensional torus \mathbb{T}^2 , i.e., $T(x_1, x_2) = (2x_1, 2x_2) \pmod{1}$. T preserves $\mathbf{P}^2 = \mathbf{P} \times \mathbf{P}$, and moreover is strong-mixing (cf. (1)).

DEFINITION 1. For a measurable function $f : \mathbb{T}^2 \rightarrow \{-1, 1\}$, we define a transformation T_f on $\mathbb{T}^2 \times \{-1, 1\}$ by

$$T_f(x, \varepsilon) = (Tx, f(x)\varepsilon), \quad (x, \varepsilon) \in \mathbb{T}^2 \times \{-1, 1\}.$$

This T_f preserves $\mu(dxd\varepsilon) := \mathbf{P}^2(dx) \times \frac{1}{2}(\delta_{-1} + \delta_1)(d\varepsilon)$, and is called a *skew product* of T by f .

We are concerned with the strong-mixing property of T_f where f is defined by means of irrational rotation. More precisely, we define $r : \mathbb{T} \rightarrow \{-1, 1\}$ by $r(x) = 1_{[0, \frac{1}{2})}(x) - 1_{[\frac{1}{2}, 1)}(x)$, and for $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \mathbb{N}; n_1 < \dots < n_k$, we consider $f : \mathbb{T}^2 \rightarrow \{-1, 1\}$ as

$$f(x_1, x_2) = r(x_1)r(x_1 + n_1x_2) \cdots r(x_1 + n_kx_2).$$

(We identify the function of \mathbb{T} with the 1-periodic function of \mathbb{R} in an obvious way. Thus r is regarded as a 1-periodic function, so that the f above is well-defined!) Then we have the following:

THEOREM. *Let f be as above. Then T_f is strong-mixing, i.e., for $\forall A, B \in \mathcal{B}(\mathbb{T}^2 \times \{-1, 1\})$*

$$\lim_{m \rightarrow \infty} \mu(A \cap T_f^{-m}B) = \mu(A)\mu(B). \quad (1)$$

2. Reduction of the problem and its motivation.

By the general theory (e.g. [6, Theorem 1.23]), the strong-mixing property of T_f (where $f : \mathbb{T}^2 \rightarrow \{-1, 1\}$ is general) is equivalent to the following convergence: Let $\{\varphi_n\}_{n=1}^\infty$ be a CONS of $L^2(\mathbb{T}^2 \times \{-1, 1\} \rightarrow \mathbb{C}, \mu)$, then

$$\begin{aligned} & \lim_{m \rightarrow \infty} \int \int_{\mathbb{T}^2 \times \{-1, 1\}} \varphi_n(T_f^m(x, \varepsilon)) \overline{\varphi_n(x, \varepsilon)} \mu(dx d\varepsilon) \\ &= \left| \int \int_{\mathbb{T}^2 \times \{-1, 1\}} \varphi_n(x, \varepsilon) \mu(dx d\varepsilon) \right|^2, \quad \forall n \in \mathbb{N}. \end{aligned}$$

In the present case, we can take, as a CONS of $L^2(\mathbb{T}^2 \times \{-1, 1\} \rightarrow \mathbb{C}, \mu)$

$$\{e^{\sqrt{-1}2\pi(p_1x_1+p_2x_2)}, e^{\sqrt{-1}2\pi(q_1x_1+q_2x_2)}\varepsilon; (p_1, p_2), (q_1, q_2) \in \mathbb{Z}^2\}.$$

For $e^{\sqrt{-1}2\pi(p_1x_1+p_2x_2)}$, the convergence above holds obviously. Therefore we obtain the following criterion for T_f to be strong-mixing:

PROPOSITION 1. T_f is strong-mixing if and only if, for $\forall (p_1, p_2) \in \mathbb{Z}^2$

$$\lim_{m \rightarrow \infty} \int_{\mathbb{T}^2} e^{\sqrt{-1}2\pi(2^m-1)(p_1x_1+p_2x_2)} f(x_1, x_2) f(2x_1, 2x_2) \cdots f(2^{m-1}x_1, 2^{m-1}x_2) dx_1 dx_2 = 0.$$

Now let us go to our problem. Our $f : \mathbb{T}^2 \rightarrow \{-1, 1\}$ was

$$f(x_1, x_2) = r(x_1)r(x_1 + n_1x_2) \cdots r(x_1 + n_kx_2).$$

For this we introduce $X^{(m)} : \mathbb{T} \rightarrow \{-1, 1\}$ by $X^{(m)}(x) := \prod_{j=1}^m r(2^{j-1}x)$. Then

$$\begin{aligned} & f(x_1, x_2) f(2x_1, 2x_2) \cdots f(2^{m-1}x_1, 2^{m-1}x_2) \\ &= X^{(m)}(x_1) X^{(m)}(x_1 + n_1x_2) \cdots X^{(m)}(x_1 + n_kx_2). \end{aligned}$$

Thus, by Proposition 1 we reduce Theorem to the following:

THEOREM 1. There exists $0 \leq \rho = \rho(n_1, \dots, n_k) < 1$ such that for $\forall p \in \mathbb{Z}$

$$\int_0^1 d\alpha \left| \int_{\mathbb{T}} e^{\sqrt{-1}2\pi(2^m-1)p\alpha} X^{(m)}(x) X^{(m)}(x + n_1\alpha) \cdots X^{(m)}(x + n_k\alpha) dx \right| = O(\rho^m)$$

as $m \rightarrow \infty$.

In the next section Theorem 1 will be proved. Before proceeding, we here mention the motivation for the study carried out in this paper. In [1], Sugita presented a pseudo-random number generator by means of irrational rotation, and as its theoretical justification showed the following:

FACT 1. For almost every $\alpha \in \mathbb{T}$, the stationary process $\{X^{(m)}(\cdot + n\alpha)\}_{n=0}^\infty$ on (\mathbb{T}, \mathbf{P}) converges in law to the fair coin tossing process (i.e., mean zero $\{-1, 1\}$ -valued i.i.d. process), as $m \rightarrow \infty$.

After that, in [3] he tried to simplify the proof and stated the following (, though its statement is weaker than that of Fact 1):

FACT 2. *The stationary process $\{X^{(m)}(x_1 + nx_2)\}_{n=0}^{\infty}$ on $(\mathbb{T}^2, \mathbf{P}^2)$ converges in law to the fair coin tossing process, as $m \rightarrow \infty$.*

In the process of showing this fact, he found the strong-mixing property of T_f , and utilizing this property, he clarified the reasons behind the validity of Fact 2. Actually he confirmed its ergodicity and weak-mixing property, and showed an equivalent statement to this fact:

FACT 2'. *For $\forall k \in \mathbb{N}$ and $\forall n_1, \dots, n_k \in \mathbb{N}; n_1 < \dots < n_k$*

$$\lim_{m \rightarrow \infty} \int_{\mathbb{T}^2} X^{(m)}(x_1) X^{(m)}(x_1 + n_1 x_2) \cdots X^{(m)}(x_1 + n_k x_2) dx_1 dx_2 = 0.$$

Fact 2' follows obviously from Theorem 1 with $p = 0$; however, his proof is purely of ergodic theory in comparison with ours, and it itself is of independent interest. Our method for proving the theorem is based on a method of cancellation, which is the same as in [1]. But, compared with that in [1], our cancellation is more transparent, so that our proof might be more readily acceptable. In this paper we solve his conjecture by proving the theorem, and at the same time succeed in his scheme of simplifying the proof of Fact 1!

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3. Proof of Theorem 1.

We begin with the following proposition:

PROPOSITION 2 (cf. [5]). *For each $m \in \mathbb{N}$*

$$X^{(m)}(x) = (\sqrt{-1})^{m-2} \sum_{k=1}^{2^{m-1}} \left(\prod_{i=1}^m \sin \frac{2k-1}{2^i} \pi \right) e^{\sqrt{-1} \frac{2k-1}{2^m} \pi} e^{\sqrt{-1} 2\pi (2k-1) \frac{[2^m x]}{2^m}}.$$

LEMMA 1 (cf. [2]). *Let $\widehat{X^{(m)}}(n)$ be the n th Fourier coefficient of $X^{(m)}$, i.e., $\widehat{X^{(m)}}(n) = \int_{\mathbb{T}} X^{(m)}(x) e^{-\sqrt{-1} 2n\pi x} dx$. Then*

$$\widehat{X^{(m)}}(n) = \begin{cases} 0 & \text{if } n \in 2\mathbb{Z} \\ (\sqrt{-1})^{m-2} \left(\prod_{j=1}^m \sin \frac{n\pi}{2^j} \right) \int_0^1 e^{-\sqrt{-1} 2\pi \frac{n}{2^m} (x - \frac{1}{2})} dx & \text{if } n \in 2\mathbb{Z} - 1. \end{cases}$$

PROOF. We define $d : \mathbb{T} \rightarrow \{0, 1\}$ by $d(x) = 1_{[\frac{1}{2}, 1)}(x)$, and set $r_j : \mathbb{T} \rightarrow \{-1, 1\}$, $d_j : \mathbb{T} \rightarrow \{0, 1\}$ by

$$r_j(x) = r(2^{j-1}x), \quad d_j(x) = d(2^{j-1}x).$$

Then $r_j = (-1)^{d_j}$ ($\forall j \in \mathbb{N}$), $\{d_j\}_{j=1}^\infty$ is an i.i.d. random sequence on (\mathbb{T}, \mathbf{P}) , and

$$x = \sum_{j=1}^{\infty} \frac{d_j(x)}{2^j}, \quad \forall x \in \mathbb{T}.$$

Using these facts, we actually compute $\widehat{X^{(m)}}(n)$:

$$\begin{aligned} \widehat{X^{(m)}}(n) &= \int_{\mathbb{T}} \left(\prod_{j=1}^m r_j(x) \right) e^{-\sqrt{-1}2n\pi \sum_{j=1}^{\infty} \frac{d_j(x)}{2^j}} dx \\ &= \int_{\mathbb{T}} \prod_{j=1}^m (-1)^{d_j(x)} \prod_{j=1}^m e^{-\sqrt{-1}2n\pi \frac{d_j(x)}{2^j}} e^{-\sqrt{-1} \frac{2n\pi}{2^m} \sum_{j=1}^{\infty} \frac{d_{j+m}(x)}{2^j}} dx \\ &= \left(\prod_{j=1}^m \int_{\mathbb{T}} (-1)^{d_j(x)} e^{-\sqrt{-1}2n\pi \frac{d_j(x)}{2^j}} dx \right) \int_{\mathbb{T}} e^{-\sqrt{-1} \frac{2n\pi}{2^m} \sum_{j=1}^{\infty} \frac{d_{j+m}(x)}{2^j}} dx \\ &= \left(\prod_{j=1}^m \frac{1}{2} (1 - e^{-\sqrt{-1}2n\pi \frac{1}{2^j}}) \right) \int_{\mathbb{T}} e^{-\sqrt{-1} \frac{2n\pi}{2^m} x} dx \\ &= e^{-\sqrt{-1}n\pi \sum_{j=1}^m \frac{1}{2^j}} \left(\prod_{j=1}^m \sqrt{-1} \sin \frac{n\pi}{2^j} \right) \int_{\mathbb{T}} e^{-\sqrt{-1} \frac{2n\pi}{2^m} x} dx \\ &= (\sqrt{-1})^m \left(\prod_{j=1}^m \sin \frac{n\pi}{2^j} \right) (-1)^n \int_{\mathbb{T}} e^{-\sqrt{-1} \frac{2n\pi}{2^m} (x - \frac{1}{2})} dx. \end{aligned}$$

From this the conclusion follows at once. \square

PROOF OF PROPOSITION 2. For simplicity we denote the RHS of the identity in Proposition 2 by $Y^{(m)}(x)$. Since $X^{(m)}(x)$ and $Y^{(m)}(x)$ are right continuous in x , it is enough to show that $X^{(m)} = Y^{(m)}$ in $L^2(\mathbb{T}, \mathbf{P})$.

By Lemma 1

$$\begin{aligned} X^{(m)}(x) &= \sum_{n \in 2\mathbb{Z}-1} (\sqrt{-1})^{m-2} \left(\prod_{j=1}^m \sin \frac{n\pi}{2^j} \right) \int_0^1 e^{-\sqrt{-1}2\pi \frac{n}{2^m} (x - \frac{1}{2})} dx e^{\sqrt{-1}2n\pi x} \\ &= \sum_{\substack{q \in \mathbb{Z}, \\ r \in \{1, \dots, 2^{m-1}\}}} (\sqrt{-1})^{m-2} \left(\prod_{j=1}^m \sin \frac{q2^m + 2r - 1}{2^j} \pi \right) \\ &\quad \times \int_0^1 e^{-\sqrt{-1}2\pi \frac{q2^m + 2r - 1}{2^m} (x - \frac{1}{2})} dx e^{\sqrt{-1}2(q2^m + 2r - 1)\pi x} \\ &= \sum_{r=1}^{2^{m-1}} \sum_{q \in \mathbb{Z}} (\sqrt{-1})^{m-2} \left(\prod_{j=1}^m \sin \frac{2r - 1}{2^j} \pi \right) e^{\sqrt{-1} \frac{2r-1}{2^m} \pi} \end{aligned}$$

$$\begin{aligned}
& \times \int_0^1 e^{-\sqrt{-1}2\pi qx} e^{-\sqrt{-1}2\pi \frac{2r-1}{2^m}\{x\}} dx e^{\sqrt{-1}2q\pi 2^m x} e^{\sqrt{-1}2\pi(2r-1)x} \\
& \quad [\text{where } \{x\} = \text{the fractional part of } x] \\
& = (\sqrt{-1})^{m-2} \sum_{r=1}^{2^{m-1}} \left(\prod_{j=1}^m \sin \frac{2r-1}{2^j} \pi \right) e^{\sqrt{-1} \frac{2r-1}{2^m} \pi} e^{\sqrt{-1}2(2r-1)\pi x} \\
& \quad \times \sum_{q \in \mathbb{Z}} \int_0^1 e^{-\sqrt{-1}2\pi qx} e^{-\sqrt{-1}2\pi \frac{2r-1}{2^m}\{x\}} dx e^{\sqrt{-1}2q\pi 2^m x} \\
& = (\sqrt{-1})^{m-2} \sum_{r=1}^{2^{m-1}} \left(\prod_{j=1}^m \sin \frac{2r-1}{2^j} \pi \right) e^{\sqrt{-1} \frac{2r-1}{2^m} \pi} e^{\sqrt{-1}2(2r-1)\pi x} e^{-\sqrt{-1}2\pi \frac{2r-1}{2^m}\{2^m x\}} \\
& = (\sqrt{-1})^{m-2} \sum_{r=1}^{2^{m-1}} \left(\prod_{j=1}^m \sin \frac{2r-1}{2^j} \pi \right) e^{\sqrt{-1} \frac{2r-1}{2^m} \pi} e^{\sqrt{-1}2\pi \frac{2r-1}{2^m}(2^m x - \{2^m x\})} \\
& = (\sqrt{-1})^{m-2} \sum_{r=1}^{2^{m-1}} \left(\prod_{j=1}^m \sin \frac{2r-1}{2^j} \pi \right) e^{\sqrt{-1} \frac{2r-1}{2^m} \pi} e^{\sqrt{-1}2\pi(2r-1) \frac{\lfloor 2^m x \rfloor}{2^m}} \\
& = Y^{(m)}(x). \quad \square
\end{aligned}$$

DEFINITION 2. Let $k, m \in \mathbb{N}$ and $p \in \mathbb{Z}$. We define $S_p^{(m)} : \mathbb{Z}^k \rightarrow \mathbb{R}$ by

$$\begin{aligned}
S_p^{(m)}(l_1, \dots, l_k) & := \left(\prod_{i=1}^m \sin \frac{2l_1 - 1 + \dots + 2l_k - 1 - p}{2^i} \pi \right) \\
& \quad \times \left(\prod_{i=1}^m \sin \frac{2l_1 - 1}{2^i} \pi \right) \times \dots \times \left(\prod_{i=1}^m \sin \frac{2l_k - 1}{2^i} \pi \right), \quad (l_1, \dots, l_k) \in \mathbb{Z}^k.
\end{aligned}$$

Clearly $S_p^{(m)}$ is symmetric and $S_p^{(m)}(\dots, l_j + 2^{m-1}, \dots) = S_p^{(m)}(\dots, l_j, \dots)$. Also $S_p^{(m)} = 0$ if $p \equiv k \pmod{2}$.

LEMMA 2. Let $k, m \in \mathbb{N}$ and $p \in \mathbb{Z}$. Then for $\forall (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$

$$\begin{aligned}
& \int_{\mathbb{T}} e^{\sqrt{-1}2\pi(2^m-1)px} X^{(m)}(x) X^{(m)}(x + \alpha_1) \dots X^{(m)}(x + \alpha_k) dx \\
& = (\sqrt{-1})^{(m-2)(k+1)} (-1)^m \int_0^1 \sum_{1 \leq l_1, \dots, l_k \leq 2^{m-1}} S_p^{(m)}(l_1, \dots, l_k) e^{\sqrt{-1} \frac{p}{2^m} \pi} e^{\sqrt{-1}2\pi p(1 - \frac{1}{2^m})y} \\
& \quad \times \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1) \frac{[y+2^m \alpha_i]}{2^m}} dy.
\end{aligned}$$

PROOF. First let $p \equiv k \pmod{2}$. In this case, since $X^{(m)}(x + 1/2) = -X^{(m)}(x)$,

$$\begin{aligned} \text{LHS} &= \int_{\mathbb{T}} e^{\sqrt{-1}2\pi(2^m-1)p(x+\frac{1}{2})} X^{(m)}\left(x + \frac{1}{2}\right) X^{(m)}\left(x + \frac{1}{2} + \alpha_1\right) \cdots X^{(m)}\left(x + \frac{1}{2} + \alpha_k\right) dx \\ &\quad [\text{by the shift invariance of } dx] \\ &= (-1)^{p+1+k} \text{LHS} \\ &= -\text{LHS}, \end{aligned}$$

so that LHS = 0. On the other hand, since $S_p^{(m)} = 0$, RHS = 0. Hence we have the conclusion in the case of $p \equiv k \pmod{2}$.

Next let $p \not\equiv k \pmod{2}$. Since, by Proposition 2

$$\begin{aligned} &X^{(m)}(x)X^{(m)}(x + \alpha_1) \cdots X^{(m)}(x + \alpha_k) \\ &= (\sqrt{-1})^{(m-2)(k+1)} \sum_{1 \leq l_0, l_1, \dots, l_k \leq 2^{m-1}} \left(\prod_{i=1}^m \sin \frac{2l_0 - 1}{2^i} \pi \right) \times \cdots \times \left(\prod_{i=1}^m \sin \frac{2l_k - 1}{2^i} \pi \right) \\ &\quad \times e^{\sqrt{-1}\left(\frac{2l_0-1}{2^m} + \cdots + \frac{2l_k-1}{2^m}\right)\pi} e^{\sqrt{-1}2\pi(2l_0-1)\frac{[2^m x]}{2^m}} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[2^m x + 2^m \alpha_i]}{2^m}}, \end{aligned}$$

it turns out that

$$\begin{aligned} \text{LHS} &= (\sqrt{-1})^{(m-2)(k+1)} \sum_{1 \leq l_0, l_1, \dots, l_k \leq 2^{m-1}} \left(\prod_{i=1}^m \sin \frac{2l_0 - 1}{2^i} \pi \right) \\ &\quad \times \cdots \times \left(\prod_{i=1}^m \sin \frac{2l_k - 1}{2^i} \pi \right) e^{\sqrt{-1}\left(\frac{2l_0-1}{2^m} + \cdots + \frac{2l_k-1}{2^m}\right)\pi} \\ &\quad \times \int_0^1 e^{\sqrt{-1}2\pi(2^m-1)px} e^{\sqrt{-1}2\pi(2l_0-1)\frac{[2^m x]}{2^m}} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[2^m x + 2^m \alpha_i]}{2^m}} dx. \end{aligned}$$

Here we further compute the integral $\int_0^1 \cdots dx$ in the last line as

$$\begin{aligned} &= \sum_{j=1}^{2^m} \int_{\frac{j-1}{2^m}}^{\frac{j}{2^m}} e^{\sqrt{-1}2\pi p(2^m x - x)} e^{\sqrt{-1}2\pi(2l_0-1)\frac{[2^m x]}{2^m}} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[2^m x + 2^m \alpha_i]}{2^m}} dx \\ &= \sum_{j=1}^{2^m} \int_0^1 e^{\sqrt{-1}2\pi p(1-\frac{1}{2^m})(y+j-1)} e^{\sqrt{-1}2\pi(2l_0-1)\frac{j-1}{2^m}} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[y+2^m \alpha_i]+j-1}{2^m}} \frac{dy}{2^m} \\ &\quad [\text{by the change of variables } 2^m x - (j-1) = y] \\ &= \frac{1}{2^m} \sum_{j=1}^{2^m} e^{\sqrt{-1}2\pi(-p+2l_0-1+\cdots+2l_k-1)\frac{j-1}{2^m}} \int_0^1 e^{\sqrt{-1}2\pi p(1-\frac{1}{2^m})y} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[y+2^m \alpha_i]}{2^m}} dy \end{aligned}$$

$$\begin{aligned}
&= 1_{-p+2l_0-1+\dots+2l_k-1 \in 2^m \mathbb{Z}} \int_0^1 e^{\sqrt{-1}2\pi p(1-\frac{1}{2^m})y} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[y+2^m\alpha_i]}{2^m}} dy \\
&\quad \left[\text{because, for } v \in \mathbb{Z} \frac{1}{2^m} \sum_{j=1}^{2^m} e^{\sqrt{-1}2\pi v \frac{j-1}{2^m}} = 1_{v \in 2^m \mathbb{Z}} = \begin{cases} 0 & \text{if } v \notin 2^m \mathbb{Z}, \\ 1 & \text{if } v \in 2^m \mathbb{Z} \end{cases} \right].
\end{aligned}$$

Substituting the last expression into the above, we have

$$\begin{aligned}
\text{LHS} &= (\sqrt{-1})^{(m-2)(k+1)} \sum_{\substack{1 \leq l_0, l_1, \dots, l_k \leq 2^{m-1}; \\ -p+2l_0-1+\dots+2l_k-1 \in 2^m \mathbb{Z}}} \left(\prod_{i=1}^m \sin \frac{2l_0-1}{2^i} \pi \right) \\
&\quad \times \dots \times \left(\prod_{i=1}^m \sin \frac{2l_k-1}{2^i} \pi \right) e^{\sqrt{-1} \left(\frac{2l_0-1}{2^m} + \dots + \frac{2l_k-1}{2^m} \right) \pi} \\
&\quad \times \int_0^1 e^{\sqrt{-1}2\pi p(1-\frac{1}{2^m})y} \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{[y+2^m\alpha_i]}{2^m}} dy.
\end{aligned}$$

Here, since $p+1 \equiv k \pmod{2}$, we see that for $\forall (l_1, \dots, l_k) \in \{1, \dots, 2^{m-1}\}^k$

$$1 \leq \exists! l_0 \leq 2^{m-1} \text{ such that } -p+2l_0-1+2l_1-1+\dots+2l_k-1 \in 2^m \mathbb{Z}.$$

For this l_0

$$\begin{aligned}
&\left(\prod_{i=1}^m \sin \frac{2l_0-1}{2^i} \pi \right) e^{\sqrt{-1} \left(\frac{2l_0-1}{2^m} + \dots + \frac{2l_k-1}{2^m} \right) \pi} \\
&= (-1)^m \left(\prod_{i=1}^m \sin \frac{2l_1-1+\dots+2l_k-1-p}{2^i} \pi \right) e^{\sqrt{-1} \frac{p}{2^m} \pi}.
\end{aligned}$$

Therefore substituting this into the above, we have the conclusion in the case of $p \not\equiv k \pmod{2}$. \square

In the following, let $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \mathbb{N}; n_1 < \dots < n_k$ be fixed.

DEFINITION 3. For $p \in \mathbb{Z}$ and $m \in \mathbb{N}$, we define $A_p^{(m)} : \mathbb{Z}^k = \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_k \rightarrow \mathbb{C}$ and

$\rho_p(m) \in [0, \infty)$ by

$$\begin{aligned}
A_p^{(m)}(K_1, \dots, K_k) &:= \sum_{1 \leq l_1, \dots, l_k \leq 2^{m-1}} S_p^{(m)}(l_1, \dots, l_k) \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1)\frac{K_i}{2^m}}, \\
&\quad (K_1, \dots, K_k) \in \mathbb{Z}^k,
\end{aligned}$$

$$\rho_p(m) := \max_{\substack{t_1 \in \{0, \dots, n_1\}, \\ \dots \\ t_k \in \{0, \dots, n_k\}}} \frac{1}{2^m} \sum_{j=1}^{2^m} |A_p^{(m)}(n_1(j-1)+t_1, \dots, n_k(j-1)+t_k)|.$$

LEMMA 3. For $\forall p \in \mathbb{Z}$ and $\forall m \in \mathbb{N}$

$$\int_0^1 d\alpha \left| \int_{\mathbb{T}} e^{\sqrt{-1}2\pi(2^m-1)p x} X^{(m)}(x) X^{(m)}(x+n_1\alpha) \cdots X^{(m)}(x+n_k\alpha) dx \right| \leq \rho_p(m).$$

PROOF. By Lemma 2

$$\begin{aligned} & \int_0^1 d\alpha \left| \int_{\mathbb{T}} e^{\sqrt{-1}2\pi(2^m-1)p x} X^{(m)}(x) X^{(m)}(x+n_1\alpha) \cdots X^{(m)}(x+n_k\alpha) dx \right| \\ & \leq \int_0^1 d\alpha \int_0^1 dy \left| \sum_{1 \leq l_1, \dots, l_k \leq 2^m-1} S_p^{(m)}(l_1, \dots, l_k) \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1) \frac{[y+2^m n_i \alpha]}{2^m}} \right| \\ & = \int_0^1 dy \int_0^1 d\alpha |A_p^{(m)}([y+n_1 2^m \alpha], \dots, [y+n_k 2^m \alpha])| \\ & = \int_0^1 dy \sum_{j=1}^{2^m} \int_{\frac{j-1}{2^m}}^{\frac{j}{2^m}} d\alpha |A_p^{(m)}([y+n_1 2^m \alpha], \dots, [y+n_k 2^m \alpha])| \\ & = \int_0^1 dy \sum_{j=1}^{2^m} \int_0^1 \frac{d\beta}{2^m} |A_p^{(m)}([y+n_1 \beta] + n_1(j-1), \dots, [y+n_k \beta] + n_k(j-1))| \\ & \quad \text{[by the change of variables } 2^m \alpha - (j-1) = \beta] \\ & = \int_0^1 \int_0^1 dy d\beta \frac{1}{2^m} \sum_{j=1}^{2^m} |A_p^{(m)}(n_1(j-1) + [y+n_1 \beta], \dots, n_k(j-1) + [y+n_k \beta])| \\ & = \sum_{\substack{t_1 \in \{0, \dots, n_1\}, \\ t_k \in \{0, \dots, n_k\}}} \iint_{\substack{(y, \beta) \in [0, 1) \times [0, 1); \\ [y+n_1 \beta] = t_1, \dots, [y+n_k \beta] = t_k}} dy d\beta \\ & \quad \times \frac{1}{2^m} \sum_{j=1}^{2^m} |A_p^{(m)}(n_1(j-1) + t_1, \dots, n_k(j-1) + t_k)| \\ & \quad \text{[because } [y+n_i \beta] \in \{0, \dots, n_i\} \text{ (} i = 1, \dots, k\text{)]} \\ & \leq \max_{\substack{t_1 \in \{0, \dots, n_1\}, \\ \dots, \\ t_k \in \{0, \dots, n_k\}}} \frac{1}{2^m} \sum_{j=1}^{2^m} |A_p^{(m)}(n_1(j-1) + t_1, \dots, n_k(j-1) + t_k)| \\ & = \rho_p(m). \end{aligned}$$

□

DEFINITION 4. We define $s_0, s_1 : \mathbb{Z} \rightarrow \mathbb{Z}$ by

$$s_1(r) = \frac{r + 1_{r \in 2\mathbb{Z}-1}}{2} = \begin{cases} \frac{r}{2} & \text{if } r \text{ is even} \\ \frac{r+1}{2} & \text{if } r \text{ is odd,} \end{cases}$$

$$s_0(r) = \frac{r - 1_{r \in 2\mathbb{Z}-1}}{2} = \begin{cases} \frac{r}{2} & \text{if } r \text{ is even} \\ \frac{r-1}{2} & \text{if } r \text{ is odd.} \end{cases}$$

LEMMA 4. *Let $m \geq 2$ and $(K_1, \dots, K_k) \in \mathbb{Z}^k$. Then*

$$\begin{aligned} A_p^{(m)}(K_1, \dots, K_k) &= \sum_{\delta \in \{0,1\}} \frac{(\sqrt{-1})^{k-1}}{2} \\ &\times \left(\prod_{i=1}^k (-1)^{K_i} (-1)^{(k-1)(1-\delta)} e^{-\sqrt{-1}\pi \frac{p}{2^m} (2\delta-1)} A_p^{(m-1)}(s_\delta(K_1), \dots, s_\delta(K_k)) \right). \end{aligned}$$

PROOF. Let $m \geq 2$ and $(K_1, \dots, K_k) \in \mathbb{Z}^k$. First, noting that

$$\begin{aligned} &\{1, \dots, 2^{m-1}\}^k \\ &= \{(l_1 + \varepsilon_1 2^{m-2}, \dots, l_k + \varepsilon_k 2^{m-2}); (l_1, \dots, l_k) \in \{1, \dots, 2^{m-2}\}^k, (\varepsilon_1, \dots, \varepsilon_k) \in \{0, 1\}^k\} \end{aligned}$$

and

$$\begin{aligned} &S_p^{(m)}(l_1 + \varepsilon_1 2^{m-2}, \dots, l_k + \varepsilon_k 2^{m-2}) \\ &= S_p^{(m-1)}(l_1, \dots, l_k) \sin\left(\frac{2l_1 - 1 + \dots + 2l_k - 1 - p}{2^m} \pi + \frac{\varepsilon_1 + \dots + \varepsilon_k}{2} \pi\right) \\ &\quad \times \sin\left(\frac{2l_1 - 1}{2^m} \pi + \frac{\varepsilon_1}{2} \pi\right) \times \dots \times \sin\left(\frac{2l_k - 1}{2^m} \pi + \frac{\varepsilon_k}{2} \pi\right), \end{aligned}$$

we see that

$$\begin{aligned} A_p^{(m)}(K_1, \dots, K_k) &= \sum_{1 \leq l_1, \dots, l_k \leq 2^{m-2}} \sum_{\varepsilon_1, \dots, \varepsilon_k \in \{0,1\}} S_p^{(m)}(l_1 + \varepsilon_1 2^{m-2}, \dots, l_k + \varepsilon_k 2^{m-2}) \\ &\quad \times \prod_{i=1}^k e^{\sqrt{-1}2\pi(2(l_i + \varepsilon_i 2^{m-2}) - 1) \frac{K_i}{2^m}} \\ &= \sum_{1 \leq l_1, \dots, l_k \leq 2^{m-2}} S_p^{(m-1)}(l_1, \dots, l_k) \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i - 1) \frac{K_i}{2^m}} \\ &\quad \times \sum_{\varepsilon_1, \dots, \varepsilon_k \in \{0,1\}} \prod_{i=1}^k \sin\left(\frac{2l_i - 1}{2^m} \pi + \frac{\varepsilon_i}{2} \pi\right) \\ &\quad \times \sin\left(\frac{2l_1 - 1 + \dots + 2l_k - 1 - p}{2^m} \pi + \frac{\varepsilon_1 + \dots + \varepsilon_k}{2} \pi\right) \prod_{i=1}^k (-1)^{\varepsilon_i K_i}. \end{aligned}$$

Next, using the identity

$$\sum_{\varepsilon \in \{0,1\}} \sin\left(X + \frac{\varepsilon}{2}\pi\right) e^{\sqrt{-1}\frac{\varepsilon}{2}\pi} (-1)^{\varepsilon K} = \sqrt{-1}(-1)^K e^{-\sqrt{-1}(-1)^K X}, \quad \forall X \in \mathbb{R}, \forall K \in \mathbb{Z},$$

we compute the summation $\sum_{\varepsilon_1, \dots, \varepsilon_k \in \{0,1\}} \dots$ in the last line as

$$\begin{aligned} &= \Im\left(e^{\sqrt{-1}\pi \frac{2l_1-1+\dots+2l_k-1-p}{2m}} \sum_{\varepsilon_1, \dots, \varepsilon_k \in \{0,1\}} \prod_{i=1}^k \left(\sin\left(\frac{2l_i-1}{2m}\pi + \frac{\varepsilon_i}{2}\pi\right) e^{\sqrt{-1}\frac{\varepsilon_i}{2}\pi} (-1)^{\varepsilon_i K_i}\right)\right) \\ &= \Im\left(e^{\sqrt{-1}\pi \frac{2l_1-1+\dots+2l_k-1-p}{2m}} \prod_{i=1}^k \left(\sum_{\varepsilon \in \{0,1\}} \sin\left(\frac{2l_i-1}{2m}\pi + \frac{\varepsilon}{2}\pi\right) e^{\sqrt{-1}\frac{\varepsilon}{2}\pi} (-1)^{\varepsilon K_i}\right)\right) \\ &= \Im\left(e^{\sqrt{-1}\pi \frac{2l_1-1+\dots+2l_k-1-p}{2m}} (\sqrt{-1})^k (-1)^{K_1+\dots+K_k} e^{-\sqrt{-1}\pi \sum_{i=1}^k (-1)^{K_i} \frac{2l_i-1}{2m}}\right) \\ &= (-1)^{K_1+\dots+K_k} \Im\left(e^{\sqrt{-1}\pi \left(\frac{k}{2} + \sum_{i=1}^k (1-(-1)^{K_i}) \frac{2l_i-1}{2m} - \frac{p}{2m}\right)}\right) \\ &= (-1)^{K_1+\dots+K_k} \frac{1}{2\sqrt{-1}} \left\{ e^{\sqrt{-1}\pi \left(\frac{k}{2} + \sum_{i=1}^k (1-(-1)^{K_i}) \frac{2l_i-1}{2m} - \frac{p}{2m}\right)} \right. \\ &\quad \left. - e^{-\sqrt{-1}\pi \left(\frac{k}{2} + \sum_{i=1}^k (1-(-1)^{K_i}) \frac{2l_i-1}{2m} - \frac{p}{2m}\right)} \right\}. \end{aligned}$$

Substituting the last expression into the above, we have

$$\begin{aligned} &A_p^{(m)}(K_1, \dots, K_k) \\ &= \sum_{1 \leq l_1, \dots, l_k \leq 2^{m-2}} S_p^{(m-1)}(l_1, \dots, l_k) e^{\sqrt{-1}2\pi \sum_{i=1}^k K_i \frac{2l_i-1}{2m}} \\ &\quad \times (-1)^{K_1+\dots+K_k} \frac{1}{2\sqrt{-1}} \left\{ (\sqrt{-1})^k e^{-\sqrt{-1}\pi \frac{p}{2m}} e^{\sqrt{-1}2\pi \sum_{i=1}^k \frac{1-(-1)^{K_i}}{2} \frac{2l_i-1}{2m}} \right. \\ &\quad \left. - (-\sqrt{-1})^k e^{\sqrt{-1}\pi \frac{p}{2m}} e^{-\sqrt{-1}2\pi \sum_{i=1}^k \frac{1-(-1)^{K_i}}{2} \frac{2l_i-1}{2m}} \right\} \\ &= \frac{(\sqrt{-1})^{k-1}}{2} (-1)^{K_1+\dots+K_k} \sum_{1 \leq l_1, \dots, l_k \leq 2^{m-2}} S_p^{(m-1)}(l_1, \dots, l_k) \\ &\quad \times \left\{ e^{-\sqrt{-1}\pi \frac{p}{2m}} e^{\sqrt{-1}2\pi \sum_{i=1}^k \left(K_i + \frac{1-(-1)^{K_i}}{2}\right) \frac{2l_i-1}{2m}} \right. \\ &\quad \left. + (-1)^{k-1} e^{\sqrt{-1}\pi \frac{p}{2m}} e^{\sqrt{-1}2\pi \sum_{i=1}^k \left(K_i - \frac{1-(-1)^{K_i}}{2}\right) \frac{2l_i-1}{2m}} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{(\sqrt{-1})^{k-1}}{2} (-1)^{K_1+\dots+K_k} \\
 &\quad \times \left\{ e^{-\sqrt{-1}\pi \frac{p}{2^m}} \sum_{1 \leq l_1, \dots, l_k \leq 2^{m-2}} S_p^{(m-1)}(l_1, \dots, l_k) \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1) \frac{s_1(K_i)}{2^{m-1}}} \right. \\
 &\quad \left. + (-1)^{k-1} e^{\sqrt{-1}\pi \frac{p}{2^m}} \sum_{1 \leq l_1, \dots, l_k \leq 2^{m-2}} S_p^{(m-1)}(l_1, \dots, l_k) \prod_{i=1}^k e^{\sqrt{-1}2\pi(2l_i-1) \frac{s_0(K_i)}{2^{m-1}}} \right\} \\
 &= \frac{(\sqrt{-1})^{k-1}}{2} (-1)^{K_1+\dots+K_k} \left\{ e^{-\sqrt{-1}\pi \frac{p}{2^m}} A_p^{(m-1)}(s_1(K_1), \dots, s_1(K_k)) \right. \\
 &\quad \left. + (-1)^{k-1} e^{\sqrt{-1}\pi \frac{p}{2^m}} A_p^{(m-1)}(s_0(K_1), \dots, s_0(K_k)) \right\} \\
 &= \sum_{\delta \in \{0,1\}} \frac{(\sqrt{-1})^{k-1}}{2} \left(\prod_{i=1}^k (-1)^{K_i} \right) (-1)^{(k-1)(1-\delta)} e^{-\sqrt{-1}\pi \frac{p}{2^m} (2\delta-1)} \\
 &\quad \times A_p^{(m-1)}(s_\delta(K_1), \dots, s_\delta(K_k)),
 \end{aligned}$$

which is just the desired expression. \square

LEMMA 5. *There exists an $m_0 \in \mathbb{N}$ such that for $\forall p \in \mathbb{Z}$ and $\forall m > m_0$*

$$\rho_p(m) \leq \left(1 - \frac{1}{2^{2m_0-1}} (1 - |\sin \frac{\pi p}{2^{m-m_0+2}}|) \right) \rho_p(m - m_0).$$

PROOF. Choose $\eta_0, \mu_0 \in \mathbb{N}$ such that $2^{\mu_0} \leq n_k \eta_0 < 2^{\mu_0+1}$ and $n_{k-1} \eta_0 < 2^{\mu_0}$, and set $h_0, m_0 \in \mathbb{N}$ by

$$\begin{aligned}
 h_0 &:= 2^{\left\lceil \frac{\log n_k}{\log 2} \right\rceil + 1} \eta_0 + 1, \\
 m_0 &:= \left\lceil \frac{\log n_k}{\log 2} \right\rceil + \mu_0 + 3.
 \end{aligned}$$

Then, for $\forall t_i \in \{0, \dots, n_i\}$ ($1 \leq i \leq k$)

$$\begin{aligned}
 n_i(h_0 - 1) + t_i &< 2^{m_0-2} \quad (1 \leq \forall i \leq k-1), \\
 2^{m_0-2} &\leq n_k(h_0 - 1) + t_k < 2^{m_0-1},
 \end{aligned}$$

i.e., for $\exists \varepsilon_{i0}, \dots, \varepsilon_{i m_0-3} \in \{0, 1\}$ ($1 \leq i \leq k$)

$$n_i(h_0 - 1) + t_i = \varepsilon_{i0} + \varepsilon_{i1}2 + \dots + \varepsilon_{i m_0-3} 2^{m_0-3}, \quad 1 \leq i \leq k-1, \quad (2)$$

$$n_k(h_0 - 1) + t_k = \varepsilon_{k0} + \varepsilon_{k1}2 + \dots + \varepsilon_{k, m_0-3} 2^{m_0-3} + 2^{m_0-2}. \quad (3)$$

In the following, let $t_i \in \{0, \dots, n_i\}$ ($1 \leq i \leq k$) be fixed arbitrarily.

1° Noting that for $m > m_0 \geq 1$

$$\{0, 1, \dots, 2^m - 1\} = \{h - 1 + 2^{m_0}(g - 1); h \in \{1, \dots, 2^{m_0}\}, g \in \{1, \dots, 2^{m-m_0}\}\},$$

we see

$$\begin{aligned} & \frac{1}{2^m} \sum_{j=1}^{2^m} |A_p^{(m)}(n_1(j-1) + t_1, \dots, n_k(j-1) + t_k)| \\ &= \frac{1}{2^m} \sum_{h=1}^{2^{m_0}} \sum_{g=1}^{2^{m-m_0}} |A_p^{(m)}(n_1(h-1 + 2^{m_0}(g-1)) + t_1, \dots, n_k(h-1 + 2^{m_0}(g-1)) + t_k)| \\ &= \frac{1}{2^{m_0}} \sum_{h=1}^{2^{m_0}} \frac{1}{2^{m-m_0}} \sum_{g=1}^{2^{m-m_0}} |A_p^{(m)}(2^{m_0}n_1(g-1) + n_1(h-1) + t_1, \dots, 2^{m_0}n_k(g-1) \\ & \quad + n_k(h-1) + t_k)|. \end{aligned}$$

2° By Lemma 4

$$\begin{aligned} & A_p^{(m)}(2^{m_0}n_1(g-1) + n_1(h-1) + t_1, \dots, 2^{m_0}n_k(g-1) + n_k(h-1) + t_k) \\ &= \sum_{\delta_1, \dots, \delta_{m_0} \in \{0,1\}} \left(\frac{(\sqrt{-1})^{k-1}}{2} \right)^{m_0} \prod_{i=1}^k (-1)^{\sum_{a=1}^{m_0} s_{\delta_{a-1}} \dots s_{\delta_1} (n_i(h-1) + t_i)} \\ & \quad \times (-1)^{(k-1)(1-\delta_{m_0} + \dots + 1 - \delta_1)} e^{-\sqrt{-1} \pi p \left(\frac{2\delta_1 - 1}{2^m} + \frac{2\delta_2 - 1}{2^{m-1}} + \dots + \frac{2\delta_{m_0} - 1}{2^{m-m_0+1}} \right)} \\ & \quad \times A_p^{(m-m_0)}(n_1(g-1) + s_{\delta_{m_0}} \dots s_{\delta_1} (n_1(h-1) + t_1), \dots, n_k(g-1) \\ & \quad + s_{\delta_{m_0}} \dots s_{\delta_1} (n_k(h-1) + t_k)), \end{aligned}$$

where

$$s_{\delta_{a-1}} \dots s_{\delta_1} = \begin{cases} \text{id} & \text{if } a = 1, \\ s_{\delta_{a-1}} \circ \dots \circ s_{\delta_1} & \text{if } a > 1. \end{cases}$$

3° Note that for $0 \leq r \leq n2^{m_0}$, $0 \leq s_{\delta_{m_0}} \dots s_{\delta_1}(r) \leq n$ ($\delta_1, \dots, \delta_{m_0} \in \{0, 1\}$). By 2°, this implies that for each $h = 1, \dots, 2^{m_0}$

$$\begin{aligned} & \frac{1}{2^{m-m_0}} \sum_{g=1}^{2^{m-m_0}} |A_p^{(m)}(2^{m_0}n_i(g-1) + n_i(h-1) + t_i)| \\ & \leq \frac{1}{2^{m-m_0}} \sum_{g=1}^{2^{m-m_0}} \sum_{\delta_1, \dots, \delta_{m_0} \in \{0,1\}} \frac{1}{2^{m_0}} |A_p^{(m-m_0)}(n_i(g-1) + s_{\delta_{m_0}} \dots s_{\delta_1} (n_i(h-1) + t_i))| \\ & = \frac{1}{2^{m_0}} \sum_{\delta_1, \dots, \delta_{m_0} \in \{0,1\}} \frac{1}{2^{m-m_0}} \sum_{g=1}^{2^{m-m_0}} |A_p^{(m-m_0)}(n_i(g-1) + s_{\delta_{m_0}} \dots s_{\delta_1} (n_i(h-1) + t_i))| \end{aligned}$$

$$\begin{aligned} &\leq \max_{\substack{\tau_1 \in \{0, \dots, n_1\}, \\ \tau_k \in \{0, \dots, n_k\}}} \frac{1}{2^{m-m_0}} \sum_{g=1}^{2^{m-m_0}} |A_p^{(m-m_0)}(n_i(g-1) + \tau_i)| \\ &= \rho_p(m-m_0). \end{aligned}$$

Here and in the sequel, for simplicity we write $A_p^{(m)}(K_1, \dots, K_k)$ as $A_p^{(m)}(K_i)$.

4° Let $\delta^{(1)} = (0, \dots, 0, 0, 1)$, $\delta^{(2)} = (0, \dots, 0, 1, 0) \in \{0, 1\}^{m_0}$. Then, by (2) and (3)

$$\begin{aligned} s_{\delta_{m_0-1}^{(1)}} \cdots s_{\delta_1^{(1)}}(n_i(h_0-1) + t_i) &= 0, \quad 1 \leq \forall i \leq k, \\ s_{\delta_{m_0}^{(1)}} \cdots s_{\delta_1^{(1)}}(n_i(h_0-1) + t_i) &= 0, \quad 1 \leq \forall i \leq k, \\ s_{\delta_{m_0-1}^{(2)}} \cdots s_{\delta_1^{(2)}}(n_i(h_0-1) + t_i) &= \begin{cases} 0 & 1 \leq \forall i \leq k-1, \\ 1 & i = k \end{cases} \\ s_{\delta_{m_0}^{(2)}} \cdots s_{\delta_1^{(2)}}(n_i(h_0-1) + t_i) &= 0, \quad 1 \leq \forall i \leq k, \end{aligned}$$

and hence

$$\begin{aligned} &\prod_{i=1}^k (-1)^{\sum_{a=1}^{m_0} s_{\delta_{a-1}^{(1)}} \cdots s_{\delta_1^{(1)}}(n_i(h_0-1)+t_i)} (-1)^{(k-1)(1-\delta_{m_0}^{(1)}+\dots+1-\delta_1^{(1)})} \\ &= \prod_{i=1}^k (-1)^{\sum_{a=1}^{m_0-1} s_0^{a-1}(n_i(h_0-1)+t_i)} (-1)^{(k-1)(m_0-1)}, \\ &\prod_{i=1}^k (-1)^{\sum_{a=1}^{m_0} s_{\delta_{a-1}^{(2)}} \cdots s_{\delta_1^{(2)}}(n_i(h_0-1)+t_i)} (-1)^{(k-1)(1-\delta_{m_0}^{(2)}+\dots+1-\delta_1^{(2)})} \\ &= (-1) \prod_{i=1}^k (-1)^{\sum_{a=1}^{m_0-1} s_0^{a-1}(n_i(h_0-1)+t_i)} (-1)^{(k-1)(m_0-1)}. \end{aligned}$$

By 2°, this implies

$$\begin{aligned} &A_p^{(m)}(2^{m_0}n_i(g-1) + n_i(h_0-1) + t_i) \\ &= \left(\frac{(\sqrt{-1})^{k-1}}{2} \right)^{m_0} \prod_{i=1}^k (-1)^{\sum_{a=1}^{m_0-1} s_0^{a-1}(n_i(h_0-1)+t_i)} (-1)^{(k-1)(m_0-1)} \\ &\quad \times \left(e^{-\sqrt{-1}\pi p(\frac{-1}{2^m} + \dots + \frac{-1}{2^{m-m_0+2}} + \frac{1}{2^{m-m_0+1}})} \right. \\ &\quad \left. - e^{-\sqrt{-1}\pi p(\frac{-1}{2^m} + \dots + \frac{-1}{2^{m-m_0+3}} + \frac{1}{2^{m-m_0+2}} + \frac{-1}{2^{m-m_0+1}})} \right) A_p^{(m-m_0)}(n_i(g-1)) \\ &+ \sum_{\delta \in \{0, 1\}^{m_0} \setminus \{\delta^{(1)}, \delta^{(2)}\}} \left(\frac{(\sqrt{-1})^{k-1}}{2} \right)^{m_0} \prod_{i=1}^k (-1)^{\sum_{a=1}^{m_0} s_{\delta_{a-1}} \cdots s_{\delta_1}(n_i(h_0-1)+t_i)} \end{aligned}$$

$$\begin{aligned} & \times (-1)^{(k-1)(1-\delta_{m_0}+\dots+1-\delta_1)} e^{-\sqrt{-1}\pi p(\frac{2\delta_1-1}{2^m}+\dots+\frac{2\delta_{m_0}-1}{2^{m-m_0+1}})} \\ & \times A_p^{(m-m_0)}(n_i(g-1) + s_{\delta_{m_0}} \cdots s_{\delta_1}(n_i(h_0-1) + t_i)), \end{aligned}$$

and by taking the absolute value

$$\begin{aligned} & |A_p^{(m)}(2^{m_0}n_i(g-1) + n_i(h_0-1) + t_i)| \\ & \leq \frac{1}{2^{m_0}} 2 \left| \sin \frac{\pi p}{2^{m-m_0+2}} \right| |A_p^{(m-m_0)}(n_i(g-1))| \\ & \quad + \frac{1}{2^{m_0}} \sum_{\delta \in \{0,1\}^{m_0} \setminus \{\delta^{(1)}, \delta^{(2)}\}} |A_p^{(m-m_0)}(n_i(g-1) + s_{\delta_{m_0}} \cdots s_{\delta_1}(n_i(h_0-1) + t_i))|. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{1}{2^{m-m_0}} \sum_{g=1}^{2^{m-m_0}} |A_p^{(m)}(2^{m_0}n_i(g-1) + n_i(h_0-1) + t_i)| \\ & \leq \frac{1}{2^{m_0}} 2 \left| \sin \frac{\pi p}{2^{m-m_0+2}} \right| \frac{1}{2^{m-m_0}} \sum_{g=1}^{2^{m-m_0}} |A_p^{(m-m_0)}(n_i(g-1))| \\ & \quad + \frac{1}{2^{m_0}} \sum_{\delta \in \{0,1\}^{m_0} \setminus \{\delta^{(1)}, \delta^{(2)}\}} \frac{1}{2^{m-m_0}} \sum_{g=1}^{2^{m-m_0}} |A_p^{(m-m_0)}(n_i(g-1) \\ & \qquad \qquad \qquad + s_{\delta_{m_0}} \cdots s_{\delta_1}(n_i(h_0-1) + t_i))| \\ & \leq \left(1 - \frac{1}{2^{m_0-1}} \left(1 - \left| \sin \frac{\pi p}{2^{m-m_0+2}} \right| \right) \right) \rho_p(m-m_0). \end{aligned}$$

5° Collecting 1°, 3° and 4°, we have

$$\begin{aligned} & \frac{1}{2^m} \sum_{j=1}^{2^m} |A_p^{(m)}(n_i(j-1) + t_i)| \\ & \leq \frac{1}{2^{m_0}} \sum_{h \in \{1, \dots, 2^{m_0}\} \setminus \{h_0\}} \rho_p(m-m_0) \\ & \quad + \frac{1}{2^{m_0}} \left(1 - \frac{1}{2^{m_0-1}} \left(1 - \left| \sin \frac{\pi p}{2^{m-m_0+2}} \right| \right) \right) \rho_p(m-m_0) \\ & = \left(1 - \frac{1}{2^{2m_0-1}} \left(1 - \left| \sin \frac{\pi p}{2^{m-m_0+2}} \right| \right) \right) \rho_p(m-m_0), \end{aligned}$$

from which the conclusion follows at once. \square

PROOF OF THEOREM 1. Let $p \in \mathbb{Z}$ be fixed. Since $|\sin \frac{\pi p}{2^{m-m_0+2}}| < 1/2$ for $m \geq \lceil \frac{\log |p| \vee 1}{\log 2} \rceil + 2 + m_0$ (where m_0 is an integer in Lemma 5), Lemma 5 tells us that

$$\rho_p(m) \leq \left(1 - \frac{1}{4^{m_0}}\right) \rho_p(m - m_0).$$

This yields that for $\forall q \geq 0, 0 \leq \forall r < m_0$

$$\rho_p\left(\left\lceil \frac{\log |p| \vee 1}{\log 2} \right\rceil + 2 + qm_0 + r\right) \leq \left(1 - \frac{1}{4^{m_0}}\right)^q \rho_p\left(\left\lceil \frac{\log |p| \vee 1}{\log 2} \right\rceil + 2\right).$$

If we express $m \geq 0$ as $m = qm_0 + r$ ($q \geq 0, 0 \leq r < m_0$), then $q > m/m_0 - 1$ and so

$$\left(1 - \frac{1}{4^{m_0}}\right)^q \leq \left(1 - \frac{1}{4^{m_0}}\right)^{\frac{m}{m_0} - 1}.$$

Combining this observation with the above, we have

$$\rho_p\left(\left\lceil \frac{\log |p| \vee 1}{\log 2} \right\rceil + 2 + m\right) \leq \left(1 - \frac{1}{4^{m_0}}\right)^{\frac{m}{m_0} - 1} \rho_p\left(\left\lceil \frac{\log |p| \vee 1}{\log 2} \right\rceil + 2\right), \quad \forall m \geq 0.$$

Consequently, by Lemma 3, we conclude that the assertion of Theorem 1 is valid with $\rho = (1 - 1/4^{m_0})^{\frac{1}{m_0}}$. \square

References

- [1] H. SUGITA, Pseudo-random number generator by means of irrational rotation, *Monte Carlo Methods and Appl.* **1** (1995), 35–57.
- [2] H. SUGITA, Pseudo-random number generator by means of irrational rotation — A proof by Fourier series expansion (in Japanese), preprint (Dec. 1997).
- [3] H. SUGITA, Pseudo-random number generator by means of irrational rotation — An ergodic consideration (in Japanese), preprint (Oct. 1998).
- [4] H. SUGITA and S. TAKANOBU, Weyl transformation and binary transformation (in Japanese), preprint (Jul. 1998).
- [5] H. SUGITA and S. TAKANOBU, Pseudo-random number generator by means of irrational rotation — Exponential decay of multi-term correlations (in Japanese), preprint (Jun. 1999).
- [6] P. WALTERS, *An introduction to ergodic theory*, Springer (1982).

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