ON THE STRUCTURE OF BALANCED INCOMPLETE BLOCK DESIGNS[1]

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1. Summary. In this paper there are developed for the first time analytical methods for the investigation of the structure of unsymmetrical balanced incomplete block designs. Two unsymmetrical balanced incomplete block designs are proved to be impossible, and for such designs in general, inequalities are found for the number of treatments common to two blocks.

2. Introduction. In the balanced incomplete block design v varieties or treatments are compared in such a manner that each treatment is assigned to r experimental units. The units themselves are arranged into b more or less homogeneous blocks, each containing k experimental units. Any two treatments are required to occur together in the same block λ times, the treatments occurring in a given block being all different. Hence the design depends on the five parameters, v, b, r, k, λ. Clearly the following conditions are necessary:

$$bk = vr, \quad r(k - 1) = (v - 1)\lambda. \tag{2.1}$$

Fisher [1] also showed that for the existence of an actual combinatorial solution it is necessary that

$$b \geq v, \text{ or } k \leq r. \tag{2.2}$$

The work of Yates [2], Fisher and Yates [3], Bose [4], and Bhattacharya [5], [6], [7] provided solutions for all of the balanced incomplete block designs with $r \leq 10$, except the designs shown in the following table.

TABLE I

Reference number in Fisher and Yates's 1938 table	Parameters				
	v	b	r	k	λ
(8)	15	21	7	5	2
(10)	22	22	7	7	2
(12)	21	28	8	6	2
(14)	29	29	8	8	2
(28)	36	45	10	8	2
(30)	46	46	10	10	2
(24)	46	69	9	6	1
(31)	51	85	10	6	1

Hussain [8], [9] proved the nonexistence of the designs (10) and (14), Nandi [10] showed the impossibility of (8), and Shrikhande [11] proved the nonexistence

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of (30). Chowla and Ryser [12], in a sequel to a paper by Bruck and Ryser [13], gave general results, of which the impossibility of (10), (14) and (30) are special cases. (Also see Schützenberger [16].)

Of the designs listed in Table I there remain to be examined (12), (28), (24), and (31). It is the object of this paper to show that (12) and (28) are impossible, and to give a proof alternative to Nandi's of the impossibility of (8). The investigations will incidentally throw much light on the structure of balanced incomplete block designs in general.

Before proceeding further, it is desirable to establish firmly that proofs of the impossibility of (12) and (28) are really needed. Designs which have $v = b$ and $r = k$ are called "symmetrical" designs. Associated with every symmetrical design is a "derived" design, which has the following relation to the symmetrical design. If the parameters of the symmetrical design are v, b, r, k, and λ, then the parameters of the derived design, which are indicated by asterisks, are

$$(2.3) \qquad v^* = v - r, \qquad b^* = b - 1, \qquad r^* = r, \qquad k^* = k - \lambda, \qquad \lambda^* = \lambda.$$

If a solution of a symmetrical design exists, then a solution of the derived design may be obtained by deleting a block and all of the treatments in the block from the symmetrical design. Such a solution of the derived design is said to be "adjoinable," since the symmetrical design can be built up from it by suitably adjoining k new treatments, λ to each block, and a block consisting of the new treatments. There do exist, however, in certain cases nonadjoinable solutions for the class of designs given by (2.3).

An instructive example is due to Bhattacharya [5]. Associated with the symmetrical design $v = b = 25, r = k = 9, \lambda = 3$, is the derived design $v = 16, b = 24, r = 9, k = 6, \lambda = 3$. In this case a solution exists for the symmetrical design, and hence there exists an adjoinable solution for the derived design. Since it is known that every two blocks of a symmetrical design have λ treatments in common, it follows that no two blocks of an adjoinable derived design can have more than λ treatments in common. If a solution exists for the derived design which contains two blocks which have more than λ treatments in common, then clearly the solution is nonadjoinable. Bhattacharya gave the following solution of the derived design for the above case which contains two blocks (starred) which have four treatments in common, and two blocks (underscored) which have zero treatments in common.

(2.4)

(1, 2, 7, 8, 14, 15)	(3, 5, 7, 8, 11, 13)	(2, 3, 8, 9, 13, 16)
(3, 5, 8, 9, 12, 14)	(1, 6, 7, 9, 12, 13)*	(2, 5, 7, 10, 13, 15)
(3, 4, 7, 10, 12, 16)	(3, 4, 6, 13, 14, 15)	(4, 5, 7, 9, 12, 15)
(2, 4, 9, 10, 11, 13)	(3, 6, 7, 10, 11, 14)	$\underline{(1, 2, 3, 4, 5, 6)}$
(1, 4, 7, 8, 11, 16)	(2, 4, 8, 10, 12, 14)	(5, 6, 8, 10, 15, 16)
(1, 6, 8, 10, 12, 13)*	(1, 2, 3, 11, 12, 15)	(2, 6, 7, 9, 14, 16)
(1, 4, 5, 13, 14, 16)	(2, 5, 6, 11, 12, 16)	(1, 3, 9, 10, 15, 16)
(4, 6, 8, 9, 11, 15)	(1, 5, 9, 10, 11, 14)	$\underline{(11, 12, 13, 14, 15, 16)}$.

The above considerations show that the existence of a symmetrical design implies the existence of the corresponding derived design. Also the nonexistence of a derived design implies the nonexistence of the corresponding symmetrical design. But the nonexistence of a symmetrical design does not imply the nonexistence of the corresponding derived design, since a nonadjoinable solution may nevertheless exist. In particular the nonexistence of designs (14) and (30) of Fisher's tables does not rule out the possible existence of nonadjoinable solutions for (12) and (28). In the next section there will be established a fundamental theorem which besides being useful for establishing the impossibility of the two last mentioned designs, has intrinsic interest in as much as it gives a helpful insight into the structural nature of balanced incomplete block designs.

3. A fundamental theorem. Before considering the theorem, we shall prove the following useful Lemma.

LEMMA 3.1. *If* $|A|$ *is the determinant defined by*

$$(3.1)\qquad |A| = \begin{vmatrix} \alpha & \beta & \cdots\beta & e_{1,v+1} & \cdots e_{1,v+t} \\ \beta & \alpha & \cdots\beta & e_{2,v+1} & \cdots e_{2,v+t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta & \beta & \cdots\alpha & e_{v,v+1} & \cdots e_{v,v+t} \\ e_{v+1,1} & e_{v+1,2} & \cdots e_{v+1,v} & e_{v+1,v+1} & \cdots e_{v+1,v+t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{v+t,1} & e_{v+t,2} & \cdots e_{v+t,v} & e_{v+t,v+1} & \cdots e_{v+t,v+t} \end{vmatrix},$$

then

$$(3.2)\qquad |A| = [\alpha + (v-1)\beta]^{-t+1}(\alpha - \beta)^{v-t-1}\,|B_t|,$$

where B_t *is of order* $t \times t$, *and the elements of* B_t *are*

$$b_{ju} = (\alpha + (v-1)\beta)(\alpha - \beta)e_{v+j,v+u} - (\alpha + (v-1)\beta) \cdot \sum_{i=1}^{v} e_{i,v+u}\,e_{v+j,i} + \beta \sum_{i=1}^{v} e_{i,v+u} \sum_{i=1}^{v} e_{v+j,i}\,.$$

To prove the lemma let the following operations be carried out on the rows and columns of $|A|$:

(i) Multiply the last t columns by

$$[\alpha + (v-1)\beta][\alpha - \beta],$$

and write an offsetting factor outside.

(ii) Add rows $1, 2, \cdots, v-1$ to row v.

(iii) Take the factor $[\alpha + (v-1)\beta]$ out of row v.

(iv) Multiply row v by β and subtract this product from rows $1, 2, \cdots, v-1$.

(v) Take the factor $(\alpha - \beta)$ out of rows $1, 2, \cdots, v-1$.

(vi) Subtract rows $1, 2, \cdots, v - 1$ from row v.

(vii) Subtract suitable multiples of columns $1, 2, \cdots, v$ from columns $v + 1, v + 2, \cdots, v + t$ so as to make the elements which are both in the first v rows and also in the last t columns equal to zero, and the lemma follows.

Consider the "incidence" matrix N of the design, that is,

$$N = \begin{bmatrix} n_{11} & \cdots & n_{1b} \\ \vdots & & \\ n_{v1} & \cdots & n_{vb} \end{bmatrix}, \tag{3.3}$$

where the rows represent treatments, the columns represent blocks, and $n_{ij} = 1$ or 0 according as the ith treatment does or does not occur in the jth block. Since every treatment is replicated r times,

$$\sum_{j=1}^{b} n_{ij} = r, \qquad (i = 1, \cdots, v), \tag{3.4}$$

and since every treatment must occur λ times with every other treatment,

$$\sum_{j=1}^{b} n_{ij} n_{uj} = \lambda, \qquad (i, u = 1, \cdots, v; \quad i \neq u). \tag{3.5}$$

Hence,

$$NN' = \begin{bmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & & \vdots \\ \lambda & \lambda & \cdots & r \end{bmatrix}, \tag{3.6}$$

where N' denotes the transpose of N. Clearly,

$$| NN' | = rk(r - \lambda)^{v-1}. \tag{3.7}$$

Choose any $t \leq b$ blocks of the design. Let the submatrix of N which corresponds to these t blocks be denoted by N_0. Let s_{ju} be the number of treatments common to the jth and uth chosen blocks ($j, u = 1, 2, \cdots, t$). Then the $t \times t$ symmetric matrix

$$S_t = N_0' N_0 = (s_{ju}) \tag{3.8}$$

is defined to be the *structural matrix of the t chosen blocks*. The jth row or column of S_t corresponds to the jth chosen block and the successive elements of the jth row or column give the number of treatments which this block has in common with the 1st, 2nd, $\cdots$, tth chosen blocks.

Let the columns of N be permuted so that the first t columns correspond to the t chosen blocks. Then let the incidence matrix be extended by adjoining t

new rows, so that the jth adjoined row consists of zero elements except the jth, which is unity. We thus get

$$N_1 = \begin{bmatrix} N & \\ I_t & 0 \end{bmatrix}, \tag{3.9}$$

where I_t is the identity matrix of order t, and 0 is the $t \times (b - t)$ zero matrix. Then

$$N_1 N_1' = \begin{bmatrix} NN' & N_0 \\ N_0' & I_t \end{bmatrix}. \tag{3.10}$$

By application of Lemma 3.1, we obtain

$$| N_1 N_1' | = kr^{-t+1}(r - \lambda)^{v-t-1} | C_t | , \tag{3.11}$$

where

$$c_{jj} = (r - k)(r - \lambda), \qquad c_{ju} = \lambda k - r s_{ju} \qquad (j \neq u), \ (j, u = 1, \cdots, t), \tag{3.12}$$

and (2.1) has been used in replacing $(r + (v - 1)\lambda)$ by rk.

The matrix C_t given by (3.11) is a symmetric matrix whose elements are in (1, 1) correspondence with the elements of the structural matrix S_t of the chosen blocks. In fact we can write

$$C_t = \lambda k E_t + r(r - \lambda)I_t - rS_t , \tag{3.13}$$

where E_t is the singular $t \times t$ matrix all of whose elements are unity.

The matrix C_t is defined as the *characteristic matrix of the t chosen blocks.* The jth row or the jth column of C_t corresponds to the jth chosen block of the design.

When P is a matrix with real elements of order $s \times t$, $t \geq s$, it is well known that $| PP' | \geq 0$. Hence if $b > v + t$, then $| N_1 N_1' | \geq 0$. Further, since the elements of N_1 are integers, if $b = v + t$, then $| N_1 N_1' |$ is a perfect integral square. Finally if $b < v + t$, then $| N_1 N_1' | = 0$. Hence we get the following fundamental theorem.

THEOREM 3.1. *If C_t is the characteristic matrix of any set of t blocks chosen from a balanced incomplete block design with parameters v, b, r, k, λ then*

(i) $| C_t | \geq 0$ *if* $t < b - v$,

(ii) $| C_t | = 0$ *if* $t > b - v$, *and*

(iii) $k(r)^{-b+v+1} (r - \lambda)^{2v-b-1} | C_{b-v} |$ *is a perfect integral square.*

To illustrate the kind of information which is contained in this theorem, consider the design with parameters $r = 9$, $k = 6$, $b = 24$, $v = 16$, and $\lambda = 3$. Let the treatments be denoted by letters, and consider whether it is possible to fill up four blocks in such a way that each block will have three treatments in common with each of the other three blocks. One way in which this can be done is as follows:

(3.14) (ABCDEF), (ABCGHI), (ABDGJK), (ACDGLM).

We can now ask whether these four blocks can form part of the completed design. To answer this question, apply Theorem 3.1. Now,

$$c_{jj} = (r - \lambda)(r - k) = 18,$$

and

$$c_{ju} = \lambda k - r\mathcal{S}_{ju} = -9, \qquad (u \neq j).$$

Hence,

$$| C_4 | = 27^3 (-9) < 0,$$

and by (i) of Theorem 3.1 it follows that (3.14) is impossible, and in fact that any set of four blocks of the type considered cannot form a part of the completed design.

Now we shall indicate some simple consequences of Theorem 3.1. By letting $t = 1$ it is easy to prove Fisher's inequality (2.2). By letting $t = 2$ we obtain

$$(3.15) \qquad | C_2 | = (r - \lambda)^2(r - k)^2 - (\lambda k - \mathcal{S}_{12}r)^2 \geq 0,$$

whence we obtain the

Corollary 3.1. $\frac{1}{r}[2\lambda k + r(r - \lambda - k)] \geq \mathcal{S}_{ju} \geq -(r - \lambda - k)$. For the symmetrical designs, that is, the designs with $r = k$, it follows from Corollary 3.1 that $\mathcal{S}_{ju} = \lambda$, so that any two blocks of such a design have exactly λ treatments in common, a result which was first noticed by Fisher. For example, for the design with parameters $r = 9$, $k = 6$, $b = 24$, $v = 16$, and $\lambda = 3$, Corollary 3.1 gives the bounds

$$(3.16) \qquad 4 \geq \mathcal{S}_{ju} \geq 0,$$

and Bhattacharya's solution (2.4) given before actually contains two blocks (starred) which have four treatments in common, and also another two blocks (underscored) with no treatments in common.

4. The structure of balanced incomplete block designs of the series $v = \frac{1}{2}k(k + 1)$, $b = \frac{1}{2}(k + 1)(k + 2)$, $r = k + 2$, and $\lambda = 2$. It is the object of this section to develop several lemmas about the relations between blocks of any design belonging to this series. The first two lemmas do not depend on Theorem 3.1, but subsequent lemmas are based on it.

Consider an initial block B_1, which contains the k treatments $a_1, \cdots, a_k$. It is desired to know how the a_j are distributed among the remaining $(b - 1)$ blocks. Let there be n_i blocks which contain i of the treatments a_j. Then the following relations are necessary:

$$(4.1) \qquad \begin{aligned} &\text{(i)} \quad \sum_{i=0}^{k} n_i = b - 1 = \tfrac{1}{2}k(k + 3), \\ &\text{(ii)} \quad \sum_{k=0}^{i} in_i = k(r - 1) = k(k + 1), \end{aligned}$$

and

$$\text{(iii)} \quad \sum_{i=0}^{k} i(i-1)n_i = k(k-1).$$

Consider

$$Q = \sum_{i=0}^{k} (i-1)(i-2)n_i, \tag{4.2}$$

where n_i, $(i = 0, \cdots, k)$, is a positive or zero integer. Now

$$Q = \sum_{i=0}^{k} i(i-1)n_i - 2\sum_{i=0}^{k} in_i + 2\sum_{i=0}^{k} n_i = 0. \tag{4.3}$$

Since $i \geq 0$ and $n_i \geq 0$, it follows from (4.3) that each term of Q is zero. Hence

$$n_i = 0 \text{ for } i = 0 \text{ and } k \geq i > 2. \tag{4.4}$$

From (4.4) and (i) and (ii) of (4.1) we obtain

LEMMA 4.1. *Any block of the design has two treatments in common with* $\frac{1}{2}k(k-1)$ *other blocks, and one treatment in common with* $2k$ *other blocks.*

Next consider two initial blocks, B_1 and B_2, which contain treatments as follows:

$$\begin{matrix} B_1: \theta_1 \cdots \theta_\gamma a_1 \cdots a_{k-\gamma} \\ B_2: \theta_1 \cdots \theta_\gamma b_1 \cdots b_{k-\gamma} \end{matrix}. \tag{4.5}$$

The treatments θ_i $(i = 1, \cdots, \gamma; \gamma = 1, 2)$ are the γ treatments which B_1 and B_2 have in common. It is desired to determine how the treatments of B_1 and B_2 may be distributed among the remaining $(b - 2)$ blocks.

The remaining $(b - 2)$ blocks are of several types depending on how the treatments of B_1 and B_2 occur in them. If $\gamma = 2$, the θ_1 and θ_2 occur together twice in B_1 and B_2 and cannot occur together again in any other block. The types of blocks are defined in

DEFINITION 4.1. *Type 1. The block contains two treatments from each of* B_1 *and* B_2. *It is of subtype* 11 *or* 12 *according as one* θ_i *does not, or does occur as one of the two treatments.*

Type 2. The block contains two treatments from one of B_1 *and* B_2, *but only one treatment from the other. It is of subtype* 21 *or* 22 *according as one* θ_i *does not, or does occur as one of the treatments.*

Type 3. The block contains one treatment from each of B_1 *and* B_2. *It is of subtype* 31 *or* 32 *according as one* θ_i *is not, or is the treatment.*

Consider the pairs which must be formed among the treatments of B_1 and B_2. Certain pairs occur in B_1 and B_2, leaving the following pairs to occur in the remaining $(b - 2)$ blocks:

	Type of pair	Number of pairs
(i)	a_ib_j	$n_1 = 2(k-\gamma)^2$
(ii)	a_ia_j or b_kb_l	$n_2 = (k-\gamma)(k-\gamma-1)$
(iii)	θ_ia_j or θ_kb_l	$n_3 = 2\gamma(k-\gamma)$

Denote the number of blocks of type li ($l = 1, 2, 3$; $i = 1, 2$) by x_{li}. Then from the above considerations the following equations are necessary.

$$
(4.6) \quad
\begin{array}{ll}
\text{(a)} & 4x_{11} + x_{12} + 2x_{21} + x_{31} = n_1, \\
\text{(b)} & 2x_{11} + x_{21} = n_2, \\
\text{(c)} & 2x_{12} + x_{22} = n_3, \\
\text{(d)} & x_{12} + x_{22} + x_{32} = \gamma k, \\
\text{(e)} & 4x_{11} + 2x_{12} + 3x_{21} + x_{22} + 2x_{31} = 2(k+1)(k-\gamma), \\
\text{(f)} & x_{11} + x_{12} + x_{21} + x_{22} + x_{31} + x_{32} = \tfrac{1}{2}k(k+3) - 1.
\end{array}
$$

The equations of (4.6) may be solved to determine the number of blocks of types 11, $\cdots$, 32. Then remembering that $\sum_{i=1}^{2} x_{li}$ is the number of blocks of type l, we obtain

LEMMA 4.2. *With respect to 2 initial blocks which have* γ, ($\gamma = 1, 2$) *treatments in common, there exist* $[(k - \gamma)(k - \gamma + 1) + k\cdot\gamma - \frac{1}{2}k(k + 3) - 1]$ *blocks of type* 1, $2\gamma(k - \gamma)$ *blocks of type* 2, *and* $[k(2 - \gamma) - 2\gamma(1 - \gamma)]$ *blocks of type* 3.

Now consider several structural matrices for 5 blocks. The first structural matrix to be considered is

$$
(4.7) \qquad S_5^{(1)} = \begin{bmatrix} k & 1 & 1 & 2 & 2 \\ & k & 1 & 2 & 2 \\ & & k & 1 & 1 \\ & & & k & s_{45} \\ & & & & k \end{bmatrix},
$$

which is a symmetric matrix. The element s_{45} is unknown, and it is desired to know what values are admissible for s_{45}, if the five blocks which have $S_5^{(1)}$ for their structural matrix are to form a part of the completed design. Of course, the admissible value is 1 or 2, or both.

Associated with $S_5^{(1)}$ is the characteristic determinant, $|C_5^{(1)}|$. Consider the elements of $C_5^{(1)}$. For the series of designs under consideration, $r - k = 2$ and $r - \lambda = k$. Hence, $c_{jj} = 2k$ and $c_{ju} = k - 2$ or -4, according as $s_{ju} = 1$ or 2, where j and u refer to the jth and uth blocks of the set of 5 blocks being considered. The element c_{45} is unknown and it is desired to know whether $k - 2$ or -4 or both are admissible for c_{45} if the 5 blocks being considered are to form a part of the completed design.

Evaluation of $|C_5^{(1)}|$ by Lemma 3.1 yields

$$
(4.8) \qquad |C_5^{(1)}| = 2(k+2)^2(2k - c_{45})[2(k-1)c_{45} + (k^2 - 28)].
$$

Now by Theorem 3.1 it is necessary that $|C_5^{(1)}| \geq 0$. If $c_{45} = -4$ we obtain

$$
(4.9) \qquad k - 10 \geq 0,
$$

whence it follows that c_{45} cannot be -4 and hence s_{45} cannot be 2 unless $k \geq 10$. If $c_{45} = (k - 2)$ we obtain

$$
(4.10) \qquad k - 4 \geq 0,
$$

whence it follows that c_{45} cannot be $(k - 2)$ and hence s_{45} cannot be 1 unless $k \geq 4$. These results are contained in

LEMMA 4.3.

(i) *If* $k < 4$, *then there cannot exist* 5 *blocks with* $S_5^{(1)}$ *as structural matrix.*

(ii) *If* $4 \leq k \leq 9$, *then in* $S_5^{(1)}$, $s_{45} = 1$.

(iii) *If* $k \geq 10$, *then both values of* s_{45} *are admissible in* $S_5^{(1)}$.

Let the second structural matrix to be considered be

$$(4.11) \qquad S_5^{(2)} = \begin{bmatrix} k & 1 & 1 & 2 & 2 \\ & k & 1 & 1 & 1 \\ & & k & 1 & 1 \\ & & & k & s_{45} \\ & & & & k \end{bmatrix}.$$

Using Lemma 3.1 we find that the value of the determinant of the associated characteristic matrix is

$$(4.12) \qquad | C_5^{(2)} | = 4(k + 2)^2(2k - c_{45})[(k - 1)c_{45} + 2(k - 4)],$$

which by Theorem 3.1 is nonnegative. Reasoning as for $S_5^{(1)}$ we obtain

LEMMA 4.4.

(i) *If* $k < 3$, *then there cannot exist* 5 *blocks with* $S_5^{(2)}$ *as structural matrix.*

(ii) *If* $k \geq 3$, *then in* $S_5^{(2)}$, $s_{45} = 1$.

Consider a third structural matrix

$$(4.13) \qquad S_5^{(3)} = \begin{bmatrix} k & 1 & 1 & 1 & 2 \\ & k & 1 & 2 & 1 \\ & & k & 2 & 1 \\ & & & k & s_{45} \\ & & & & k \end{bmatrix}.$$

Using Lemma 3.1 we find that the value of the determinant of the associated characteristic matrix is

$$(4.14) \qquad \begin{aligned} | C_5^{(3)} | = 4(k + 2)^2[-(k - 1)c_{45}^2 \\ - (k - 2)(k + 8)c_{45} + (k - 2)(k^2 - k - 18)], \end{aligned}$$

which by Theorem 3.1 is nonnegative. Reasoning as for $S_5^{(1)}$, and observing by placement of the treatments in the blocks that the design with $k = 2$ cannot contain five blocks with the structural matrix $S_5^{(3)}$, we obtain

LEMMA 4.5.

(i) *If* $k < 3$, *then there cannot exist* 5 *blocks with* $S_5^{(3)}$ *as structural matrix.*

(ii) *If* $k \geq 3$, *then in* $S_5^{(3)}$, $s_{45} = 2$.

5. The impossibility of balanced incomplete block designs (8) and (28) of Fisher and Yates's table. In this section the proof of the impossibility of the designs (8) and (28) of Table 1 is completed. These designs belong to the series considered in Section 4 and correspond respectively to $k = 5$ and $k = 8$.

For $t = b$ we obtain from (3.8) the structural matrix S_b of the design. From Lemmas 4.1 and 4.2 it follows that there exist two rows (blocks) of S_b which are as follows:

$$(5.1)\quad \begin{array}{ccc|c|c|c|c} k & 1 & 1 & 1\ldots1 & 1\ldots1 & 2\ldots2 & 2\ldots2 \\ & k & 1 & 1\ldots1 & 2\ldots2 & 2\ldots2 & 1\ldots1, \\ & & k & & & & \end{array}$$

where the partitions break up the matrix S_b into submatrices A, B, C, D, and E, in left to right order. According to Lemma 4.1, rows 1 and 2 both contain $\frac{1}{2}k(k-1)$ 2's and $2k$ 1's. Now since blocks 1 and 2 intersect in one treatment, it follows from Lemma 4.2 that there exist $\frac{1}{2}(k-1)(k-2)$ blocks of type 1, $2(k-1)$ blocks of type 2, and k blocks of type 3. Hence, it is necessary that A contain 3 columns, that B, C, and E each contain $(k-1)$ columns, and that D contain $\frac{1}{2}(k-1)(k-2)$ columns.

Consider how the third row of S_b may be filled up. By Lemma 4.1 it must contain $\frac{1}{2}k(k-1)$ 2's and $2k$ 1's. Since block 3 intersects block 1 in one treatment, it follows by considering blocks 1 and 3 as initial blocks that the number of blocks of types 1, 2, and 3 must be as given in the preceding paragraph. Also block 3 intersects block 2 in one treatment, so the same result holds for blocks 2 and 3 as initial blocks. Unfortunately these conditions are met by numerous arrangements of the 1's and 2's in row 3. In fact, it follows from Lemmas 4.1 and 4.2 that if there are $(k-j-1)$ 2's in row 3 of B, then there are j 2's in row 3 of C, $[\frac{1}{2}(k-1)(k-2)-j]$ 2's in row 3 of D, and j 2's in row 3 of E, $(j = 0, \cdots, k-1)$.

Consider S_{k+2}, the structural matrix for the following $(k+2)$ blocks: the blocks of A, the j blocks of C which have 2 in row 3, and the $(k-j-1)$ blocks of E which have 1 in row 3, that is,

$$(5.2)\qquad S_{k+2} = \left[\begin{array}{ccc|c|c} k & 1 & 1 & 1\cdots1 & 2\cdots2 \\ & k & 1 & 2\cdots2 & 1\cdots1 \\ & & k & 2\cdots2 & 1\cdots1 \\ \hline & & & F & G \\ \hline & & & & H \end{array}\right],$$

where F and H have k in the main diagonal, and are symmetric matrices. The other elements of F, G, and H are so far unknown and will be determined below. Comparison of the structure of S_{k+2} with the structures of $S_5^{(1)}$ of (4.7), $S_5^{(2)}$ of (4.11), and $S_5^{(3)}$ of (4.13) shows that Lemmas 4.3, 4.4, and 4.5 apply. Hence the elements of F are 1, the elements of G are 2, and the elements of H are 1, for $k < 10$.

Corresponding to S_{k+2} is the characteristic matrix C_{k+2}. It is useful to compute

(5.3) $|C_{k+2}| =$

$$\left|\begin{array}{ccc:cccc:cccc}
(2k) & (k-2) & (k-2) & (k-2) & & \cdots & (k-2) & (-4) & & \cdots & (-4) \\
 & (2k) & (k-2) & (-4) & & \cdots & (-4) & (k-2) & & \cdots & (k-2) \\
 & & (2k) & (-4) & & \cdots & (-4) & (k-2) & & \cdots & (k-2) \\
\hdashline
 & & & (2k) & (k-2) & \cdots & (k-2) & (-4) & & \cdots & (-4) \\
 & & & (k-2) & (2k) & \cdots & (k-2) & & & & \\
 & & & \vdots & \vdots & & \vdots & \vdots & & & \vdots \\
 & & & (k-2) & (k-2) & \cdots & (2k) & (-4) & & \cdots & (-4) \\
\hdashline
 & & & & & & & (2k) & (k-2) & \cdots & (k-2) \\
 & & & & & & & (k-2) & (2k) & \cdots & (k-2) \\
 & & & & & & & \vdots & \vdots & & \vdots \\
 & & & & & & & (k-2) & (k-2) & \cdots & (2k)
\end{array}\right|.$$

Using Lemma 3.1 repeatedly we obtain

$$|C_{k+2}| = j(j-k+2)(k-6)(k+2)^{k+1}. \tag{5.4}$$

Now by (ii) of Theorem 3.1, $|C_{k+2}| = 0$. From (5.4) it is clear that $|C_{k+2}| = 0$ when and only when $j = 0$ or $(k-2)$, or $k = 6$.

Let $k = 8$. Then from (5.4), either $j = 0$ or $j = 6$. If $j = 0$, then consider $S_9^{(1)}$ for blocks 1, 2, 3, and any 6 blocks of E of 5.1. Then from (3.11),

$$|(N_1^{(1)})(N_1^{(1)})'| = 2^{83}, \tag{5.5}$$

where the 9 chosen blocks of $N_1^{(1)}$ are the blocks which have $S_9^{(1)}$ as structural matrix.

If $j = 6$, then consider $S_9^{(2)}$ for blocks 1, 2, 3, and the six blocks of C of (5.1) for which the third row contains 2. Then

$$|(N_1^{(2)})(N_1^{(2)})'| = 2^{85}, \tag{5.6}$$

where the 9 chosen blocks of $N_1^{(2)}$ are the blocks which have $S_9^{(2)}$ as structural matrix.

The determinant $|(N_1^{(i)})(N_1^{(i)})|$, $i = 1, 2$, is not a perfect integral square. But from (iii) of Theorem 3.1, it must be a perfect integral square. Hence the

THEOREM 5.1. *The balanced incomplete block design with parameters* $r = 10$, $k = 8$, $b = 45$, $v = 36$, *and* $\lambda = 2$ *is impossible.*

Although a similar argument might be given for $k = 5$, an easy proof is as follows. Consider

$$S_4 = \begin{bmatrix} 5 & 1 & 1 & 1 \\ & 5 & 2 & 2 \\ & & 5 & s_{34} \\ & & & 5 \end{bmatrix}, \tag{5.7}$$

where s_{34} is unknown but is either 1 or 2. The corresponding characteristic determinant is

$$|C_4| = (7)(10 - c_{34})(13\, c_{34} + 38). \tag{5.8}$$

By Theorem 3.1, it is necessary that $|C_4| \geq 0$. It follows that $c_{34} = k - 2 = 3$. Hence, $s_{34} = 1$ and blocks 1, 3, and the four blocks of C of (5.1) have the structural matrix

$$S_6 = \begin{bmatrix} 5 & 1 & \cdots & 1 \\ 1 & 5 & \cdots & 1 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & 1 & \cdots & 5 \end{bmatrix}. \tag{5.9}$$

The corresponding characteristic determinant is

$$|C_6| = 7^5 \cdot 5^2, \tag{5.10}$$

and from (3.11),

$$|N_1 N_1'| = 5^{11}, \tag{5.11}$$

where the 6 chosen blocks of N_1 are the blocks which have S_6 as structural matrix. The determinant $|N_1 N_1'|$ is not a perfect integral square, which contradicts (iii) of Theorem 3.1. Hence, the

THEOREM 5.2. *The balanced incomplete block design with parameters* $r = 7$, $k = 5$, $b = 21$, $v = 15$, *and* $\lambda = 2$ *is impossible.* This result was obtained by Nandi [10] by a different method.

6. The impossibility of the balanced incomplete block design (12) of Fisher and Yates's table. This design is the member of the series of section 4 which has $k = 6$. From (5.4) it is seen that $(k - 6)$ is a factor of $|C_{k+2}|$. Hence the argument used for $k = 8$ will not apply for $k = 6$.

Consider (5.1) in which two rows of S_b are given. Assume that there do not exist five blocks having for their structural matrix

$$S_5^{(4)} = \begin{bmatrix} 6 & 1 & \cdots & 1 \\ 1 & 6 & \cdots & 1 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & 1 & \cdots & 6 \end{bmatrix}. \tag{6.1}$$

This assumption will be contradicted. For the assumption to be true it is necessary for row 3 of C to contain exactly three 2's. For if it contains less than three 2's then it contains at least three 1's, which we may for definiteness take to be in columns 1, 2, and 3 of C. But then blocks 1 and 3 of A, and 1, 2, and 3 of C form $S_5^{(4)}$, by Lemma 4.4. If row 3 of C contains more than three 2's, then by Lemma 4.3, blocks 1 of A and any 4 blocks of C which have 2 in row 3 form $S_5^{(4)}$. Hence, there exist three blocks such that the first three rows of S_b are as shown below.

$$
\begin{array}{ccc|ccccc|ccccc|c}
6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
 & 6 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & \cdots \\
 & & 6 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & \cdots \\
 & & & 6 & & & & & & & & & & \\
 & & & & 6 & & & & & & & & & \\
 & & & & & 6 & & & & & & & &
\end{array} \tag{6.2}
$$

Denote the element in row i and column j of submatrix B of S_b by (i, j). Then for $S_5^{(4)}$ not to exist, it is necessary in B that

$$(4, 2) = (4, 3) = (5, 3) = 2. \tag{6.3}$$

But then blocks 1, 2, 4, 5, and 6 of S_b form $S_5^{(2)}$ with $s_{45} = 2$, which contradicts Lemma 4.4. Hence, the

LEMMA 6.1. *If the design exists then there exist five blocks having* $S_5^{(4)}$ *of* (6.1) *for their structural matrix.*

Without loss of generality, let $S_5^{(4)}$ be the leading principal minor matrix of order 5 in S_b. Let S_b be partitioned as in (5.1). Then row 3 of B contains at least two 1's and cannot contain more than three 2's. Hence, row 3 of C cannot contain fewer than two 2's. If row 3 of C contains u 2's, then by Lemma 4.2, row 3 of E contains $(5 - u)$ 1's, $(u = 2, \cdots, 5)$.

CASE 1. Row 3 of C contains either two or three 2's. Then row 3 of E contains at least two 1's. Let $S_7^{(1)}$ be the structural matrix for the three blocks of A, any two blocks from C which have 2 in row 3, and two blocks from E which have 1 in row 3. Then

$$
S_7^{(1)} = \left[\begin{array}{ccc|cc|cc}
6 & 1 & 1 & 1 & 1 & 2 & 2 \\
 & 6 & 1 & 2 & 2 & 1 & 1 \\
 & & 6 & 2 & 2 & 1 & 1 \\
 & & & 6 & 1 & 2 & 2 \\
 & & & & 6 & 2 & 2 \\
 & & & & & 6 & 1 \\
 & & & & & & 6
\end{array}\right], \tag{6.4}
$$

where the partitions separate the blocks from A, C, and E, in that order. The elements in rows 4, 5, and 6 and not in the main diagonal of $S_7^{(1)}$ are uniquely determined by Lemmas 4.3, 4.4, and 4.5.

CASE 2. Row 3 of C contains either four or five 2's. Let $S_7^{(2)}$ be the structural matrix for the three blocks of A and any four blocks of C which contain 2 in row 3. Then

$$S_7^{(2)} = \left[\begin{array}{ccc|cccc} 6 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 6 & 1 & 2 & 2 & 2 & 2 \\ & & 6 & 2 & 2 & 2 & 2 \\ & & & 6 & 1 & 1 & 1 \\ & & & & 6 & 1 & 1 \\ & & & & & 6 & 1 \\ & & & & & & 6 \end{array}\right], \tag{6.5}$$

where the partition separates the blocks from A and C.

An easy computation shows that (iii) of Theorem 3.1, that is, the perfect integral square condition, does not rule out either Case 1 or Case 2.

We shall however use the notion of "rational congruence" to prove the impossibility of this design. Let a symmetric matrix A and a matrix B be nonsingular matrices of order n with rational elements, and let x and y be column vectors with n variables each. Then if there exists a transformation $x = By$ which carries the quadratic form $f = x'Ax$ into the form $g = y'B'ABy$, we say that f and g are rationally congruent forms, and likewise we say that the matrices A and $C = B'AB$ are rationally congruent matrices.

Consider the Hasse invariant

$$c_p(f) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p, \tag{6.7}$$

where p is a prime, D_i is the leading principal minor determinant of order i in the coefficient matrix A of f, and $(a, b)_p$ is Pall's [14] generalization of the Hilbert norm-residue symbol. For properties of this symbol, see, for example, [11]. Let i = the index of the form, and d = the square-free integer part of the determinant of the form. Then we have

THEOREM 6.1. *Two forms f and g are rationally congruent if and only if they have the same values for their invariants n, i, d, and c_p for every p.* These invariants are not independent of each other but satisfy certain relations which will not be stated here. For a proof of this important theorem consult the book by Jones [15].

Instead of considering the rational congruence of quadratic forms, we may consider the rational congruence of their coefficient matrices. Thus if $f = x'Ax$, we may write $c_p(A)$ instead of $c_p(f)$.

Now consider $C_7^{(1)}$ and $C_7^{(2)}$, which correspond respectively to $S_7^{(1)}$ of (6.4) and $S_7^{(2)}$ of (6.5). Multiply the last two rows and columns of $C_7^{(2)}$ by -1. The result is $C_7^{(1)}$. Hence $C_7^{(2)}$ is rationally congruent to $C_7^{(1)}$, and it follows that consideration of Case 1 only is sufficient.

Let N_1 be the matrix of (3.9) which has the 7 blocks of $S_7^{(1)}$ of (6.4) as chosen blocks. Now we may regard N_1N_1' and I, the identity matrix of order b, as the coefficient matrices of quadratic forms. Since

$$N_1^{-1}(N_1N_1')(N_1^{-1})' = I, \tag{6.8}$$

N_1N_1' and I are rationally congruent. From (6.7), $c_p(I) = +1$ for p odd. Hence it is necessary that

(6.9) $$c_p(N_1N_1') = +1.$$

We may evaluate $c_p(N_1N_1')$ for a general N_1 by a method similar to that in [11] or [13] and obtain

(6.10) $$\begin{aligned} c_p(N_1N_1') &= (-1, r-\lambda)_p^{\frac{1}{2}(v+2)(v-1)}(r-\lambda, r)_p^{v-1} \\ &\cdot (r-\lambda, k)_p^{v-1}(v, r)_p\,(v, k)_p\,(v, r-\lambda)_p(-1, -D_b)_p \prod_{j=0}^{b-v-1} (D_{v+j}, -D_{v+j+1})_p\,, \end{aligned}$$

for p an odd prime. For the N_1 under consideration,

(6.11) $$c_p(N_1N_1') = (3, 2)_p(7, 2)_p(5, -1)_p$$

for p an odd prime, and for $p = 3$,

(6.12) $$c_3(N_1N_1') = -1.$$

From (6.9) and (6.12), Theorem 6.1 is contradicted, and hence we obtain

THEOREM 6.2. *The balanced incomplete block design with parameters* $r = 8$, $k = 6$, $b = 28$, $v = 21$, *and* $\lambda = 2$ *is impossible.*

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