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On the Strong Rainbow Connection of a Graph

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Abstract. A path in an edge-colored graph, where adjacent edges may be colored the same, is a rainbow path if no two edges of it are colored the same. For any two vertices u and v of G, a rainbow $u-v$ geodesic in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is strongly rainbow connected if there exists a rainbow $u-v$ geodesic for any two vertices u and v in G. The strong rainbow connection number of G, denoted by $src(G)$, is the minimum number of colors that are needed in order to make G strongly rainbow connected. In this paper, we first give a sharp upper bound for $src(G)$ in terms of the number of edge-disjoint triangles in a graph G, and give a necessary and sufficient condition for the equality. We next investigate the graphs with large strong rainbow connection numbers. Chartrand *et al.* obtained that $src(G)=m$ if and only if G is a tree, we will show that $src(G)\neq m-1$, and characterize the graphs G with $src(G)=m-2$ where m is the number of edges of G.

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1. Introduction

All graphs in this paper are finite, undirected and simple. Let G be a nontrivial connected graph on which there is a coloring $c: E(G)\rightarrow\{1,2,\cdots,n\}$, $n\in\mathbb{N}$, of the edges of G, where adjacent edges may be colored the same. A path is a *rainbow path* if no two edges of it are colored the same. An edge-colored graph G is *rainbow connected* if any two vertices are connected by a rainbow path. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, we define the *rainbow connection number* of a connected graph G, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected. Let c be a rainbow coloring of a connected graph G. For any two vertices u and v of G, a *rainbow $u-v$ geodesic* in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is *strongly rainbow connected* if there exists a rainbow $u-v$ geodesic for any pair of vertices u and v in G. In this case, the coloring c is called a *strong rainbow coloring*

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of G. Similarly, we define the *strong rainbow connection number* of a connected graph G, denoted by $src(G)$, as the smallest number of colors that are needed in order to make G strongly rainbow connected. A strong rainbow coloring of G using $src(G)$ colors is called a *minimum strong rainbow coloring* of G. Clearly, we have $diam(G) \leq rc(G) \leq src(G) \leq m$ where $diam(G)$ denotes the diameter of G and m is the number of edges of G.

The topic of rainbow connection number is fairly interesting and recently a series of papers have been written about it. The reader can see [7] for a monograph and [8] for a survey on this topic. The strong rainbow connection number is also interesting and, by definition, the investigation of it is more challenging than that of rainbow connection number. However, there are very few papers that have been written about it. In this paper, we do research on it. In [3], Chartrand *et al.* determined the precise strong rainbow connection numbers for some special graph classes including trees, complete graphs, wheels, complete bipartite (multipartite) graphs.

Recently, Ananth and Nasre [1] derived the following hardness result about the strong rainbow connection number.

Theorem 1.1. [1] *For every integer $k \geq 3$, deciding whether $src(G) \leq k$, is NP-hard even when G is bipartite.*

So, for a general graph G, it is almost impossible to give the precise value for $src(G)$. And we aim to give upper bounds for it according to some graph parameters. In this paper, we will derive a sharp upper bound for $src(G)$ in terms of the number of edge-disjoint triangles (if exist) in a graph G, and give a necessary and sufficient condition for the equality (Theorem 3.1).

In [4], the authors investigated the graphs with small rainbow connection numbers, they showed a sufficient condition that guarantees $rc(G) = 2$ and gave a threshold function for a random graph $G = G(n, p)$ to have $rc(G(n, p)) \leq 2$.

Theorem 1.2. [4] *Any non-complete graph with $\delta(G) \geq n/2 + \log n$ has $rc(G) = 2$.*

Theorem 1.3. [4] *$p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc(G(n, p)) \leq 2$.*

In [3], the authors derived that the problem of considering graphs with $rc(G) = 2$ is equivalent to that of considering graphs with $src(G) = 2$.

Proposition 1.1. [3] *$rc(G) = 2$ if and only if $src(G) = 2$.*

In Section 4.2 of [7], Li and Sun did research on graphs with large rainbow connection numbers, and showed that $rc(G) \neq m-1$ and characterized the graphs with $rc(G) = m-2$. In this paper, we aim to investigate the graphs with large strong rainbow connection numbers. In [3], Chartrand *et al.* obtained that $src(G) = m$ if and only if G is a tree. We will show that $src(G) \neq m-1$ and characterize the graphs with $src(G) = m-2$ by showing that $src(G) = m-2$ if and only if G is a 5-cycle or belongs to one of two graph classes (Theorem 4.1).

We use $V(G)$, $E(G)$ for the set of vertices and edges of G, respectively. For any subset X of $V(G)$, let $G[X]$ denote the subgraph induced by X, and $E[X]$ the edge set of $G[X]$; similarly, for any subset E_1 of $E(G)$, let $G[E_1]$ denote the subgraph induced by E_1. Let $\mathscr{G}$ be a set of graphs, then $V(\mathscr{G}) = \bigcup_{G \in \mathscr{G}} V(G)$, $E(\mathscr{G}) = \bigcup_{G \in \mathscr{G}} E(G)$. A *rooted tree* $T(x)$ is a tree T with a specified vertex x, called the *root* of T. The path xTv is the unique $x-v$

path in T, each vertex on the path xTv, including the vertex v itself, is called an *ancestor* of v, an ancestor of a vertex is *proper* if it is not the vertex itself, the immediate proper ancestor of a vertex v other than the root is its *parent* and the vertices with parent v are its *children* or *sons*. We let P_n and C_n denote the path and cycle with n vertices, respectively. If $P: u_1, u_2, \cdots, u_t$ is a path, then the $u_i - u_j$ *section* of P, denoted by u_iPu_j, is the path $u_i, u_{i+1}, \cdots, u_j$. Similarly, for a cycle $C: v_1, \cdots, v_t, v_1$, we define the $v_i - v_j$ section, denoted by v_iCv_j, of C, and C contains two $v_i - v_j$ sections. Note the fact that if P is a $u_1 - u_t$ geodesic, then u_iPu_j is also a $u_i - u_j$ geodesic where $1 \leq i, j \leq t$. We use $l(P)$ to denote the length of a path P. For a set S, $|S|$ denotes the cardinality of S. In a graph G with at least one cycle, the length of a shortest cycle is called its *girth*, denoted by $g(G)$. In an edge-colored graph G, we use $c(e)$ to denote the color of an edge e, and for a subgraph G_1 of G, we use $c(G_1)$ to denote the set of colors of the edges in G_1. We follow the notation and terminology of [2].

2. Basic results

We first give a necessary condition for an edge-colored graph to be strongly rainbow connected. If G contains at least two cut edges, then for any two cut edges $e_1 = u_1u_2$, $e_1 = v_1v_2$, there must exist some $1 \leq i_0, j_0 \leq 2$, such that any $u_{i_0} - v_{j_0}$ path must contain the edges e_1, e_2. So we have:

Observation 2.1. If G is strongly rainbow connected under some edge-coloring, and e_1, e_2 are two cut edges, then $c(e_1) \neq c(e_2)$.

The following lemma will be useful in our discussion.

Lemma 2.1. *If $src(G) = m-1$ or $m-2$, then $3 \leq g(G) \leq 5$.*

Proof. Let $C: v_1, \cdots, v_k, v_{k+1} = v_1$ be a minimum cycle of G with $k = g(G)$, and $e_i = v_iv_{i+1}$ for each $1 \leq i \leq k$, we suppose $k \geq 6$. We give the cycle C the same strong rainbow coloring as in [3]: If k is even, let $k = 2\ell$ for some integer $\ell \geq 3$, $c(e_i) = i$ for $1 \leq i \leq \ell$ and $c(e_i) = i - \ell$ for $\ell + 1 \leq i \leq k$; If k is odd, let $k = 2\ell + 1$ for some integer $\ell \geq 3$, $c(e_i) = i$ for $1 \leq i \leq \ell + 1$ and $c(e_i) = i - \ell - 1$ for $\ell + 2 \leq i \leq k$. We color each other edge with a fresh color. This procedure costs $\lceil \frac{k}{2} \rceil + (m-k) = m - (k - \lceil \frac{k}{2} \rceil) \leq m - 3$ colors totally.

We only consider the case $k = 2\ell (\ell \geq 3)$, since the case for $k = 2\ell + 1 (\ell \geq 3)$ can be done similarly. Let $P: u = u_1, \cdots, v = u_t$ be a $u - v$ geodesic of G. If there are two edges of P, say e'_1, e'_2, with the same color, then they must be in C. Without loss of generality, let $e'_1 = v_1v_2$. We first consider the case that $e'_1 = v_1v_2$, and $v_1 = u_{i_1}, v_2 = u_{i_1+1}$ for some $1 \leq i_1 \leq t$. Then we must have $e'_2 = v_{\ell+1}v_{\ell+2}$ where $v_{\ell+1} = u_{j_1}$, $v_{\ell+2} = u_{j_1+1}$ for some $i_1 + 1 \leq j_1 \leq t$ or $v_{\ell+2} = u_{j_2}$, $v_{\ell+1} = u_{j_2+1}$ for some $i_1 + 1 \leq j_2 \leq t$. If $v_{\ell+1} = u_{j_1}$, $v_{\ell+2} = u_{j_1+1}$ for some $i_1 + 1 \leq j_1 \leq t$, then the section $v_2Pv_{\ell+1}$ of P is a $v_2 - v_{\ell+1}$ geodesic, and so it is not longer than the section $C': v_2, v_3, \cdots, v_{\ell+1}$ of C, then the length of $v_2Pv_{\ell+1}$, $l(v_2Pv_{\ell+1}) \leq \ell - 1$, is smaller than the length of the section $C'': v_2, v_1, v_k, \cdots, v_{\ell+1}$ of C. So the sections $v_2Pv_{\ell+1}$ and C' will produce a smaller cycle than C (this produces a contradiction), or $v_2Pv_{\ell+1}$ is the same as C' (but in this case, the section $C''': v_1, v_k, \cdots, v_{\ell+2}$ of C is shorter than $v_1Pv_{\ell+2}$ which now is a $v_1 - v_{\ell+2}$ geodesic, this also produces a contradiction). If $v_{\ell+2} = u_{j_2}$, $v_{\ell+1} = u_{j_2+1}$ for some $i_1 + 1 \leq j_2 \leq t$, then the section $v_1Pv_{\ell+2}$ of P is a $v_1 - v_{\ell+2}$ geodesic, so it is not longer than the length of the section $\overline{C'}: v_1, v_k, v_{k-1}, \cdots, v_{\ell+2}$ of C and its length, $l(v_1Pv_{\ell+2}) \leq \ell - 1$, is smaller than that of the section $\overline{C''}: v_1, v_2, \cdots, v_{\ell+2}$ of C. So the sections $v_1Pv_{\ell+2}$ and $\overline{C'}$

will produce a smaller cycle than C, this also produces a contradiction. So P is strongly rainbow. The remaining two subcases correspond to the case that $v_1 = u_{i_1+1}$, $v_2 = u_{i_1}$, and with a similar argument, a contradiction will be produced. Then the conclusion holds. ∎

Note that we have proved the above lemma by contradiction: we first chose a smallest cycle C of a graph G, then gave it a strong rainbow coloring the same as in [3], and gave a fresh color to any other edge. Then for any $u-v$ geodesic P, we derived that either one section of P was the same as one section of C and then found a shorter path than the geodesic, or one section of P and one section of C produced a smaller cycle than C, each of these two cases would produce a contradiction. This technique will be useful in the sequel.

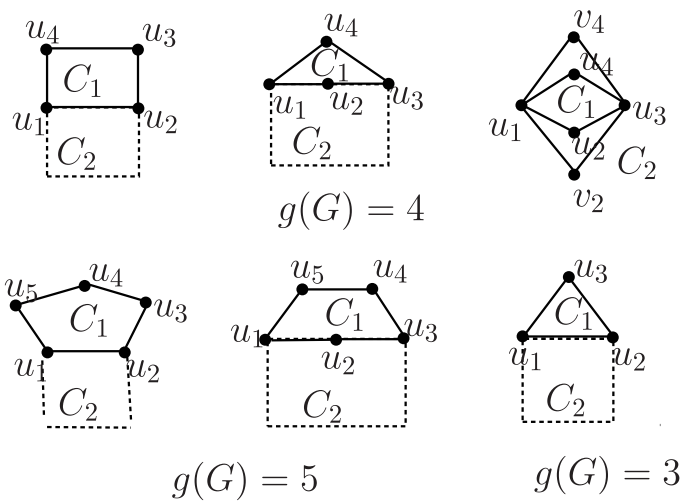

Figure 1. The graphs for Observation 2.2.

The following observation is obvious and we omit its proof.

Observation 2.2. Let G be a connected graph with at least one cycle, and $3 \leq g(G) \leq 5$. Let C_1 be the smallest cycle of G, and C_2 be the second smallest cycle (if exists) of G. If C_1 and C_2 have at least two common vertices, then we have:

(1) if $g(G) = 3$, then C_1 and C_2 have one common edge as shown in Figure 1;

(2) if $g(G) = 4$, then C_1 and C_2 have one common edge, or two common adjacent edges, or C_1 and C_2 are two edge-disjoint 4-cycles, as shown in Figure 1;

(3) if $g(G) = 5$, then C_1 and C_2 have one common edge, or two common adjacent edges, as shown in Figure 1.

The following observation is easy and very useful in the sequel.

Observation 2.3. For any two vertices u, $v \in G$, we have the following:

(1) if T is a triangle in a graph G, then any $u-v$ geodesic P contains at most one edge of T;

(2) if $g(G) = 4$ and C_1 is the smallest cycle of G, then any $u-v$ geodesic P contains at most one edge or two adjacent edges of C_1;

(3) if $g(G) = 5$ and C_1 is the smallest cycle of G, then any $u-v$ geodesic P contains at most one edge or two adjacent edges of C_1.

3. A sharp upper bound for $src(G)$ in terms of edge-disjoint triangles

In this section, we give an upper bound for $src(G)$ in terms of their edge-disjoint triangles (if exist) in a graph G, and give a necessary and sufficient condition for the equality.

Recall that a *block* of a connected graph G is a maximal connected subgraph without any cut vertex. Thus, every block of a graph G is either a maximal 2-connected subgraph or a bridge (cut edge). We now introduce a new graph class. For a connected graph G, we say $G \in \overline{\mathscr{G}}_t$, if it satisfies the following conditions: C_1: each block of G is a bridge or a triangle; C_2: G contains exactly t triangles; C_3: each triangle contains at least one vertex of degree two in G.

By definition, each graph $G \in \overline{\mathscr{G}}_t$ is formed by (edge-disjoint) triangles and paths (may be trivial), these triangles and paths fit together in a treelike structure, and G contains no cycles but the t (edge-disjoint) triangles. For example, see Figure 2, here $t = 2$, u_1, u_2, u_6 are vertices of degree 2 in G. If a tree is obtained from a graph $G \in \overline{\mathscr{G}}_t$ by deleting one vertex of degree 2 for each triangle, then we call this tree a D_2-*tree*, denoted by T_G, of G. For example, in Figure 2, T_G is a D_2-tree of G. Clearly, the D_2-tree is not unique, since in this example, we can obtain another D_2-tree by deleting u_1 instead of u_2. On the other hand, we can say that any element of $\overline{\mathscr{G}}_t$ can be obtained from a tree by adding t new vertices of degree 2. It is easy to show that the number of edges of T_G is $m-2t$ where m is the number of edges of G.

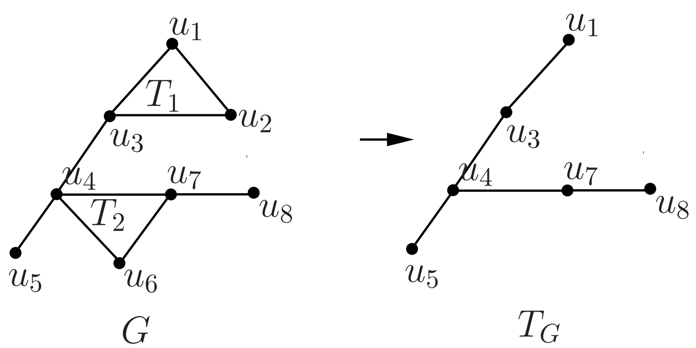

Figure 2. An example of $G \in \overline{\mathscr{G}}_t$ with $t = 2$.

Theorem 3.1. *If G is a graph with m edges and t edge-disjoint triangles, then*

$$src(G) \le m - 2t,$$

the equality holds if and only if $G \in \overline{\mathscr{G}}_t$.

Proof. Let $\mathscr{T} = \{T_i : 1 \le i \le t\}$ be a set of t edge-disjoint triangles in G. We color each triangle with a fresh color, that is, the three edges of each triangle receive the same color, then we give each other edge a fresh color. For any two vertices u, v of G, let P be any $u-v$ geodesic, then P contains at most one edge from each triangle by Observation 2.3, and so P

is strongly rainbow under the above coloring. As this procedure costs $m-2t$ colors totally, we have $src(G) \leq m-2t$.

Claim 1. If the equality holds, then for any set $\mathscr{T}$ of edge-disjoint triangles of G, we have $|\mathscr{T}| \leq t$.

Proof. We suppose that there is a set $\mathscr{T}'$ of t' edge-disjoint triangles in G with $t' > t$. Then, with a similar procedure, we have $src(G) \leq m-2t' < m-2t$, a contradiction. ∎

Claim 2. If the equality holds, then G contains no cycle but the above t (edge-disjoint) triangles.

Proof. We suppose that there is at least one cycle distinct with the above t triangles. Let $\mathscr{C}$ be the set of these cycles and C_1 be the smallest element of $\mathscr{C}$ with $|C_1| = k$. We will consider two cases:

Case 1. $E(C_1) \cap E(\mathscr{T}) = \emptyset$, that is, C_1 is edge-disjoint from each of the above t triangles. Clearly, C_1 has at most one common vertex with each of them. In this case $k \geq 4$ by Claim 1, and we give G an edge-coloring as follows: we first color the edges of C_1 the same as in [3] (this was shown in the proof of Lemma 2.1); then we color each triangle with a fresh color; for the remaining edges, we give each one a fresh color. Recall the fact that any geodesic contains at most one edge from each triangle and with a similar procedure to the proof of Lemma 2.1, we know that the above coloring is strongly rainbow, as this procedure costs $\lceil \frac{k}{2} \rceil + t + (m-k-3t) = (m-2t) + (\lceil \frac{k}{2} \rceil - k) < m-2t$ colors totally, we have $src(G) < m-2t$, this produces a contradiction.

Case 2. $E(C_1) \cap E(\mathscr{T}) \neq \emptyset$, that is, C_1 has common edges with the above t triangles, in this case $k \geq 3$. By the choice of C_1, we know that $|E(C_1) \cap E(T_i)| \leq 1$ for each $1 \leq i \leq t$. We will consider two subcases according to the parity of k.

Subcase 2.1. $k = 2\ell$ for some $\ell \geq 2$. For example, see the graph (α) of Figure 3, here $\mathscr{T} = \{T_1, T_2, T_3\}$, $V(C_1) = \{u_i : 1 \leq i \leq 6\}$, $E(C_1) \cap E(T_1) = \{u_1u_2\}$, $E(C_1) \cap E(T_2) = \{u_4u_5\}$. Without loss of generality, we assume that there exists a triangle, say T_1, which contains the edge u_1u_2, and let $V(T_1) = \{u_1, u_2, w_1\}$, $G' = G \backslash E(T_1)$. If there exists some triangle, say T_2, which contains the edge $u_{\ell+1}u_{\ell+2}$, we let $V(T_2) = \{u_{\ell+1}, u_{\ell+2}, w_2\}$.

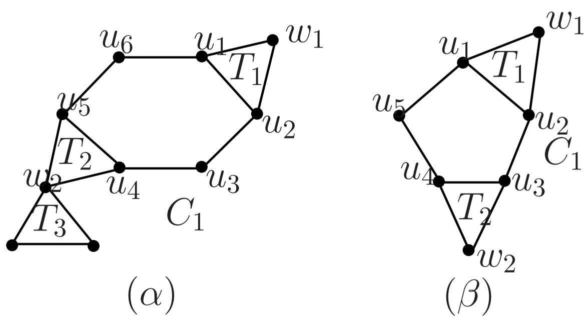

Figure 3. The graphs for the two examples in Theorem 3.1.

We first consider the case for $\ell = 2$, see Figure 4. We first give each triangle of G' a fresh color; for the remaining edges of G', we give each of them a fresh color; for the edges of T_1, let $c(u_1w_1) = c(u_2u_3)$, $c(u_2w_1) = c(u_1u_4)$, $c(u_1u_2) = c(u_3u_4)$. Then it is easy to show that there is a $u-v$ geodesic which contains at most one edge from any two edges with the same color for $u,v \in G$, and so the above coloring is strongly rainbow. As this procedure costs $m-2t-1 < m-2t$ colors totally, we have $src(G) < m-2t$, a contradiction.

We next consider the case for $\ell \geq 3$. Let $G'' = G\backslash(E(T_1) \cup E(T_2))$. We give G an edge-coloring as follows: We first give each triangle of G'' a fresh color; then give a fresh color to each of the remaining edges of G''; for the edges of T_1 and T_2, let $c(u_1w_1) = c(u_2u_3)$, $c(u_2w_1) = c(u_1u_k)$, $c(u_1u_2) = c(u_{\ell+1}u_{\ell+2}) = c$, $c(w_2u_{\ell+1}) = c(u_{\ell+2}u_{\ell+3})$, $c(w_2u_{\ell+2}) = c(u_\ell u_{\ell+1})$ where c is a new color. Then it is easy to show that there is a $u-v$ geodesic which contains at most one edge from any two edges with the same color for $u,v \in G$, and so the above coloring is strongly rainbow. As this procedure costs $m-2t-1 < m-2t$ colors totally, we have $src(G) < m-2t$, a contradiction. ∎

Subcase 2.2. $k = 2\ell+1$ for some $\ell \geq 1$.

We first consider the case for $\ell \geq 2$. For example, see the graph (β) of Figure 3, here $\mathscr{T} = \{T_1, T_2\}$, $V(C_1) = \{u_i : 1 \leq i \leq 5\}$, $E(C_1) \cap E(T_1) = \{u_1u_2\}$, $E(C_1) \cap E(T_2) = \{u_3u_4\}$. Without loss of generality, we assume that there exists a triangle, say T_1, which contains the edge u_1u_2, and let $V(T_1) = \{u_1, u_2, w_1\}$. If there exists some triangle, say T_2, which contains the edge $u_{\ell+1}u_{\ell+2}$, we let $V(T_2) = \{u_{\ell+1}, u_{\ell+2}, w_2\}$ and $G' = G\backslash(E(T_1) \cup E(T_2))$.

We give G an edge-coloring as follows: We first give each triangle of G' a fresh color; then give a fresh color to each of the remaining edges of G'; for the edges of T_1 and T_2, let $c(u_1w_1) = c(u_2u_3)$, $c(u_2w_1) = c(u_1u_k)$, $c(u_{\ell+1}w_2) = c(u_{\ell+2}u_{\ell+3})$ and let $c(u_1u_2) = c(u_{\ell+1}u_{\ell+2}) = c(w_2u_{\ell+2})$ be a fresh color. With a similar procedure to the proof of Lemma 2.1, we can show that G is strongly rainbow connected, and so $src(G) \leq (t-1)+(m-3t) = (m-2t)-1 < m-2t$, this produces a contradiction.

For the case for $\ell = 1$, that is, C_1 is a triangle, see Figure 4, we color the three edges (if exist) with color 1, these edges are shown in the figure; the remaining edges of these three triangles (if exist) all receive color 2; each other triangle receives a fresh color; for the remaining edges, we give each one a fresh color. It is easy to show that the above coloring is strongly rainbow, and so we have $src(G) < m-2t$ in this case, a contradiction. So the claim holds.

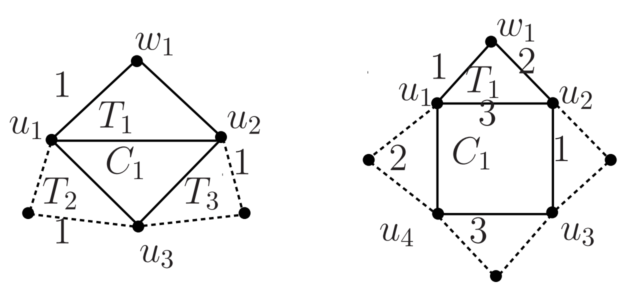

Figure 4. The edge-colorings for the case that C_1 is a triangle and the case that C_1 a 4-cycle in Theorem 3.1.

Claim 3. If the equality holds, then $G \in \overline{\mathscr{G}}_t$.

Proof. To show $G \in \overline{\mathscr{G}}_t$, it suffices to show that each triangle contains at least one vertex of degree 2 in G. Suppose that this does not hold, without loss of generality, let T_1 be the triangle with $deg_G(v_i) \geq 3$, where $V(T_1) = \{v_i : 1 \leq i \leq 3\}$. By Claim 2, it is easy to show that $E(T_1)$ is an edge-cut of G. Let H_i be the subgraph of $G \backslash E(T_1)$ containing the vertex v_i $(1 \leq i \leq 3)$. By the assumption of T_1, we know that each H_i is nontrivial. We now give G an edge-coloring: for the $t-1$ (edge-disjoint) triangles of $G \backslash E(T_1)$, we give each of them a fresh color; for the remaining edges of $G \backslash E(T_1)$ (by Claim 2, each of them must be a cut edge), we give each of them a fresh color; for the edges of $E(T_1)$, let $c(v_1v_3) \in c(H_2)$, $c(v_1v_2) \in c(H_3)$, $c(v_2v_3) \in c(H_1)$. It is easy to show that, with the above coloring, G is strongly rainbow connected, and we have $src(G) < m - 2t$, a contradiction, and so the claim holds. ∎

Claim 4. If $G \in \overline{\mathscr{G}}_t$, then the equality holds.

Proof. Let T_G be a D_2-tree of G. The result clearly holds for the case $|E(T_G)| = 1$. So now we assume that $|E(T_G)| \geq 2$. We will show that, for any strong rainbow coloring of G, $c(e_1) \neq c_(e_2)$ where $e_1, e_2 \in T_G$, that is, each edge of T_G receives a distinct color, and so the edges of T_G cost $m - 2t$ colors totally. Recall that $|E(T_G)| = m - 2t$, then $src(G) \geq m - 2t$, by the above claim, Claim 4 holds. ∎

For any two edges, say e_1, e_2, of T_G, let $e_1 = u_1u_2$, $e_2 = v_1v_2$. Without loss of generality, we assume that $d_{T_G}(u_1, v_2) = \max\{d_{T_G}(u_i, v_j) : 1 \leq i, j \leq 2\}$ where $d_{T_G}(u, v)$ denotes the distance between u and v in T_G. As T_G is a tree, the (unique) $u_1 - v_2$ geodesic, say P, in T_G must contain the edges e_1, e_2. Moreover, it is easy to show that P is also a unique $u_1 - v_2$ geodesic in G, and so $c(e_1) \neq c_(e_2)$ under any strong rainbow coloring.

By Claims 3 and 4, the equality holds if and only if $G \in \overline{\mathscr{G}}_t$. Then our result holds. ∎

In [5, 6], Li and Sun investigated the rainbow connection numbers of line graphs. As an application to Theorem 3.1, we consider the strong rainbow connection numbers of line graphs of connected cubic graphs. Recall that the *line graph* of a graph G is the graph $L(G)$ whose vertex set is $V(L(G)) = E(G)$ and two vertices e_1, e_2 of $L(G)$ are adjacent if and only if they are adjacent in G. The star, denoted by $S(v)$, at a vertex v of graph G, is the set of all the edges incident to v. Let $\langle S(v) \rangle$ be the subgraph of $L(G)$ induced by $S(v)$, clearly, it is a clique of $L(G)$. A *clique decomposition* of G is a collection $\mathscr{C}$ of cliques such that each edge of G occurs in exactly one clique in $\mathscr{C}$. An *inner vertex* of a graph is a vertex with degree at least 2. For a graph G, we use $\overline{V_2}$ to denote the set of all the inner vertices of G. Let $\mathscr{K}_0 = \{\langle S(v) \rangle : v \in V(G)\}$, $\mathscr{K} = \{\langle S(v) \rangle : v \in \overline{V_2}\}$. It is easy to show that $\mathscr{K}_0$ is a clique decomposition of $L(G)$ and each vertex of the line graph belongs to at most two elements of $\mathscr{K}_0$. We know that each element $\langle S(v) \rangle$ of $\mathscr{K}_0 \setminus \mathscr{K}$, a single vertex of $L(G)$, is contained in the clique induced by u that is adjacent to v in G. So $\mathscr{K}$ is a clique decomposition of $L(G)$.

Corollary 3.1. *Let $L(G)$ be the line graph of a connected cubic graph G with n vertices. Then $src(L(G)) \leq n$.*

Proof. Since G is a connected cubic graph, each vertex of G is an inner vertex and the clique $\langle S(v) \rangle$ in $L(G)$ corresponding to each vertex v is a triangle. We know that $\mathscr{K} = \{\langle S(v) \rangle : v \in \overline{V_2}\} = \{\langle S(v) \rangle : v \in V\}$ is a clique decomposition of $L(G)$. Let $\mathscr{T} = \mathscr{K}$. Then $\mathscr{T}$ is a set of n edge-disjoint triangles that cover all the edges of $L(G)$. As there are $3n$ edges in $L(G)$, by Theorem 3.1 we have $src(L(G)) \leq 3n - 2n = n$. ∎

4. Graphs with large strong rainbow connection numbers

In this section, we will give our result on graphs with large strong rainbow connection numbers. We first introduce two graph classes. Let C be the unique cycle of a unicyclic graph G, $V(C) = \{v_1, \cdots, v_k\}$ and $\mathscr{T}_G = \{T_i : 1 \le i \le k\}$ where T_i is the unique tree containing the vertex v_i in subgraph $G \backslash E(C)$. We say that T_i and T_j are *adjacent* (*nonadjacent*) if v_i and v_j are adjacent (nonadjacent) in C. Then let
$\mathscr{G}_1 = \{G : G$ is a unicyclic graph, $k = 3$, $\mathscr{T}_G$ contains at most two nontrivial elements$\}$,
$\mathscr{G}_2 = \{G : G$ is a unicyclic graph, $k = 4$, $\mathscr{T}_G$ contains two nonadjacent trivial elements and the other two (nonadjacent) elements are paths.$\}$.

Theorem 4.1. *Let G be a connected graph with m edges. Then we have:*
(i) $src(G) \neq m-1$,
(ii) $src(G) = m-2$ *if and only if G is a 5-cycle or belongs to $\mathscr{G}_1$ or $\mathscr{G}_2$.*

Proof. In [3], the authors obtained that $src(G) = m$ if and only if G is a tree. So $src(G) \le m-1$ if and only if G is not a tree. In order to derive our conclusion, we need the following claim:
Claim 5. If $src(G) = m-1$ or $m-2$, then G is a unicyclic graph.

Proof. Suppose that G contains at least two cycles. Let C_1 be the smallest cycle of G and C_2 be the second smallest one. Let $|C_i| = k_i (i = 1,2)$. By Lemma 2.1, we have $3 \le k_1 \le 5$ and $k_2 \ge k_1$. We will consider two cases according to the value of $|E(C_1) \cap E(C_2)|$. ∎

Case 1. $|E(C_1) \cap E(C_2)| = 0$, that is, C_1 and C_2 have no common edge. There are three subcases:
Subcase 1.1. $k_1 = 3$, that is, C_1 is a triangle.

By Observation 2.2, we must have $|V(C_1) \cap V(C_2)| \le 1$. We first give C_2 a strong rainbow coloring using $\lceil \frac{k_2}{2} \rceil$ colors the same as in [3]; then give a fresh color to C_1, that is, the edges of C_1 receive a same color; for the remaining edges, we give each of them a fresh color. With a similar procedure to that of Lemma 2.1 and by Observation 2.3, we can show that the above coloring is strongly rainbow. As this costs $1 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 3)$ colors totally, we have $src(G) \le 1 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 3) = (m-2) - (k_2 - \lceil \frac{k_2}{2} \rceil) \le m-3$, a contradiction.
Subcase 1.2. $k_1 = 4$, that is, C_1 is a 4-cycle.

If $|V(C_1) \cap V(C_2)| \le 1$, we first give C_2 a strong rainbow coloring using $\lceil \frac{k_2}{2} \rceil$ colors the same as in [3]; then we give two fresh colors to C_1 in the same way; for the remaining edges, we give each of them a fresh color. With a similar procedure to that of Lemma 2.1 and by Observation 2.3, we can show that the above coloring is strongly rainbow. As this costs $2 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 4)$ colors totally, we have $src(G) \le 2 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 4) = (m-2) - (k_2 - \lceil \frac{k_2}{2} \rceil) \le m-3$, a contradiction.

Otherwise, by Observation 2.2, it must be the graph of the three graphs with $g(G) = 4$ on the right-hand side in Figure 1. We let $c(u_1u_2) = c(u_3u_4) = a, c(u_2u_3) = c(u_1u_4) = b, c(u_1v_2) = c(u_3v_4) = c, c(v_2u_3) = c(u_1v_4) = d$, where a,b,c,d are four distinct colors; for the remaining edges, we give each of them a fresh color. This procedure costs $m-4$ colors totally. As now both C_1 and C_2 are the smallest cycle of G, by Observation 2.3, any geodesic contains at most one of the two edges with the same color, and so $src(G) \le m-4$, a contradiction.
Subcase 1.3. $k_1 = 5$, that is, C_1 is a 5-cycle.

By Observation 2.2, we must have $|V(C_1) \cap V(C_2)| \leq 1$. We first give C_2 a strong rainbow coloring using $\lceil \frac{k_2}{2} \rceil$ colors the same as in [3]; then we give three fresh colors to C_1 in the same way; for the remaining edges, we give each of them a fresh color. With a similar procedure to that of Lemma 2.1 and by Observation 2.3, we can show that the above coloring is strongly rainbow. As this procedure costs $3 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 5)$ colors totally, we have $src(G) \leq 3 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 5) = (m-2) - (k_2 - \lceil \frac{k_2}{2} \rceil) \leq m - 3$, a contradiction.

Note that for each above subcase, by Observation 2.3, the cycle produced during the procedure while we use the similar technique to that of Lemma 2.1 cannot be the cycle C_1 and must be smaller than C_2, then a contradiction will be produced.

Case 2. $|E(C_1) \cap E(C_2)| \geq 1$, that is, C_1 and C_2 have at least one common edge, and so C_1 and C_2 have at least two common vertices. There are also three subcases:

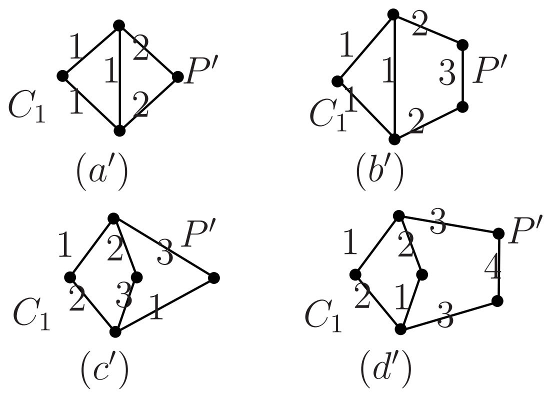

Figure 5. The graphs for Case 2 of the claim.

Subcase 2.1. $k_1 = 3$, that is, C_1 is a triangle. By Observation 2.2, C_1 and C_2 have one common edge as shown in Figure 1. Let $V(C_1) = \{u_i : 1 \leq i \leq 3\}$ and $V(C_2) = \{v_i : 1 \leq i \leq k_2\}$ and $v_{k_2+1} = v_1$, where $v_1 = u_1, v_2 = u_2$. Let P' be the subpath of C_2 that does not contain the edge v_1v_2. We now give G an edge-coloring as follows:

For the cases $l(P') = 2, 3$, we first color the edges of $C_1 \cup C_2$ as shown in Figure 5 (graphs a' and b'); then we give each other edge of G a fresh color. This procedure costs $m-3$ colors totally. Then it is easy to show that any geodesic cannot contain two edges with the same color, and so $src(G) \leq m-3$, which produces a contradiction.

For the remaining case, that is, $l(P') \geq 4$ and $k_2 \geq 5$, we first give the cycle C_1 a color, say a, that is, the three edges of C_1 receive the same color. Then in C_2, if $k_2 = 2\ell$ for some $\ell \geq 2$, then let $c(v_2v_3) = c(v_{\ell+2}v_{\ell+3})$ be a new color, say b; if $k_2 = 2\ell + 1$ for some $\ell \geq 2$, then let $c(v_2v_3) = c(v_{\ell+3}v_{\ell+4})$ be a new color, say b. For the remaining edges, we give each of them a fresh color. This procedure costs $m-3$ colors totally. For any two vertices u, v, if P is a $u-v$ geodesic, by Observation 2.3, P cannot contain two edges with color a; for the two edges with color b, with a similar argument to that of Lemma 2.1 (Note that now, by Observation 2.3, the cycle produced during the procedure cannot be C_1 and must be shorter than C_2, then a contradiction will be produced), we can show that P contains at most one of them. So P is strongly rainbow and $src(G) \leq m-3$, which produces a contradiction.

Subcase 2.2. $k_1 = 4$, that is, C_1 is a 4-cycle. By Observation 2.2, C_1 and C_2 have one common edge, or two common adjacent edges, as shown in Figure 1.

If C_1 and C_2 have one common edge, say u_1u_2 (see the graph of the three graphs with $g(G) = 4$ on the left-hand side in Figure 1), we let $V(C_2) = \{v_i : 1 \le i \le k_2\}$, where $v_1 = u_1, v_2 = u_2$. We let $c(v_2v_3) = c(u_4v_1) = a$, $c(v_2u_3) = c(v_1v_{k_2}) = b$, $c(v_1v_2) = c(u_3u_4) = c$. For the remaining edges, we give each of them a fresh color. This procedure costs $m-3$ colors totally. For any two vertices u, v, P is a $u-v$ geodesic, then by Observation 2.3, P contains at most one of the two edges with color c; for the two edges with color a (b), it is easy to show that there exists one $u-v$ geodesic which contains at most one of them. So we have $src(G) \le m-3$, which produces a contradiction.

Otherwise, then C_1 and C_2 have two common adjacent edges, say u_1u_2, u_2u_3 (see the graph of the three graphs with $g(G) = 4$ in the middle of Figure 1). We let $V(C_2) = \{v_i : 1 \le i \le k_2\}$, where $v_1 = u_1, v_2 = u_2, v_3 = u_3$. Let P' be the subpath of C_2 which does not contain the edges u_1u_2, u_2u_3.

For the cases $l(P') = 2, 3$, we first color the edges of $C_1 \cup C_2$ as shown in Figure 5 (graphs c' and d'); then we give each other edge of G a fresh color. This procedure costs $m-3$ colors totally. Then it is easy to show that any geodesic cannot contain two edges with the same color, and so we have $src(G) \le m-3$, which produces a contradiction.

For the case $l(P') \ge 4$, that is $k_2 \ge 6$, we let $c(u_4v_1) = c(v_3v_4) = a$, $c(v_1v_2) = c(v_3u_4) = b$; for the edge v_2v_3, we give a similar treatment to that of Subcase 2.1 and let $c(v_2v_3) = c$; we then give each other edge of G a fresh color. This procedure costs $m-3$ colors totally. For any two vertices u, v, let P be a $u-v$ geodesic, then by Observation 2.3, P contains at most one of the two edges with color b. For the two edges with color a, it is easy to show that there exists a $u-v$ geodesic which contains at most one of them. With a similar argument to that of Lemma 2.1 (Note that now, by Observation 2.3, the cycle produced during the procedure cannot be C_1 and must be shorter than C_2, then a contradiction will be produced), we can show that any geodesic contains at most one edge with color c. So we have $src(G) \le m-3$, which produces a contradiction.

Subcase 2.3. $k_1 = 5$, that is, C_1 is a 5-cycle. By Observation 2.2, C_1 and C_2 have one common edge, or two common adjacent edges, as shown in Figure 1. The following discussion will use Observation 2.3.

If C_1 and C_2 have one common edge, say u_1u_2 (see the graph of the two graphs with $g(G) = 5$ on the left-hand side in Figure 1), we let $V(C_2) = \{v_i : 1 \le i \le k_2\}$, where $v_1 = u_1, v_2 = u_2$, and let $c(u_4u_5) = c(v_2v_3) = a$, $c(v_1u_5) = c(v_2u_3) = b$, and $c(v_1v_2) = c(u_3u_4) = c$; for the remaining edges, we give each of them a fresh color. This procedure costs $m-3$ colors totally. With a similar argument to the above, we can show that $src(G) \le m-3$, which produces a contradiction.

Otherwise, then C_1 and C_2 have two common adjacent edges, say u_1u_2, u_2u_3 (see the graph of the two graphs with $g(G) = 5$ on the right-hand side in Figure 1). We let $c(v_1u_5) = c(v_3v_4) = a$, $c(v_1v_2) = c(v_3u_4) = b$, and $c(v_2v_3) = c(u_4u_5) = c$; for the remaining edges, we give each of them a fresh color. This procedure costs $m-3$ colors totally. With a similar argument to above, we can show that $src(G) \le m-3$, which produces a contradiction.

With the above discussion, Claim 5 holds.

Let G be a unicyclic graph and C be its unique cycle, $|C| = k$ where $3 \le k \le 5$. We now investigate the strong rainbow connection number of G.

Case 1. $k = 3$.

Subcase 1.1. All T_is are nontrivial. We first give each edge of $G\backslash E(C)$ a fresh color, then let $c(v_1v_2) \in c(T_3)$, $c(v_2v_3) \in c(T_1)$, $c(v_1v_3) \in c(T_2)$. It is easy to show that, with this coloring, G is strongly rainbow connected, and so $src(G) \leq m-3$.
Subcase 1.2. At most two T_is are nontrivial, that is, $G \in \mathscr{G}_1$. At first we consider the case that there are exactly two T_is which are nontrivial, say T_1 and T_2. We first give each edge of $G\backslash E(C)$ a fresh color, then let $c(v_1v_2) = c(v_2v_3) = c(v_1v_3)$. It is easy to show that, with this coloring, G is strongly rainbow connected, and now $src(G) \leq m-2$. On the other hand, by Observation 2.1 and the definition of a rainbow geodesic, we know that in a strong rainbow coloring, $c(T_1) \cap c(T_2) = \emptyset$ and $c(v_1v_2)$ does not belong to $c(T_1) \cup c(T_2)$. So we have $src(G) = m-2$. With a similar argument, we can derive that $src(G) = m-2$ for the case that at most one T_i is nontrivial. So $src(G) = m-2$ if $G \in \mathscr{G}_1$.
Case 2. $k = 4$.
Subcase 2.1. There are at least three nontrivial T_is, say T_1, T_3, T_4. We first give each edge of $G\backslash E(C)$ a fresh color, then let $c(v_1v_2) \in c(T_4)$, $c(v_3v_4) \in c(T_1)$, $c(v_1v_4) \in c(T_3)$ and we give the edge v_2v_3 a fresh color. It is easy to show that, with this coloring, G is strongly rainbow connected, and so $src(G) \leq m-3$.
Subcase 2.2. There are exactly two nontrivial T_is, say T_{i_1} and T_{i_2}.
Subsubcase 2.2.1. T_{i_1} and T_{i_2} are adjacent, say T_1 and T_2. We first give each edge of $G\backslash E(C)$ a fresh color, then let $c(v_2v_3) \in c(T_1)$, $c(v_1v_4) \in c(T_2)$ and we color the edges v_1v_2 and v_3v_4 with the same new color. It is easy to show that, with this coloring, G is strongly rainbow connected, and so $src(G) \leq m-3$.

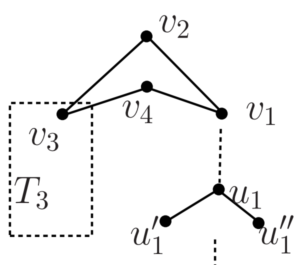

Figure 6. The graph for Subsubcase 2.2.2.

Subsubcase 2.2.2. T_{i_1} and T_{i_2} are nonadjacent, say T_1 and T_3. We can consider T_i as a rooted tree with root v_i $(i = 1,3)$. If there exists some T_i, say T_1, that contains a vertex, say u_1, with at least two sons, say u_1', u_1'' (see Figure 6). We first color each edge of $\bigcup_{i=1,3} T_i \cup \{v_1v_2\}$ with a distinct color, this costs $m-3$ colors, then we let $c(v_1v_4) = c(v_1v_2), c(v_2v_3) = c(u_1u_1'), c(v_3v_4) = c(u_1u_1'')$. It is easy to show that this coloring is strongly rainbow and we have $src(G) \leq m-3$. If G also belongs to $\mathscr{G}_2$, we first give each edge of $G\backslash E(C)$ a fresh color, then let $c(v_1v_2) = c(v_3v_4) = a$ and $c(v_2v_3) = c(v_1v_4) = b$ where a and b are two new colors. It is easy to show that, with this coloring, G is strongly rainbow connected, and so $src(G) \leq m-2$. On the other hand, $src(G) \geq m-2 = diam(G)$, and so $src(G) = m-2$.
Subcase 2.3. There is at most one nontrivial T_i. Then with a similar argument to Subsubcase 2.2.2, we can derive that $src(G) = m-2$ if G also belongs to $\mathscr{G}_2$.

By the discussions of Subsubcase 2.2.2 and Subcase 2.3, we can derive that $src(G) = m-2$ if $G \in \mathscr{G}_2$.

Case 3. $k = 5$.

If there is at least one nontrivial T_i, say T_1, then we give each edge of $G \backslash E(C)$ a fresh color, and let $v_3 v_4 \in c(T_1)$, $c(v_1 v_2) = c(v_4 v_5) = a$ and $c(v_2 v_3) = c(v_1 v_5) = b$ where a and b are two new colors. It is easy to show that, with this coloring, G is strongly rainbow connected, and now we have $src(G) \leq m-3$. On the other hand, we know $src(G) = m-2 = 3$ if $G \cong C_5$ from [3].

By Lemma 2.1 and Claim 5, we derive that if $src(G) = m-1$ or $m-2$, then G is a unicyclic graph with a unique cycle of length at most 5. By the discussion from the above Case 1 to Case 3, we know that if G is a unicyclic graph with a unique cycle of length at most 5, then $src(G) \neq m-1$. So $src(G) \neq m-1$ for any graph G. Furthermore, we have $src(G) = m-2$ if and only if G is a 5-cycle or belongs to one of $\mathscr{G}_i$s $(1 \leq i \leq 2)$. So the theorem holds. ∎

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