ANNALES DE L'INSTITUT FOURIER

MORIHIKO SAITO

On the structure of Brieskorn lattice

Annales de l'institut Fourier, tome 39, nº 1 (1989), p. 27-72

<http://www.numdam.org/item?id=AIF_1989__39_1_27_0>

Ann. Inst. Fourier, Grenoble
39, 1 (1989), 27-72

ON THE STRUCTURE OF BRIESKORN LATTICE

by Morihiko SAITO

Introduction.

Let $f : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ be a holomorphic function with an isolated singularity ($n \geq 1$), and $f : X \to S$ a good represent of f, called the Milnor fibration of f, i.e. X is the intersection of a sufficiently small ball in $\mathbf{C}^{n+1}$ with the pull-back by f of a much smaller disc S in $\mathbf{C}$. Then the *filtered Gauss-Manin system* $\int_f (\mathcal{O}_X, F)$ is defined to be the filtered direct image, cf. §2, and we have the natural isomorphism

$$(*) \qquad F_{-n}M = \Omega_{X,0}^{n+1}/df \wedge d\Omega_{X,0}^{n-1},$$

where $M \; := \left(\int_f^0 \mathcal{O}_X\right)_0$ (the stalk at 0) and F on M is induced by the filtration F on $\int_f \mathcal{O}_X$. The right hand side of $(*)$ was first studied by Brieskorn [B] and we call it the Brieskorn lattice of M, and denote it by M_0. In fact, he defined the regular singular connection (called the Gauss-Manin connection) on M_0, which calculates the Milnor monodromy. We can easily verify that the connection ∇ is compatible with the action of ∂_t on M as left $\mathcal{D}_{S,0}$-module, where t is the coordinate of S. More precisely, the inverse of the action of $\nabla_{\partial/\partial t}$(resp.$\partial_t$) is well-defined as a $\mathbf{C}$-endomorphism of M_0(resp.M) and they coincide on $F_{-n}M$ by the isomorphism (*)

Key-words : Gauss-Manin system – Microlocalization – Mixed Hodge structure – Vanishing cycle – b-function.

A.M.S. Classification : 32C40.

After Brieskorn, many people have studied the structure of this lattice with the connection, by looking for a basis of M_0 over $\mathbb{C}\{t\}$ (or $\mathbb{C}\{\{\partial_t^{-1}\}\}$, cf. 1.8) such that the differential equation associated to the connection and the basis becomes as simple as possible. In this paper, we change the view point and try to understand the "shape" of this lattice as a subspace of M. Here the key point is that we know completely the structure of the regular holonomic $\mathcal{D}_{S,0}$-modules on which the action of ∂_t is invertible (i.e. bijective), cf. §1. In particular, we have a basis $\{u_i\}$ of M over $\mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t]$ corresponding to a choice of a basis of the Milnor cohomology which is compatible with the monodromy decomposition (i.e. the decomposition by the eigenvalue of the monodromy). We assume further that the basis of the Milnor cohomology is compatible with the Hodge filtration, i.e. gives a splitting of the Hodge filtration. Then we can find a basis $\{v_i\}$ of M_0 over $\mathbb{C}\{\{\partial_t^{-1}\}\}$ such that the base change matrix between $\{u_i\}$ and $\{v_i\}$ becomes as simple as possible, cf. 3.4. Up to now, we used only the $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module structure of M_0 and M. To study the action of t on M_0, we have to add some condition for the choice of the basis of the Milnor cohomology, i.e. the above splitting of the Hodge filtration is compatible with N the logarithm of the unipotent part of the monodromy, divided by $2\pi i$. Then the main result of this paper is that the corresponding basis $\{v_i\}$ behaves very well with respect to the action of t, i.e.

THEOREM. — *There are two matrices A_0 and A_1 with complex coefficient such that*

$$(**)\qquad tv = A_0 v + A_1 \partial_t^{-1} v$$

where $v = {}^t(v_1, \dots, v_\mu)$ with $\{v_i\}$ as above. Moreover A_0 is nilpotent, A_1 is semi-simple and the eigenvalues of A_1 coincide with the exponents (or the singularity spectra added by one), i.e. each v_i is an eigenvector of A_1 with eigenvalue α_i and $\{\alpha_1, \dots, \alpha_\mu\}$ are the exponents of f in the sense of [S7].

Because the basis $\{v_i\}$ is not intrinsic, we can replace it by the corresponding $\mathbb{C}$-linear section v of the natural projection :

$$\mathrm{pr} : M_0 \to M_0/\partial_t^{-1}M_0 \cong \Omega_{X,0}^{n+1}/df \wedge \Omega_{X,0}^n =: \Omega_f$$

such that $\mathrm{Im}\, v = \Sigma \mathbb{C} v_i$. Then the above theorem means :

THEOREM $'$. — *To each splitting of the Hodge filtration satisfying the conditions as above, there correspond a* $\mathbb{C}$*-linear section* v *of* pr *and* $\mathbb{C}$*-linear endomorphisms* A_0 *and* A_1 *of* Ω_f *such that*

$$(***) \qquad tv = vA_0 + \partial_t^{-1} vA_1.$$

Let Ω_f^α be the α-eigenspace of Ω_f with respect to the action of A_1, and put $V^\alpha\Omega_f = \oplus_{\beta\ge\alpha}\Omega_f^\beta$. Then

PROPOSITION. — *We have* $A_0V^\alpha\Omega_f \subset V^{\alpha+1}\Omega_f$ *and* $\mathrm{Gr}_V A_0$ *can be identified with* $-N$. *In particular,* T_u *and* T_s *are identified with* $\exp(-2\pi i \mathrm{Gr}_V A_0)$ *and* $\exp(-2\pi i A_1)$ *respectively, where* $T = T_sT_u$ *is the Jordan decomposition of the monodromy. (But* A_0 *and* A_1 *do not commute in general.)*

Here note that A_0 represents the multiplication of f on Ω_f by definition, cf. [V2]. In the proof of the proposition we use the section v or the basis $\{v_i\}$ to construct the identification between Ω_f and the Milnor cohomology (hence it depends on the choice of the splitting of the Hodge filtration on the Milnor cohomology).

Combining the main theorem with a result of Malgrange [M3], we can deduce the existence of the primitive form in the sense of K. Saito [Sk], because its definition is purely *microlocal.* (We also show that there exists a primitive form whose associated exponents are different from the usual one.) But, to get a nontrivial result on the associated period mapping (e.g. the determination of its image or the construction of the inverse mapping), we need a completely new idea, because the period map behaves very wildly around the points of the discriminant corresponding to non rational double singularities and the support of the microlocal Gauss-Manin system is just the conormal of the discriminant, cf. [Ph]. For a moment his theory works well only in the case of rational double point and simple elliptic singularity where the non-negativity of the degree of the $\mathbb{C}^*$-action on the base space is essentially used, cf. also [Lo] etc.

As another application of the theorem, we get some information about the b-function $b(s)$ of f. By [M2], $(s+1)^{-1}b(s)$ is the minimal polynomial of the action of $-\partial_t t$ on $\widetilde{M_0}/t\widetilde{M_0}$, where $\widetilde{M_0} = \sum(\partial_t t)^i M_0$ the saturation of M_0. Let M^α be the subspace of M annihilated by a sufficiently high power of $\partial_t t - \alpha$. Then for each section v in Theorem' there exists subspaces G^α of M^α and $\mathbb{C}$-linear maps $c_{\beta\alpha} : G^\alpha \to M^\beta$ for $0 < \alpha \leq \beta < n+1$ such that $c_{\alpha\alpha}$ is the natural inclusion and $\mathrm{Im}\sum_\beta c_{\beta\alpha} = v(\Omega_f^\alpha)$, i.e. $c_{\beta\alpha}$'s correspond to the transformation matrix for the two bases $\{u_i\}$ and $\{v_i\}$. Then $\widetilde{M}_0$ is compatible with the infinite direct sum decomposition $M = \widehat{\oplus} M^\alpha$, i.e. $\widetilde{M}_0 = \widehat{\oplus}\widetilde{M}_0^\alpha$ with $\widetilde{M}_0^\alpha = \widetilde{M}_0 \cap M^\alpha$, and $\widetilde{M}_0^\beta$ is spanned by the images of $N^i\partial_t^{-j}c_{\beta-j,\alpha}$ for $i \geq 0, j \geq 0, \alpha \leq \beta - j$, where $N = -(\partial_t t - \beta)$ on M^β. Note that $(s+1)^{-1}b(s)$ is the product of the minimal polynomial of the action of $-\partial_t t$ on $\widetilde{M}_0^\alpha / \partial_t^{-1}\widetilde{M}_0^{\alpha-1}$.

In the case $n = 1$, the situation becomes simple and we get

$$(s+1)^{-1}b(s)$$
$$= \left[\prod_{\mathrm{Ker}\, f\cap\Omega_f^\alpha \neq \Omega_f^\alpha}(s+1-\alpha)\prod_{\mathrm{Ker}\, f\cap\Omega_f^\alpha\neq 0}(s+2-\alpha)\right]_{\mathrm{red}}\prod_{\alpha\in\Delta}(s+\alpha)$$

$$\dim \widetilde{M}_0/M_0 = \sum \dim\,(\Omega_f^\alpha/\mathrm{Ker}\, f\cap\Omega_f^\alpha)(\geq \dim \Omega_f/\mathrm{Ker}\, f = \dim \mathrm{Im} f),$$

if the decomposition $\Omega_f = \oplus\Omega_f^\alpha$ is associated to v corresponding to a splitting of the Hodge filtration orthogonal with respect to the duality of mixed Hodge structures (cf. 2.8). Here $[\prod(s+\alpha_i)^{m_i}]_{\mathrm{red}} = \prod(s+\alpha_i)$ if $\alpha_i \neq \alpha_j (i \neq j)$ and $m_i \geq 1$, and Δ is the set of rational numbers α such that $0 < \alpha < 1$ and the monodromy is not semisimple on its $\exp(-2\pi i\alpha)$-eigenspace. This formula was inspired by a work of Cassou-Noguès [CN2].

In general $c_{\beta\alpha}$ depends holomorphically on the parameter of a μ-constant deformation, and we can determine the b-function of the generic μ-constant deformation of a quasi-homogeneous polynomial in the case $n = 1$, cf. (4.2.6). In some special case, it was obtained by Cassou-Noguès using another method [CN1]. I am also informed that (4.2.6) is known to Briançon-Granger-Maisonobe, cf. [BGM].

The contents of this paper are as follows.

In §1 we review some elementary facts in the theory of regular holonomic $\mathcal{D}$-modules of one variable for non-specialists of $\mathcal{D}$-modules (cf. also [BM][Bo]). In §2 we review the theory of filtered (micro-local) Gauss-Manin system (cf. also [SKK][K2][Ph], etc.). We also explain the relation between the micro-local duality and the "higher residue pairing" of K. Saito (cf. also [O]). Some of the facts in §1 and §2 (e.g. 2.2-4) were explained to Scherk-Steenbrink and used in [SS]. In §4 we introduce the notion of (A)-and (B)-lattice and prove the formal part of the main theorem, cf. 3.4-6. For the existence of the splitting of the Hodge filtration we prove 3.7 and use [V1][S1][SS]. In §4 the relation with b-function and primitive form is explained.

This paper is the revised version of [S3] and the most part of the results was obtained during my stay at Institut Fourier in 1982/83. I would like to thank the staff of the institute for the hospitality, and CNRS for financial support. I thank Professor Cassou-Noguès for useful communications.

1. Regular holonomic $\mathcal{D}$-modules of one variable.

1.1. Let S be an open disc with a coordinate t, and put $S^* = S\backslash\{0\}$, $\mathcal{D} = \mathcal{D}_{S,0}$ and $\mathcal{O} = \mathcal{O}_{S,0}$. Then a $\mathcal{D}$-module M is called regular holonomic, if M is finite over $\mathcal{D}$ and its localization $M[t^{-1}]$ by t is a regular singular meromorphic connection. Here the last condition is equivalent to the existence of a saturated lattice L of $M[t^{-1}]$ (or M), where a *lattice* and *saturated* mean respectively that L is a finite $\mathcal{O}$-submodule generating $M[t^{-1}]$ (or M) over $\mathcal{O}[t^{-1}]$ (or $\mathcal{D}$), and L is stable by $t\partial_t$. Because any regular holonomic $\mathcal{D}$-module is uniquely extended to a $\mathcal{D}_S$-Module whose restriction to S^* is finite free over $\mathcal{O}_{S^*}$, M will be sometimes identified with the extended $\mathcal{D}_S$-Module. Then we can define the de Rham functor (dual of the solution) by

$$(1.1.1) \qquad DR(M) = C(\partial_t : M \to M)[-1] \in D^b_c(\mathbf{C}_S),$$

where we use an old definition of DR so that $DR(\mathcal{O}) = \mathbf{C}_S$ for example. By definition the restriction of $DR(M)$ to S^* is a local system and its monodromy is called the monodromy of M. We denote by $M_{\mathrm{rh}}(\mathcal{D})$ the category of regular holonomic $\mathcal{D}$-modules. Then

1.2. Lemma. — *The simple objects of $M_{\mathrm{rh}}(\mathcal{D})$ are (up to isomorphism) :*

$$(a)\mathcal{O} = \mathcal{D}/\mathcal{D}\partial_t, \quad (b)\mathcal{B} = \mathcal{D}/\mathcal{D}t, \quad (c)M(\alpha) = \mathcal{D}/\mathcal{D}(\partial_t t - \alpha)(\alpha \in \Lambda'),$$

where $\Lambda' = \Lambda\backslash\{1\}$ and Λ is a subset of $\mathbb{C}$ such that the composition $\Lambda \to \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ is bijective and $\Lambda \ni 1$. Moreover for simple objects M, N we have $\mathrm{Ext}^1(M,N) = \mathbb{C}$ if $(M,N) = (\mathcal{O},\mathcal{B}), (\mathcal{B},\mathcal{O})$ or $(M(\alpha), M(\alpha))$ and it is zero otherwise.

We can verify by induction on the length :

1.3. LEMMA. — *The indecomposable objects of $M_{\mathrm{rh}}(\mathcal{D})$ are (up to isomorphism) :*

I. $\mathcal{D}/\mathcal{D}(\partial_t t)^i$, II. $\mathcal{D}/\mathcal{D}(t\partial_t)^i$, III. $\mathcal{D}/\mathcal{D}(\partial_t t)^{i-1}\partial_t$, IV. $\mathcal{D}/\mathcal{D}(t\partial_t)^{i-1}t$,

V. $\mathcal{D}/\mathcal{D}(\partial_t t - \alpha)^i (\alpha \in \Lambda')$ *where* $i \in \mathbb{N}\backslash\{0\}$.

1.4. *Remarks.* — 1) By the above proof, any indecomposable object has a unique increasing filtration G such that $\mathrm{Gr}_i^G M$ is simple and $\mathrm{Gr}_i^G M \neq 0$ iff $0 \le i \le r-1$, where r is the length of M. Moreover $\mathrm{Gr}_i^G M = \mathcal{O}$ for i even (resp. odd) and $\mathcal{B}$ for i odd (resp. even) if M is type I, III (resp. II, IV), and $\mathrm{Gr}_i^G M = M(\alpha)$ if M is type V. For the existence of such a filtration we can also use :

$$0 \to \mathcal{D}/\mathcal{D}P \xrightarrow{Q} \mathcal{D}/\mathcal{D}PQ \to \mathcal{D}/\mathcal{D}Q \to 0.$$

2) The action of t is bijective, i.e. M is a meromorphic connection, if M is type I, V, and ∂_t is bijective, i.e. M is isomorphic to its microlocalization $\mathcal{E} \otimes_{\mathcal{D}} M$, if M is type II, V. Here $\mathcal{E}$ is the germ of microdifferential operators at $(0, dt)$, cf. [SKK][Ph].

3) Put $K = DR(M)$, cf. (1.1.1), and $L = K|_{S^*}$. Then $K = \mathbf{R}j_*L, j_!L, j_*L$ respectively if M is type I, II, III, and these three coincide if M is type V.

4) Let $f : X \to S$ be a Milnor fibration (resp. a projective morphism such that X is smooth and f is smooth over S^*). Then the indecomposable submodules of $\left(\int_f^j \mathcal{O}_X\right)_0$ are type II, V if $j = 0$ and $n \ge 1$ (resp. type III, V and $\mathcal{B}$ in 1.2 (i.e. type IV with $i = 1$) for any j). The assertion in the case of Milnor fibration follows from the contractibility of the singular fiber, cf. [M1][Ph], etc. The assertion for f projective is equivalent to the local invariant cycle theorem (modulo the duality), because the latter is equivalent to $\mathrm{Gr}_0^G M = \mathcal{O}$ for any indecomposable submodule M of

$\left(\int_f^j \mathcal{O}_X\right)_0$ whose support is S and monodromy is unipotent. In fact by the exact sequence

$$\int_f^{j-1} \mathcal{O}_X \xrightarrow{\partial_t} \int_f^{j-1} \mathcal{O}_X \to R^{n+j} f_* \mathbb{C}_X \to \int_f^j \mathcal{O}_X \xrightarrow{\partial_t} \int_f^j \mathcal{O}_X$$

with the surjection of ∂_t on S^*, the local invariant cycle theorem is equivalent to the surjectivity of

$H^0DR(M)_0 \to (H^0DR(M)_{t'})^T$ (the invariant part by the monodromy T)

for any M as above, and we have

$$H^0DR(M)_0 = \mathrm{Ker}\ (\partial_t\ :\ M \to M)_0 = \mathrm{Ker}\ (\partial_t\ :\ M^1 \to M^0),\ \text{cf. } 1.6,$$
$$(H^0DR(M)_{t'})^T = \mathrm{Ker}\ (t\partial_t\ :\ M^1 \to M^1), \text{cf. } (1.6.1), \text{ where } t' \in S^*.$$

This argument (with the above proof of 1.3) was used in the first analytic proof of the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber for f projective and dim $S = 1$ as above, cf. [S2]. (Here note that the intersection complexes are type III, V or $\mathcal{B}$).

5) The above classification 1.3 was well-known to specialists more or less independently, cf. for example, [BM][Bo]. We can also verify that it is equivalent to the following :

1.5. Lemma. — *For $M \in M_{\mathrm{rh}}(\mathcal{D})$ put $M^\alpha = \cup_i \mathrm{Ker}\ ((\partial_t t - \alpha)^i\ :\ M \to M)$. Then we have the natural inclusions*

$$\oplus_\alpha M^\alpha \to M \to \prod_\alpha M^\alpha, \tag{1.5.1}$$

and M is generated by $\oplus_\alpha M^\alpha$ over $\mathcal{O}$ in $\prod_\alpha M^\alpha$, i.e. M is the completion of $\oplus_\alpha M^\alpha$ by some topology (compatible with that of $\mathcal{O}$)and we denote $M = \widehat{\oplus}_\alpha M^\alpha$.

1.6. *Remarks.* — 1) Put $M_{(\alpha)} = \widehat{\oplus}_{\beta - \alpha \in \mathbb{Z}} M^\beta$ so that $M = \oplus_{\alpha \in \Lambda} M_{(\alpha)}$. Then $M_{(\alpha)}$ is uniquely determined by $(M^\alpha\ ;\ \partial_t t - \alpha)$ if $\alpha \in \Lambda'$ and $(M^1, M^0\ ;\ \partial_t, t)$ if $\alpha = 1$, because $\partial_t\ :\ M^\alpha \to M^{\alpha-1}$ and $t\ :\ M^{\alpha-1} \to M^\alpha$ are bijective for $\alpha \neq 1$. Therefore 1.5 implies 1.3. The converse is clear.

2) The lemma 1.5 was classically well-known in the meromorphic case, cf. also [D1]. Moreover we have the isomorphism :

$$L_\infty^\lambda \xrightarrow{\sim} M^\alpha \quad \text{for } \lambda = \exp(-2\pi i\alpha) \text{ and } \alpha \in \Lambda \tag{1.6.1}$$

induced by $u \to t^{\alpha-1}\exp(-N\log t)u$, where $L_\infty = \Gamma(\widetilde{S}^*, \pi^*L)$ with $\pi : \widetilde{S}^* \to S^*$ a universal covering, $L_\infty^\lambda = \mathrm{Ker}\,(T_s - \lambda : L_\infty \to L_\infty)$, $N = (2\pi i)^{-1}\log T_u$ and $T = T_sT_u$ is the Jordan decomposition of the monodromy. Here we used the natural inclusion

$$M[t^{-1}] \to j_*(\mathcal{O}_{S^*} \otimes_{\mathbb{C}} L) \tag{1.6.2}$$

where $j : S^* \to S$ is the natural inclusion, cf. [D1]. Note that the natural morphism :

$$M \to M[t^{-1}] \tag{1.6.3}$$

induces an isomorphism $M^\alpha \xrightarrow{\sim} (M[t^{-1}])^\alpha$ for $\alpha \notin -\mathbb{N}$ and (1.6.1) is true for a general $M \in M_{\mathrm{rh}}(\mathcal{D})$. We can also prove 1.5 for a general M using (1.6.3), because the functor $M \to M^\alpha$ is *exact*, and $M = \oplus_{i\in\mathbb{N}} M^{-i}$ if supp $M = \{0\}$.

1.7. Let $M_{\mathrm{rh}}(\mathcal{D})_{\mathrm{qu}}$ be the full subcategory of $M_{\mathrm{rh}}(\mathcal{D})$, whose objects have *quasi-unipotent* monodromies, i.e. $M^\alpha = 0$ for $\alpha \notin \mathbb{Q}$ by (1.6.1). For $M \in M_{\mathrm{rh}}(\mathcal{D})_{\mathrm{qu}}$, we define

$$V^\alpha M = \sum_{\beta\geq\alpha} \mathcal{O}M^\beta \quad (= \widehat{\oplus}_{\beta\geq\alpha} M^\beta).$$

(If M is not quasi-unipotent, we have to choose some order of $\mathbb{C}$.) Let m be the order of T_s. Then $V^\alpha M = V^{i/m}M$ if $(i-1)/m < \alpha \leq i/m$ with $i \in \mathbb{Z}$, and $\mathrm{Gr}_V^\alpha M = V^\alpha M / V^{\alpha+\varepsilon} M (0 < \varepsilon \ll m^{-1})$ is well-defined. If $\alpha > 0$, $V^\alpha M$ is Deligne's extension of $M|_{S^*}$ such that the eigenvalues of the residue of the connection are contained in $[\alpha - 1, \alpha)$ [D1], cf. 1.6.2.

Note that the filtration V is exhaustive and separated, and is independent of the coordinate t; in fact, V is characterized by the following conditions (due to Kashiwara, cf. [K3]) :

(1.7.1) $V^\alpha M$ are finite $\mathcal{O}$-submodules and $M = \cup_\alpha V^\alpha M$.

(1.7.2) $t(V^\alpha M) \subset V^{\alpha+1}M$, $\partial_t(V^\alpha M) \subset V^{\alpha-1}M$ and $t(V^\alpha M) = V^{\alpha+1}M (\alpha > 0)$.

(1.7.3) $\partial_t t - \alpha$ is nilpotent on $\mathrm{Gr}_V^\alpha M$.

1.8. Let K be the subring of $\mathcal{E}$ (cf. 1.4.2) whose elements commute with ∂_t, i.e. $K = \mathbb{C}\{\{\partial_t^{-1}\}\}[\partial_t]$ and $R := \mathbb{C}\{\{\partial_t^{-1}\}\}$ is

$$\left\{ \sum_{i\geq 0} a_i \partial_t^{-i} \; : \; \sum_i a_i r^i / i! < \infty \text{ for some } r > 0 \right\}.$$

Then R is a *discreet valuation ring* and K is its *quotient field*. We can verify that if the action of ∂_t on M is bijective, $V^\alpha M$ are free R-module of rank r and M is an $\mathcal{E}$-module (hence a K-module) where r is the rank of M over K (in fact, $M = K \otimes_R V^\alpha M$ for any $\alpha \in \mathbb{Q}$). In this case, an R-submodule M_0 is free of rank r over R if M_0 is finite over R and generates M over K (i.e. $M = \sum \partial_t^i M_0$).

2. Filtered Gauss-Manin Systems.

2.1. Let $f : X \to Y$ be a morphism of complex manifolds with $\dim X = n + m$ and $\dim Y = m$. The *Gauss-Manin system* is naturally defined as a filtered complex of *right* $\mathcal{D}$-Modules :

(2.1.1) $$\mathbf{R}f_*(\Omega_X^\cdot \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y, F)[n+m],$$

cf. [S5, §2]. Let $(y_1, \ldots, y_m)$ be a local coordinate system of Y, and put $f_i = f^* y_i, \partial_i = \partial/\partial y_i$ and $\partial = (\partial_1, \ldots, \partial_m)$. Then the corresponding filtered complex of *left* $\mathcal{D}$-Modules is denoted by $\int_f (\mathcal{O}_X, F)$ and locally expressed by

(2.1.2) $$\mathbf{R}f_*(\Omega_X^\cdot[\partial], F)[n+m],$$

where the differential and the $\mathcal{D}$-Modules structure is given by

$$d(\omega \otimes P) = d\omega \otimes P - \sum df_j \wedge \omega \otimes \partial_j P$$
$$y_i(\omega \otimes P) = f_i \omega \otimes P + \omega \otimes [y_i, P]$$
$$\partial_i(\omega \otimes P) = \omega \otimes \partial_i P$$

for $\omega \in \Omega_X^k$ and $P \in \mathbb{C}[\partial]$, and the filtration by

$$F_p(\Omega_X^k[\partial]) = \sum_{|\nu| \leq p+k-m} \Omega_X^k \otimes \partial^\nu.$$

Here we define the filtered direct image to be the inductive limit of $\mathbf{R}f_*(F_p\Omega_X^{\cdot}[\partial])[n+m]$ (using the canonical flabby resolution of Godement) so that the filtration becomes exhaustive, cf. [loc. cit]. Note that the direct image f_* does not commute with the inductive limit in general, and forgetting the filtration, this definition (2.1.2) does not coincide with the usual one :

$$\int_f \mathcal{O}_X = \mathbf{R}f_*(D_{Y\leftarrow X}\overset{\mathbf{L}}{\otimes}_{\mathcal{D}_X}\mathcal{O}_X) \tag{2.1.3}$$

(for example, consider the composition of $X \to pt \to \mathbb{C}$ where X is zero dimensional and has infinitely many components, cf. [S5, 2.3.8]). But we have the following :

2.2. Lemma. — *If* Sing $f = \{x \in X : \operatorname{rank} df_x \neq m\}$ *is proper over* Y, $\int_f(\mathcal{O}_X, F)$ *coincides with* $\int_f \mathcal{O}_X$ *forgetting the filtration* F.

This follows from the next two lemmas :

2.3. Lemma. — *Let* $f : X \to Y$ *be a morphism of topological spaces, and* (K, F) *a filtered complex such that* $\mathrm{Gr}_p^F K^i$ *are flabby,* F *is exhaustive and* K *is bounded below. Assume that there exists a closed subset* Z *of* X *such that* Z *is proper over* Y *and* $\mathrm{Gr}_p^F K$ *are acyclic on* $U = X\backslash Z$ *for* $p \gg 0$. *Then the natural morphism*

$$\varinjlim f_*F_pK \to \mathbf{R}f_*K$$

is a quasi-isomorphism.

Proof. — Note that the assertion is clear, il f is proper. In this case, we can show the commutativity of f_* and $\varinjlim$, and the stability of c-soft sheaves by inductive limit.

In the general case, we take a flabby resolution $K \to I$ so that $\mathbf{R}f_*K = f_*I$, and consider the distinguished triangles :

$$\to \underline{\Gamma}_Z L \to L \to j_*j^{-1}L \xrightarrow{+1}$$

for $L = F_pK, I$ where $j : U \to X$. Then we have a quasi-isomorphism $j^{-1}F_pK \to j^{-1}I$ $(p \gg 0)$ by assumption, and $\underline{\Gamma}_Z F_pK^i, \underline{\Gamma}_Z I^i$ are also flabby. Then the assertion follows from the diagram

$$\begin{array}{cccc}
\to \varinjlim f_*\underline{\Gamma}_Z F_pK & \to \varinjlim f_*F_pK & \to \varinjlim f_*j_*j^{-1}F_pK & \xrightarrow{+1} \\
\downarrow & \downarrow & \downarrow & \\
\to f_*\underline{\Gamma}_Z I & \to f_*I & \to f_*j_*j^{-1}I & \xrightarrow{+1}
\end{array}$$

because the quasi-isomorphism $\varinjlim \underline{\Gamma}_Z F_pK \to \underline{\Gamma}_Z I$ follows from the morphism of the above triangles.

2.4. LEMMA. — *Let $f : X \to Y$ and $(\Omega^{\cdot}_X[\partial], F)$ be as in (2.1.2). Assume f is smooth, i.e.* Sing $f = \emptyset$. *Let $A^{\cdot}$ be the subcomplex of $\Omega^{\cdot}_X[\partial]$ such that $A^k = \cap_i \mathrm{Ker}\,(df_i\wedge \;:\; \Omega^k_X \to \Omega^{k+1}_X)$, where f_i is as in 2.1. Then $\Pi_i df_i\wedge$ induces an isomorphism $\Omega^{\cdot}_{X/Y} \xrightarrow{\sim} A[m]$, and the natural inclusion $(A^{\cdot}, F) \to (\Omega^{\cdot}_X[\partial], F)$ is a filtered quasi-isomorphism, where F on $A^{\cdot}$ is induced by the inclusion, i.e. $F_pA^{\cdot} = \sigma_{\geq m-p}A^{\cdot}$.*

Proof. — Let $(x_1, \dots, x_{n+m})$ be a local coordinate system of X such that $x_i = f_i$ for $i \leq m$. Then the first assertion is clear. The graded complex $\mathrm{Gr}^F(\Omega^{\cdot}_X[\partial])$ is the Koszul complex associated to $\eta_i (1 \leq i \leq m)$, $0(n$ times$) \in \mathrm{End}\,(\mathcal{O}_X[\eta])$, where $\eta = (\eta_1, \dots, \eta_m)$ and $\eta_i = \mathrm{Gr}\,\partial_i$. Therefore it is graded quasi-isomorphic to the Koszul complex associated to $0(n$times$) \in \mathrm{End}\,(\mathcal{O}_X)$ shifted by m to the right. Then we get the assertion, because the last complex is isomorphic to Gr $A^{\cdot}$ by the natural morphism.

Remark. — $\Omega^{\cdot}_{X/Y}$ is globally well-defined, but $\Omega^{\cdot}_X[\partial]$ is not. The exterior product by $df_1 \wedge \dots \wedge df_m$ corresponds to the transformation of left $\mathcal{D}_Y$-Modules to right Modules.

2.5. Let $f : X \to Y$ and y_i , f_i, etc. be as above. Put $Z = X \times Y$ and $\mathcal{B}_{X|Z} = \int_i \mathcal{O}_X$ where $i : X \to Z$ is the immersion by the graph of f. Then we have an isomorphism

(2.5.1)
$$\mathcal{B}_{X|Z} = \mathcal{D}_Z / \sum \mathcal{D}_Z(f_i - y_i) + \sum \mathcal{D}_Z(\partial/\partial x_i + \sum_j (\partial f_j/\partial x_i)\partial_j) = i_*\mathcal{O}_X[\partial]$$

so that the action of $\mathcal{D}_Z$ on $i_*\mathcal{O}_X[\partial](= \mathcal{D}_Z \Pi \delta(y_j - f_j))$ is given by

$$\begin{aligned}(2.5.2)\quad (\partial/\partial x_i)(a\otimes P)&= (\partial a/\partial x_i)\otimes P - \sum(\partial f_j/\partial x_i)a\otimes \partial_j P\\ (\partial/\partial y_j)(a\otimes P)&= a\otimes \partial_j P\\ x_i(a\otimes P)&= x_i a\otimes P\\ y_j(a\otimes P)&= f_j a\otimes P + a\otimes [y_j, P]\end{aligned}$$

for $a \in \mathcal{O}_X$ and $P \in \mathbb{C}[\partial]$, where $(a\otimes P)$ is identified with $(a\otimes P)\Pi\delta(y_j - f_j)$. Let F be the filtration of $\mathcal{B}_{X|Z}$ by the order of ∂ (i.e. induced by that on $\mathcal{D}_Z\Pi\delta(y_j - f_j)$) shifted by m. Then we get the natural isomorphism

$$(\Omega_X^{\cdot}[\partial], F) = DR_{Z/Y}(\mathcal{B}_{X|Z}, F)$$

where $(F_p DR_{Z/Y}\mathcal{B}_{X|Z})^k = \Omega_X^k \otimes F_{p+k}\mathcal{B}_{X|Z}$. Tensoring (2.5.1) by $\mathcal{E}_X$ over $\mathcal{D}_X$ (where $\mathcal{E}_X$ is defined on T^*X, cf. [KK]), we get

$$(2.5.3)\quad \mathcal{C}_{X|Z} = \mathcal{E}_Z/\sum \mathcal{E}_Z(f_i - y_i) + \sum \mathcal{E}_Z(\partial/\partial x_i + \sum_j(\partial f_j/\partial x_i)\partial_j)$$

where $\mathcal{C}_{X|Z}$ has the filtration F induced by that of $\mathcal{E}_Z$ shifted by $m = \operatorname{codim}_Z X$. We define the *micro-local filtered Gauss-Manin system* by

$$(2.5.4)\qquad \int_p (\mathcal{C}_{X|Z}, F) = \mathbb{R}p_* DR_{Z/Y}(\mathcal{C}_{X|Z}, F)[n+m],$$

where $p : Z \to Y$ is the projection and $X \times T^*Y \to T^*Y$ is also denoted by p. Here note that $DR_{Z/Y}(\mathcal{C}_{X|Z}, F)$ if filtered acyclic on the complement of $X \times T^*Y$, because $(\partial/\partial x_i)^{-1}$ exists in $F_{-1}\mathcal{E}_Z$ on $\{\xi_i \neq 0\}$ where $\xi_i = \operatorname{Gr} \partial/\partial x_i$, cf. [SKK]. Let $\overset{\circ}{T}{}^*Y$ be the complement of the zero section. Then $DR_{Z/Y}(\mathcal{C}_{X|Z}, F)$ is filtered acyclic on $(X\backslash \operatorname{Sing} f) \times \overset{\circ}{T}{}^*Y$, because $\operatorname{supp} M \cap (X\backslash \operatorname{Sing} f) \times \overset{\circ}{T}{}^* Y = \emptyset$. In particular, (2.5.4) coincides with the usual definition [K1][Ph], forgetting the filtration and restricting to $\overset{\circ}{T}{}^*Y$, if $\operatorname{Sing} f$ is *proper* over Y. From now on, we assume the following :

(2.5.5) Assumption : Sing f is *finite* (hence proper) over Y.

In particular, p is non-characteristic to $\mathcal{C}_{X|Z}$. By [KK] [K2] [Ph], the cohomologies of $\int_p \mathcal{C}_{X|Z|\overset{\circ}{T}{}^*Y}$ are zero except for degree zero and $\int_p \mathcal{C}_{X|Z|\overset{\circ}{T}{}^*Y}$ (identified with its zero-th cohomology) is coherent over $\mathcal{E}_{Y|\overset{\circ}{T}{}^*Y}$ and regular holonomic so that

$$(2.5.6)\qquad \Lambda = p_*(\operatorname{Ch}(\mathcal{C}_{X|Z}) \cap X \times \overset{\circ}{T}{}^*Y)$$

where $\Lambda = \mathrm{Ch}\left(\int_p \mathcal{C}_{X|Z}|_{\overset{\circ}{T^*}Y}\right)$ and $\mathrm{Ch}(M) =$ supp (M) for an $\mathcal{E}$-Module M. We have also

(2.5.7) $\qquad F$ on $\int_p \mathcal{C}_{X|Z}|_{\overset{\circ}{T^*}Y}$ is strict.

In fact, the non-charactericity implies $\dim \mathrm{Ch}(\mathcal{C}_{X|Z}) \cap X \times \overset{\circ}{T^*}Y \leq m$, and $\mathrm{Gr}^F DR_{Z/Y}(\mathcal{C}_{X|Z}, F)|_{X \times \overset{\circ}{T^*}Y}$ is acyclic except for degree n by the microlocalization of (2.5.2), because

$$\begin{aligned}\mathrm{Ch}(\mathcal{C}_{X|Z}) \cap X \times \overset{\circ}{T^*}Y = &\{(x,\eta) \in X \times_Y \overset{\circ}{T^*}Y \\ &: \sum_j (\partial f_j/\partial x_i)\eta_j = 0 (1 \leq i \leq n+m)\}.\end{aligned}$$

(In this case, we can also use $\dim f(X_r) \leq r$ and $\dim X_r \leq r (r < m)$, where $X_r = \{\mathrm{rank}\ df_x \leq r\}$.) We can also verify for $y \in Y$:

(2.5.8) $\quad$ rank $df_x = m-1$ for any $x \in (\mathrm{Sing}\ f)_y$, iff $\dim T_y^*Y \cap \Lambda = 1$,

where $(\mathrm{Sing}\ f)_y = \mathrm{Sing}\ f \cap f^{-1}(y)$. If $\{x\} = (\mathrm{Sing}\ f)_y$, (2.5.8) is equivalent to :

(2.5.9) $\quad f$ is a deformation of a function with an isolated singularity on a neighborhood of y, iff $\int_p \mathcal{C}_{X|Z}$ is in a *generic position* at $\overset{\circ}{T_y^*}Y \cap \Lambda$, cf. [KK].

Here the first condition means that y has a neighborhood of the form $S \times T$ such that $\dim S = 1$ and $\mathrm{pr}_2 \circ f$ is smooth on a neighborhood of x, where $\mathrm{pr}_2 : S \times T \to T$. From now on, we assume the above equivalent conditions. By restricting X and Y, we also assume for $0 < \delta, \delta' \ll \varepsilon \ll 1$:

(2.5.10) $\quad Y = S \times T$ with $S = \{t \in \mathbb{C} : |t| < \delta\}, T = \{t' \in \mathbb{C}^{m-1} : |t'| < \delta'\}$ and $X = \{(x,t') \in \mathbb{C}^{n+1} \times T : |x| < \varepsilon, f'(x) \in S\}$, where f' is defined by $f(x,t') = (f'(x,t'), t')$.

Here $t = t_1$ and $t' = (t_2, \ldots, t_m)$. We may assume

(2.5.11) $\qquad n > 0$,

because the case $n = 0$ is a deformation of A_ℓ-singularity and not so much interesting. By (2.5.10) we have a factorization $f = p' \circ i'$ where $p' : Z' = X \times_T Y \to Y$ is the second projection. Then we have

$$(2.5.12)\qquad \begin{aligned}\int_f (\mathcal{O}_X, F) &= \int_{p'} (\mathcal{B}_{X|Z'}, F) = f_*(\Omega^{\cdot}_{X/T}[\partial_1], F)[n+1]\\ \int_p (\mathcal{C}_{X|Z}, F) &= \int_{p'} (\mathcal{C}_{X|Z'}, F)\end{aligned}$$

where $\partial_i = \partial/\partial t_i$ for $1 \le i \le m$ and the filtration and the differential of $\Omega^{\cdot}_{X/T}[\partial_1]$ are given by

$$F_p(\Omega^k_{X/T}[\partial_1]) = \sum_{i \le p+k-1} \Omega^k_{X/T} \otimes \partial_1^i$$

$$d(\omega \otimes \partial_1^i) = d\omega \otimes \partial_1^i - df' \wedge \omega \otimes \partial_1^{i+1}.$$

Here note that f is a Stein morphism and $\Omega^k_{X/T}$ are f_*-acyclic. We denote also by F the induced filtration on $\int_f^k \mathcal{O}_X \ := \mathcal{H}^k \int_f \mathcal{O}_X$ and $\mathcal{H}^k(\Omega^{\cdot}_{X/T}[\partial_1])$. Then $\left(\int_f^k \mathcal{O}_X\right)_0$ is independent of ε in (2.5.10) and

$$(2.5.13)\qquad \left(\int_f^0 \mathcal{O}_X, F\right)_0 = (H^{n+1}(\Omega^{\cdot}_{X/T,0}[\partial_1]), F),$$

because $\mathcal{H}^k\Omega^{\cdot}_{X/S}$ are locally constant on the fibers of f, if f is smooth. By [Ph] we have the canonical isomorphism

$$(2.5.14)\qquad \left(\int_f^0 \mathcal{O}_X\right)_0 \xrightarrow{\ \sim\ } \left(\int_p^0 \mathcal{C}_{X|Z}\right)_\eta \quad \text{for } \eta \in \mathring{T}^*{}_0Y \cap \Lambda.$$

(Here we may assume $\eta = dt_1$ by changing the coordinates and the decomposition $Y = S \times T$). In fact, the morphism in (2.5.14) is the natural morphism associated to the micro-localization and the right hand side is a holonomic $\mathcal{D}_{Y,0}$-module by [KK, 5.1.1]. Therefore its kernel and cokernel are finite free $\mathcal{O}_{Y,0}$-modules, because its micro-localization induces an isomorphism on $\mathring{T}^*_0Y$. But the action of ∂_1 is bijective on the both terms of (2.5.14) by the exact sequence

$$H^n\Omega^{\cdot}_{X/T,0} \to \left(\int_f^0 \mathcal{O}_X\right)_0 \xrightarrow{\ \partial_1\ } \left(\int_f^0 \mathcal{O}_X\right)_0 \to H^{n+1}\Omega^{\cdot}_{X/T,0}.$$

Thus we get (2.5.14). As to the filtration F, we have

$$\left(F_{-n}\int_f^0 \mathcal{O}_X\right)_0 = \Omega^{n+1}_{X/T,0}/df'\wedge d\Omega^{n-1}_{X/T,0}, \tag{2.5.15}$$

because $\{\partial f'/\partial x_i(0\le i\le n), t_i(2\le i\le m)\}$ is a regular sequence. We also verify the stability of $\left(F_{-n}\int_f^0 \mathcal{O}_X\right)_0$ by ∂_1^{-1} so that

$$\left(F_{-n+i}\int_f^0 \mathcal{O}_X\right)_0 = \partial_1^i\left(F_{-n}\int_f^0 \mathcal{O}_X\right)_0 \quad \text{for } i\ge 0. \tag{2.5.16}$$

Then the isomorphism (2.5.14) induces

$$\left(F_q\int_f^0 \mathcal{O}_X\right)_0 = \left(F_q\int_p^0 \mathcal{C}_{X|Z}\right)_\eta \quad \text{for } q\ge -n, \tag{2.5.17}$$

because it is enough to show it for $q=-n$ and in this case we can reduce the assertion to the case $Y=S$ using the non-characteristic restriction to $S\times\{0\}$, cf. [K2][Ph]. In fact, (2.5.17) is equivalent to the assertion that its first term is an $\mathcal{E}_{Y,\eta}(0)-$ (or $\mathcal{O}_{T,0}\{\{\partial_1^{-1}\}\}-$) module, because

$$\left(F_q\int_p^0 \mathcal{C}_{X|Z}\right)_\eta \quad \text{are finite free of rank } \mu \text{ over } R, \tag{2.5.18}$$

where $R=\mathcal{O}_{T,0}\{\{\partial_1^{-1}\}\}=\mathcal{E}_{Y,\eta}\cap\mathcal{O}_{T,0}[[\partial_1^{-1}]]$, cf. [Ph].

Remark. — The right hand side of (2.5.15) is called the Brieskorn lattice. We can also verify the coincidence of ∂_1^{-1} with the inverse of the natural connection $\Delta_{\partial/\partial t_1}$ defined on it, using (2.5.2).

2.6. Let the notation and the assumption (i.e. (2.5.5)(2.5.10) and (2.5.11)) be as above. We assume further $Y=S$, i.e. $T=\{0\}$ in (2.5.10). Put

$$M=\left(\int_f^0 \mathcal{O}_X\right)_0,\ M_0=F_{-n}M(=\Omega^{n+1}_{X,0}/df\wedge d\Omega^{n-1}_{X,0}). \tag{2.6.1}$$

Then M is quasi-unipotent and has the filtration V, cf. 1.7. We denote also by F the induced filtration on $\mathrm{Gr}_V^\alpha M$. Note that the freeness of M_0 follows from the inclusion $M_0\subset V^{>0}M$ proved in [M1, K1]. (It can be also proved as in [Ka] with $\Omega^{\cdot}_{X/S}(\log D)$ replaced by $A^{\cdot}_X$, where

$A_X^p := \{\omega \in \Omega_X^p : df \wedge \omega = 0\}$.) Let $H^n(X_\infty, \mathbf{C})$ be the cohomology of the Milnor fiber, and $H^n(X_\infty, \mathbf{C})_\lambda$ the kernel of $T_s - \lambda$ where $T = T_s T_u$ is the Jordan decomposition of the Milnor monodromy, cf. [St]. Then $H^n(X_\infty, \mathbf{C})$ has the natural mixed Hodge structure whose Hodge filtration is compatible with the monodromy decomposition [loc. cit] :

$$H^n(X_\infty, \mathbf{C}) = \oplus_\lambda H^n(X_\infty, \mathbf{C})_\lambda.$$

By (1.6.1) we have the isomorphisms

(2.6.2) $$H^n(X_\infty, \mathbf{C})_\lambda = \mathrm{Gr}_V^\alpha M \quad \text{for } \alpha > 0 \text{ and } \lambda = \exp(-2\pi i\alpha).$$

Moreover this isomorphism is compatible with the Hodge filtration, i.e.

(2.6.3)
$$F^p H^n(X_\infty, \mathbf{C})_\lambda = F_{-p}\mathrm{Gr}_V^\alpha M \quad \text{for } 1 \geq \alpha > 0 \text{ and } \lambda = \exp(-2\pi i\alpha),$$

cf. [V1][S1-2] (see also the revised version of [SS] for a proof using a language which is not so much sophisticated). Note that (2.6.3) was (essentially) first found by Varchenko [V1], but he used essentially the filtration $F'_{-n+p} = t^{-p}M_0$ in $M[t^{-1}]$ instead of $F_{-n+p} = \partial_t^p M_0$ in M, cf. (2.5.16), where $t = t_1$ and $\partial_t = \partial_1$. (Here we use the natural inclusion $M_0 \to M[t^{-1}]$ induced by $M \to M[t^{-1}]$, cf. (1.6.3).) Therefore he had to take Gr^W to cancel the difference of t and ∂_t^{-1} (i.e. the action of N). Note also that the decomposition $M = \widehat{\oplus} M^\alpha$ (cf. 1.5) corresponds to the asymptotic expansion used by Varchenko [loc. cit] and the Hodge filtration F is not compatible with this decomposition, i.e. $M_0 \cap V^\alpha M \neq (M_0 \cap M^\alpha) + (M_0 \cap V^{>\alpha}M)$ in general, cf. [S1]. This is why we have to take Gr_V^α instead of $V^{>0}M[t^{-1}]/V^{>1}M[t^{-1}]$ as in the first version of [SS]. By the same reason we may not use $\partial_t^p M_0$ in $M[t^{-1}]$.

In [S4, §4] we gave some explanation of (2.6.3) from the view point of (mixed) Hodge Modules (i.e. (2.5.7) (2.5.17) are essential). This gives also a proof of (2.6.3) without using a compactification of f.

2.7. Let $Y = S \times T$ be as in 2.5. Put $(\mathcal{E}, F) = (\mathcal{E}_{Y,\eta}, F)$ with $\eta = (0, dt_1)$ and $F_p\mathcal{E} = \mathcal{E}(p)$, cf, [KK]. Let (M, F) be a filtered $\mathcal{E}$-module such that

(2.7.1) $$F_pM \text{ are free of rank } r \text{ over } R,$$

where $R = \mathcal{E} \cap \mathcal{O}_{T,0}[[\partial_1^{-1}]]$ and $K = R[\partial_1]$. Let F be the filtration of R, K by the order of ∂_t, i.e. $F_pR = R \cap F_p\mathcal{E}$. Then (M, F) is isomorphic to a direct sum of (K, F) as a filtered K-module. We define the right action of ξ on $\mathcal{E} \otimes_K M$ by

$$(P \otimes u)\xi = P\xi \otimes u - P \otimes \xi u$$

for $\xi = t_1, \partial_2, \ldots, \partial_m$, where $\partial_i = \partial/\partial t_i$ and $t' = (t_2, \ldots, t_m)$ are the coordinates of T. We consider the Koszul complex C' associated to the right action of $\partial_2, \ldots \partial_m$, where C' is shifted by $m-1$ to the left so that $C'^k = 0$ for $k < -m$ or $k > 0$. We define the filtration F on $\mathcal{E} \otimes_K M$ and C' by

$$F_p(\mathcal{E} \otimes_K M) = \mathrm{Im}\ (F_p\mathcal{E} \otimes_R F_0 M \to \mathcal{E} \otimes_K M)$$
$$F_p C'^k = \oplus F_{p+k}(\mathcal{E} \otimes_K M).$$

Let (C, F) be the mapping cone of the right action $t_1 : (C', F) \to (C', F)$. Then C is the Koszul complex associated to the right action of $t_1, \partial_2, \ldots, \partial_m$. By definition we have a natural morphism $(C, F) \to (M, F)$ as complexes of filtered $\mathcal{E}$-modules. Then

(2.7.2) $(C, F) \to (M, F)$ is a filtered quasi-isomorphism, i.e. $F_p C \to F_p M$ are quasi-isomorphisms.

In fact it is enough to show the acyclicity of $C(\mathrm{Gr}_p^F C \to \mathrm{Gr}_p^F M)$ for any (or some) p, because $F_p C^k$, $F_p M$ are finite $F_0\mathcal{E}$-modules, cf. [SKK] etc. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_T := \mathcal{O}_{T,0}$. Then it is enough to show its acyclicity after tensoring $\mathcal{O}_T/\mathfrak{m}^j$ over $\mathcal{O}_T$ for any $j > 0$, cf. [Se], because $\mathrm{Gr}_p^F C^k$, $\mathrm{Gr}_p^F M$ are finite over $\mathcal{O}_{P^*Y,\eta} = \mathrm{Gr}\,_0^F \mathcal{E}$, where P^*Y is the projective bundle associated to T^*Y, cf. [SKK].

By definition $\mathrm{Gr}_p^F C \otimes_{\mathcal{O}_T} \mathcal{O}_T/\mathfrak{m}^j$ is the Koszul complex associated to the action of $t_1 \otimes \mathrm{id} - \mathrm{id} \otimes t_1$, $\tau_1^{-1}\tau_i \otimes \mathrm{id} - \mathrm{id} \otimes \tau_1^{-1}\tau_i (i \geq 2)$ on

$$(\mathcal{O}_{P^*T,\eta}/\mathfrak{m}^j \mathcal{O}_{P^*Y,\eta}) \otimes_{\mathcal{O}_T/\mathfrak{m}^j} \mathrm{Gr}_p^F M/\mathfrak{m}^j\, \mathrm{Gr}_p^F M$$

where $\tau_i = \mathrm{Gr}\ \partial_i$. Here the action of $t_1, \tau_1^{-1}\tau_i (i \geq 2)$ on $\mathrm{Gr}_p^F M/\mathfrak{m}^j\, \mathrm{Gr}_p^F M$ is nilpotent, because it is finite dimensional. Then we may assume that their action is zero, taking the graduation associated to some filtration, because these actions are commutative. We get the assertion by the fact that $t_1, \tau_1^{-1}\tau_i (i \geq 2)$ is a regular sequence of

$$\mathcal{O}_{P^*Y,\eta}/\mathfrak{m}^j \mathcal{O}_{P^*Y,\eta} = \mathbb{C}\{t_1, \ldots, t_m, \tau_1^{-1}\tau_2, \ldots, \tau_1^{-1}\tau_m\} \otimes_{\mathcal{O}_T} \mathcal{O}_T/\mathfrak{m}^j,$$

cf. [Se]. Thus we get the canonical filtered free resolution of (M, F).

By definition $\mathbb{D}(M,F)$ the dual of (M,F) is given by

$$\mathbb{D}(M,F) = \mathrm{Hom}_{\mathcal{E}}((C,F),(\mathcal{E},F[2m]))[m], \tag{2.7.3}$$

where $F[j]$ is defined by $F[j]_p = F_{p-j}$ for $j,p \in \mathbb{Z}$. Here the right hand side is a filtered right $\mathcal{E}$-module, and for the transformation to the left module we shall use the anti-involution $P \mapsto P^*$ of $\mathcal{E}$ such that

$$t_i^* = t_i, \quad \partial_i^* = -\partial_i, \text{ cf. [SKK].} \tag{2.7.4}$$

We consider the canonical isomorphisms as right $\mathcal{E}$-modules

$$\mathrm{Hom}_K(M,K)\otimes\mathcal{E} \xrightarrow{\sim} \mathrm{Hom}_K(M,\mathcal{E}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{E}}(\mathcal{E}\otimes M,\mathcal{E})$$

induced by $\phi\otimes P \mapsto [u \mapsto \phi(u)P], \psi \mapsto [Q\otimes u \mapsto Q\psi(u)]$. We define the right action of $\xi = t_1, \partial_i (i \geq 2)$ on $\mathrm{Hom}_K(M,K)$ by

$$(\phi\xi)(u) = \phi(\xi u) - [\xi, \phi(u)]$$

for $\phi \in \mathrm{Hom}_K(M,K)$, $u \in M$, and its left action on $\mathrm{Hom}_K(M,K)\otimes_K \mathcal{E}$ by $\xi(\phi\otimes P) = \phi\otimes\xi P - \phi\xi\otimes P$. Then we can verify that the left action of ξ on $\mathrm{Hom}_K(M,K)\otimes\mathcal{E}$ corresponds by the above isomorphism to the action on $\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}\otimes M,\mathcal{E})$ induced by that on $\mathcal{E}\otimes M$. By the same argument as above, $\mathbb{D}(M,F)$ is a filtered $\mathcal{E}$-module, i.e. F is strict, where it is well-known that $\mathbb{D}M$ is a holonomic $\mathcal{E}$-module, cf. [K1].

We define $\mathbb{D}'(M,F)$ a filtered K-module with the right action of t_1, $\partial_i (i \geq 2)$ by

$$\mathbb{D}'(M,F) = \mathrm{Hom}_K((M,F),(K,F[1+m])), \tag{2.7.4}$$

where the action of t_1, etc. is defined as above. Then by the above argument we have the canonical isomorphism

$$\mathbb{D}(M,F) = \mathbb{D}'(M,F) \tag{2.7.5}$$

as filtered K-modules with the right action of t_1, $\partial_i (i \geq 2)$, because Gr^F of (2.7.5) is clear and $F_p\mathbb{D}M$ are finite over R, cf. for example, [Ph][M3]. In particular the right action of t_1, etc. is extended to the structure of right $\mathcal{E}$-module.

Let (N,F) be a filtered right $\mathcal{E}$-module. By (2.7.5) there is a one-to-one correspondence between the filtered $\mathcal{E}$-linear morphisms

$$\phi : (N,F) \to \mathbb{D}(M,F)$$

and the filtered K-linear morphisms

$$\mathbb{S} \,:\, (N,F)\otimes_K (M,F) \to (K, F[1+m])$$

satisfying

$$[\xi, \mathbb{S}(v,u)] = \mathbb{S}(v,\xi u) - \mathbb{S}(v\xi, u) \quad \text{for} \quad u\in M, v\in N, \tag{2.7.6}$$

where $\xi = t_1$, $\partial_i (i \geq 2)$. Assume now (M,F) is self dual, i.e. we have a duality isomorphism

$$\mathbb{D}(M,F) \cong (M, F[w]) \tag{2.7.7}$$

for some $w \in \mathbb{Z}$, where w is called the weight of (M,F). Then the duality isomorphisms (2.7.6) correspond bijectively to the pairings

$$\mathbb{S} \,:\, (M,F)\otimes_{\mathbb{C}} (M,F) \to (K, F[1+m-w])$$

satisfying

$$P\mathbb{S}(u,v) = \mathbb{S}(u,Pv) = \mathbb{S}(P^* u, v) \quad \text{for } P\in K, \tag{2.7.8}$$

$$[\xi, \mathbb{S}(u,v)] = \mathbb{S}(u,\xi v) - \mathbb{S}(\xi^* u, v) \quad \text{for } \xi = t_1, \partial_i (i\geq 2), \tag{2.7.9}$$

$$\mathbb{S}(F_p M, F_q M) \subset F_{p+q-1-m+w} K \tag{2.7.10}$$

so that

$$\mathbb{S} \,:\, \mathrm{Gr}_p^F M \otimes_{\mathcal{O}_T} \mathrm{Gr}_p^F M \to \mathrm{Gr}_{p+q-1-m+w}^F K$$

is non-degenerate over $\mathcal{O}_T$, where $*$ is the anti-involution of $\mathcal{E}$ satisfying (2.7.4). If furthermore M is simple holonomic, i.e. $\mathrm{Ch}(M)$ is irreducible at η and the multiplicity of M is one on some Zariski open smooth subset of $\mathrm{Ch}(M)$ near η, we have

$$\mathrm{End}_{\mathcal{E}}(M) = \mathbb{C} \tag{2.7.11}$$

and the conditions (2.7.8-9) are enough to characterize $\mathbb{S}$ uniquely up to a multiple constant.

From now on we assume $(M,F) = \displaystyle\int_p (C_{X|Z}, F)_\eta$, cf. 2.5. Then we have the micro-local Poincaré duality

$$\mathbb{D}(M,F) = (M, F[n+m]) \tag{2.7.12}$$

(using, for example, the micro-localization of the duality in [S5, §2] applied to a compactification of f). As a corollary, we get the existence and the uniqueness of the "higher residue pairing" of K. Saito [Sk] on the Gauss-Manin system of a versal deformation, because $\mathrm{Ch}(M)$ is smooth and the multiplicity is one in this case, cf. for example [Ph] etc. (This name comes from the fact that the restriction of $\mathbb{S}$ to $\mathrm{Gr}^F_{-n}M = \Omega^{n+1}_{X/T}/df' \wedge \Omega^n_{X/T}$ (cf. (2.7.10)) is the residue pairing of Grothendieck, where $\mathrm{Gr}^F_i K = \mathcal{O}_T \otimes \partial^i_1$ by definition.) The above construction was inspired by Kashiwara's interpretation of the higher residue pairings as the duality (cf.[O]) and the residue pairings (unpublished) of $\mathcal{E}$-Modules. We can also check directly that (2.7.12) corresponds to the higher residue pairing by the above construction using the facts in [loc. cit].

The above identification is compatible with base change of T. From now on we assume $T = \{0\}, m = 1$, and use the notations in §1. We have

$$\mathbb{S}(M^\alpha, M^\beta) \subset \mathbb{C} \otimes \partial_t^{-i} \text{ if } \alpha + \beta = i \text{ and } 0 \text{ otherwise,} \tag{2.7.13}$$

by (2.7.8-9), and it induces the perfect pairings

$$\begin{aligned} &\mathrm{Gr}_V\mathbb{S} : \ \mathrm{Gr}_V^{-\alpha+1}(M,F) \otimes_{\mathbb{C}} \mathrm{Gr}_V^\alpha(M,F) \to (\mathbb{C}, F[-n])\,(0<\alpha<1) \\ &\mathrm{Gr}_V\mathbb{S} : \ \mathrm{Gr}_V^1(M,F) \otimes_{\mathbb{C}} \mathrm{Gr}_V^1(M,F) \to (\mathbb{C}, F[-n-1]) \end{aligned} \tag{2.7.14}$$

where F on $\mathbb{C}$ is defined by $\mathrm{Gr}^F_p\mathbb{C} = 0$ for $p \neq 0$, and $\mathrm{Gr}^F_i(K,F) = (\mathbb{C}, F[i])$. We can also verify that the perfect pairings (2.7.14) correspond to the perfect pairing of mixed Hodge structure on the Milnor cohomology by the isomorphism (1.6.1) (in particular, $\mathbb{S}$ is compatible with the rational structure $H^n(X_\infty, \mathbb{Q})$ up to Tate twist, cf. Appendix of [S3].) In fact we can easily verify that (2.7.14) corresponds to the pairing of ϕ(i.e. Gr^α_V for $0 \le \alpha < 1$) of $\int_{\bar f}^0 (\mathcal{O}_X, F)$ where $\bar f$ is a compactification of f), because ϕ does not change by the microlocalization. Then the assertion follows from the general theory of mixed Hodge Modules, cf. [S6].

Let $H_n(X_\infty, \mathbb{C})$ be the dual of $H^n(X_\infty, \mathbb{C})$, and $\mathbb{S}^*$ the induced duality on it by $\mathbb{S}$. Let I be the intersection form on $H_n(X_\infty, \mathbb{C})$, and put $H_n(X_\infty, \mathbb{C})_{\neq 1} = \oplus_{\lambda \neq 1} H_n(X_\infty, \mathbb{C})_\lambda$. Then we have up to sign (and Tate twist, depending of the definition of I) :

$$\begin{aligned} &I = \mathbb{S}^* &&\text{on } H_n(X_\infty, \mathbb{C})_{\neq 1} \\ &I = \mathbb{S}^* \circ (N \otimes \mathrm{id}) &&\text{on } H_n(X_\infty, \mathbb{C})_1 \end{aligned} \tag{2.7.15}$$

where $N = -N^*$ on $H_n(X_\infty, \mathbf{C})$. (This follows from [S5, 5.2.3].) Note that for $f = \sum_{i=1}^{2m+1} x_i^2$, (2.7.15) implies $U_{2m} = 2^{m+1}\pi^m/\Pi_{i=1}^m(2i-1)$, where U_n is the volume of the n-dimensional unit sphere (and $U_{2m-1} = 2\pi^m/(m-1)!$).

The following lemma will be used in §4.

2.8. Lemma. — *Let H be $H^n(X_\infty, \mathbf{C})_{\neq 1}$ or $H^n(X_\infty, \mathbf{C})_1$ with the duality of mixed Hodge structures $\mathbf{S} : H \otimes H \to \mathbf{C}(-r)$ defined over $\mathbf{Q}$, where $r = n$ or $n+1$. Then the Hodge filtration F of H has a splitting $H = \oplus G^p$ (i.e. $F^pH = \oplus_{i \geq p} G^i$) such that $T_sG^i \subset G^i, NG^i \subset G^{i-1}$ and $\mathbf{S}(G^i, G^j) = 0$ for $i + j \neq r$, where $T = T_sT_u$ is the Jordan decomposition of the monodromy and $N = (2\pi i)^{-1} \otimes \log T_u$.*

Proof. — Put

$$G^p = F^p \cap \sum_q (\overline{F}^q \cap W_{p+q}) \qquad \text{for } i \in \mathbf{Z}$$

$$I^{pq} = (F^p \cap W_{p+q}) \cap (\overline{F}^q \cap W_{p+q} + \sum_{j>0} \overline{F}^{q-j} \cap W_{p+q-j-1}) \text{ for } p, q \in \mathbf{Z}$$

where W is the weight filtration of H. Then we have $H = \oplus_{p,q} I^{p,q}$ by [D2] and $G^p \supset \oplus_q I^{p,q}$. Since $\mathbf{S}$ is a duality of mixed Hodge structure, we have

$$\mathbf{S}(F^p, F^{p'}) = 0 \quad \text{for } p + p' > r (\text{same for } \overline{F}),$$
$$\mathbf{S}(W_i, W_{i'}) = 0 \quad \text{for } i + i' < 2r.$$

Then

$$\mathbf{S}(G^p, G^{p'}) = 0 \quad \text{for } p + p' \neq r,$$

because

$$\mathbf{S}(\overline{F}^q \cap W_{p+q}, \overline{F}^{q'} \cap W_{p'+q'}) = 0 \quad \text{for } p + p' < r.$$

Therefore we get $G^p = \oplus_q I^{p,q}$ and $H = \oplus_p G^p$, because

$$\dim G^p \leq \dim H - \sum_{p' \neq r-p} \dim G^{p'} \leq \sum_q \dim I^{p,q}.$$

The conditions $T_sG^p \subset G^p$ and $NG^p \subset G^{p-1}$ are clear by

$$T_sF^p \subset F^p (\text{same for } \overline{F}, W),\ NF^p \subset F^{p-1} (\text{same for } \overline{F})$$
$$NW_i \subset W_{i-2}.$$

2.9. Let $Y = S \times T$ as in 2.7, i.e. S and T are polydiscs such that dim $S = 1$. Let (M, F) be a filtered regular holonomic $\mathcal{E}_Y$-Module on $\mathring{T}^*Y (= T^*Y -$ zero section) whose support is contained in the conormal of $\{0\} \times T$. Then M is trivial along $T (= \{0\} \times T)$ and $(M(t'), F) = \mathcal{O}_{S\times\{t'\}} \otimes_{\mathcal{O}_Y} (M, F)$ is a filtered $\mathcal{E}_{S\times\{t'\}}$-Module on $\mathring{T}^*S \times \{t'\}$. Assume M has the filtration V along T (cf. [K3]) indexed by $\mathbf{Q}$ (cf. [S5]). Then it gives the filtration V of each $M(t')$ by $V^\alpha M(t') = \mathcal{O}_{S\times\{t'\}} \otimes_{\mathcal{O}_Y} V^\alpha M$. Let M^T be the intersection of the kernel of $\xi : M \to M$ for any vector field ξ on T.

Then M^T is a constant sheaf of $\mathcal{E}_{S,(0,dt)}$-modules on T (using the section $t' \mapsto (t', dt)$), where t is the coordinate of S. Since $V^\alpha M$ are stable by ξ, M^T has the filtration V, and the stalk of (M^T, V) at $t' \in T$ is isomorphic to $(M(t'), V)$ so that $V^\alpha M = \mathcal{O}_Y \otimes_{\mathcal{O}_S} V^\alpha M^T$. We assume

(2.9.1) $\quad \mathrm{Gr}_p^F \mathrm{Gr}_V^\alpha M$ are free $\mathcal{O}_T$-Modules and F on each $\mathrm{Gr}_V^\alpha M$ is a finite filtration.

Then there exists integers, a, b such that

(2.9.2) $\quad F_pM \subset V^\alpha M$ for $p + \alpha \le a$ and $F_pM \supset V^\alpha M$ for $p + \alpha \ge b$.

In particular V is also finite on each $\mathrm{Gr}_p^F M$. We check that Gr_p^F, Gr_V^α commute with the restriction $\mathcal{O}_{S\times\{t'\}}\otimes$ and dim $\mathrm{Gr}_p^F\ \mathrm{Gr}_V^\alpha M(t')$ are constant for $t' \in T$.

Let $\xi \in \mathrm{Der}\, T$ be a vector field on T. Then

$$\xi : M \to M$$

is not $\mathcal{O}_T$-linear and does not induce a well-defined map $\xi : M(t') \to M(t')$ if $\xi \ne 0$. But $\mathrm{Gr}^F \partial_t^{-1}\xi : \mathrm{Gr}_p^F M \to \mathrm{Gr}_p^F M$ is $\mathcal{O}_T$-linear and induces

$$\mathrm{Gr}^F \partial_t^{-1}\xi : \mathrm{Gr}_p^F M(t') \to \mathrm{Gr}_p^F M(t')$$

so that $\mathrm{Gr}^F \partial_t^{-1}\xi (V^\alpha \mathrm{Gr}_p^F M(t')) \subset V^{\alpha+1} \mathrm{Gr}_p^F M(t')$.

2.10. *Remark.* — In the case of Gauss-Manin system associated to a μ-constant deformation f' (cf. (2.5.10)), we can check the assumption (2.9.1) using [V3] (or [S6] and the vanishing cycle functor ϕ along f'). We have the isomorphism

$$\mathrm{Gr}_{-n}^F M(t') = \Omega_{f'_{t'}} (:= \Omega^{n+1}_{\mathbf{C}^{n+1},0} / df'_{t'} \wedge \Omega^n_{\mathbf{C}^{n+1},0})$$

so that $\mathrm{Gr}^F \partial_t^{-1}\xi$ is identified with the multiplication of $-(\xi f')|_{t'}$ on $\Omega_{f'_{t'}}$, because $\xi\delta(t-f') = -(\xi f')\partial_t\delta(t-f')$, cf. (2.5.2) applied to $f' : X \to S$. Here $f'_{t'}$ and $(\xi f')|_{t'}$ are the restriction of f' and $\xi f'$ to $\mathbb{C}^{n+1} \times \{t'\}$, cf. (2.5.10). This implies a positive answer to the problem in [S1] :

(2.10.1) the local moduli of μ-constant deformation is determined by the Brieskorn lattice (as a subspace of the Gauss-Manin system),

in the case the μ-constant stratum T is smooth. Here (2.10.1) means the injectivity of the "period map" (defined on a neighborhood of $0 \in T$) :

$$\Psi : T \ni t' \to F_{-n}M(t') \in \mathsf{L}(M(0)) := \{\text{the lattices of } M(0)\},$$

where the parallel translation given by M^T in 2.9 is used to define Ψ. Let $H = V^{>0}M(0)/V^{n+3}M(0)$ and $F^p_{t'} = F_{-n-p}M(t')/V^{n+3}M(t') \subset H (0 \le p \le 2)$, where $F_{-n}M(t') \supset V^{n+1}M(t')$ by (2.6.3), cf. (4.1.3). Then we have

$$\Phi : T \ni t' \to (F^0_{t'}, F^1_{t'}, F^2_{t'}) \in \mathrm{Flag}^3(H)$$

which is factorized by Ψ, because $F_{-n-p}M(t') = \partial_t^{-p}F_{-n}M(t')$. By definition Φ is horizontal, i.e. $\xi F^p \subset F^{p-1}$ with $F^p := F_{-n-p}M/V^{n+3}M \subset \mathcal{O}_T \otimes H (0 \le p \le 2)$, $F^3 = 0, F^{-1} = \mathcal{O}_T \otimes H$. Therefore the image of $d\Phi$ belongs to the horizontal tangential space :

$$T^h_{\Phi(t')}\mathrm{Flag}^3(H) := \oplus_{0\le p\le 2} \mathrm{Hom}\,(\mathrm{Gr}^p_{F_{t'}}, \mathrm{Gr}^{p-1}_{F_{t'}}) \subset T_{\Phi(t')}\,\mathrm{Flag}\,^3(H),$$

and the component of $d\Phi(\xi)$ for $p = 1$ is identified with

$$\mathrm{Gr}^F_1\xi \in \mathrm{Hom}\,(\mathrm{Gr}^F_{-n-1}M(t'), \mathrm{Gr}^F_{-n}M(t')).$$

This implies the injectivity of $d\Phi$ and Φ, Ψ on a neighborhood of 0, because T is a smooth subspace of a versal deformation so that $\mathrm{Der}_T \ni \xi \mapsto \xi f' \in \mathcal{O}_X / \sum_{0\le i\le n+1} (\partial f'/\partial x_i)\mathcal{O}_X$ is injective.

Note that (2.10.1) is not true, if the Brieskorn lattice $F_{-n}M(t')$ is replaced by $\mathrm{Gr}_V F_{-n}M(t')$ (i.e. the Hodge filtration F on $H^n(X_\infty, \mathbb{C})$, cf. (2.6.3)); for example $f' = x^4 + y^5 + t_2x^2y^3$, where F is constant, because it is compatible with the decomposition $H^n(X_\infty, \mathbb{C}) = \oplus H^n(X_\infty, \mathbb{C})_\lambda$ and $\dim H^n(X_\infty, \mathbb{C})_\lambda = 1$ or 0.

3. Structure of (B)-lattices.

3.1. Let the notations be as in §1. Take $M \in M_{\text{rh}}(\mathcal{D})_{\text{qu}}$ on which the action of ∂_t is bijective, cf. 1.4.2, and let M_0 be a finite $\mathcal{E}(0)$-submodule of M generating M over $\mathcal{E}$ (i.e. M_0 is a lattice of the D-module M (cf. 1.1) and stable by ∂_t^{-1}.) We define an increasing filtration F on M by $F_pM = \partial_t^p M_0$, and on M^α by

$$F_pM^\alpha = \text{Im}\,(\text{Gr}_V^\alpha F_pM \to \text{Gr}_V^\alpha M \xleftarrow{\sim} M^\alpha)$$

so that $F_pM^\alpha = \partial_t^p\ \text{Gr}_V^{\alpha+p} M_0$. Then F is a finite increasing filtration of M^α by assumption, and satisfies

$$(3.1.1) \qquad \partial_t^m\ :\ F_pM^\alpha \xrightarrow{\sim} F_{p+m}M^{\alpha-m}.$$

Let $U = \{U^p\}_{p\in\mathbf{Z}}$ be a finite decreasing filtration of M^α satisfying

$$(3.1.2) \qquad \partial_t^m\ :\ U^pM^\alpha \xrightarrow{\sim} U^{p+m}M^{\alpha-m}.$$

We say that U is *opposite* to F if

$$(3.1.3) \qquad \text{Gr}_p^F\ \text{Gr}_U^q M^\alpha = 0 \quad \text{for } p \neq q, \text{ cf. [D2]}.$$

This condition is equivalent to the splitting :

$$(3.1.14) \qquad M^\alpha = \oplus_p F_pU^pM^\alpha \text{ such that } F_pM^\alpha = \oplus_{q\le p} F_qU^qM^\alpha \text{ and } U^pM^\alpha = \oplus_{q\ge p} F_qU^qM^\alpha, \text{ cf. [loc. cit]}.$$

An opposite filtration U is called an *opposite* $(A)-$ (resp.$(B)-$) *filtration*, if

$$(3.1.5) \qquad N(U^pM^\alpha) \subset U^pM^\alpha (\text{resp.} N(U^pM^\alpha) \subset U^{p+1}M^\alpha),$$

where N is defined by $-(\partial_t t - \alpha)$ on M^α. Put

$$(3.1.6) \qquad G^\alpha = F_0U^0M^\alpha \subset M^\alpha.$$

Then (3.1.1-4) implies the decompositions :

$$(3.1.7) \qquad M^\alpha = \oplus_p \partial_t^p G^{\alpha+p},\ F_pM^\alpha = \oplus_{q\le p} \partial_t^q G^{\alpha+q}$$
$$U^pM^\alpha = \oplus_{q\ge p} \partial_t^q G^{\alpha+q},$$

and (3.1.5) is equivalent to

$$N(G^\alpha) \subset G^\alpha \oplus \partial_t G^{\alpha+1} (\text{resp.} N(G^\alpha) \subset \partial_t G^{\alpha+1}), \tag{3.1.8}$$

because $N(F_p M^\alpha) \subset F_{p+1} M^\alpha$ follows from $tM_0 \subset M_0$. We say that M_0 is an (A)-lattice (resp.a(B)-lattice), if F on each M^α has an opposite (A)-filtration (resp.(B)-filtration).

3.2. *Examples.* — 1) Let $M = \mathcal{D}/\mathcal{D}(\partial_t t - \alpha)^r (= \mathcal{E}/\mathcal{E}(\partial_t t - \alpha)^r)$ for $\alpha \in \Lambda$, cf. 1.2-4, and $\{e_i\}$ a K-basis of M such that $(\partial_t t - \alpha)e_i = e_{i+1}$ for $0 \le i < r-1$ and $(\partial_t t - \alpha)e_{r-1} = 0$, where $R = \mathbb{C}\{\{\partial_t^{-1}\}\}$ and $K = R[\partial_t]$, cf. 1.8. Put

$$M_0 = \sum R\partial_t^{-i} e_i$$
$$M_0' = \sum Re_i$$
$$M_0'' = \sum R\partial_t^{i} e_i.$$

Then M_0 is a (B)-lattice but not saturated, M_0' is a saturated (A)-lattice but not a (B)-lattice, and M_0'' is saturated but not an (A)-lattice.

2) Let $M = \oplus_{i=1,2}\mathcal{E}/\mathcal{E}(\partial_t t - \alpha_i)$ with $\{e_1, e_2\}$ a K-basis such that $(\partial_t t - \alpha_i)e_i = 0$. Then any lattice of M is a (B)-lattice (because $N = 0$) and given by

$$\sum R\partial_t^{-m_i} e_i + \mathbb{C}(\sum a_i \partial_t^{-m_i+1} e_i)$$

for $m_1, m_2 \in \mathbb{Z}, a_1, a_2 \in \mathbb{C}$ (changing $\{e_i\}$ if $\alpha_1 - \alpha_2 \in \mathbb{Z}$). In fact $u := \sum_{i=1,2;j\ge 0} a_{ij}\partial_t^{-n_i-j} e_i \in M_0$ with $a_{i0} \neq 0$ and $\alpha_1 + n_1 \neq \alpha_2 + n_2$ implies $\partial_t^{-n_i-1} e_i, \sum a_{i0}\partial_t^{-n_i} e_i \in M_0$, because we may assume $a_{1j} = 0 \quad (j > 0)$ by $RM_0 \subset M_0$, so that $(t - (\alpha_1 + n_1)\partial_t^{-1})u$ generates $R\partial_t^{-n_2-1} e_2 \subset M_0$, cf. the proof of 3.4-6.

3.3. DEFINITION. — *Let M and M_0 be as in 3.1, and*

$$\mathrm{pr} : M_0 \to \overline{M}_0 := M_0/\partial_t^{-1} M_0 (= \mathbb{C} \otimes_R M_0)$$

the natural projection. Then a $\mathbb{C}$-linear section v of pr *is called a good section, if $\mathrm{pr}_* V = v^* V$ (cf. 1.7 for the definition of V). Here $\mathrm{pr}_* V$(resp.v^*V) means the quotient* (resp. *induced) filtration, i.e.* $(\mathrm{pr}_* V)^\alpha = \mathrm{pr}\, V^\alpha$(resp. $(v^*V)^\alpha = v^{-1} V^\alpha$).

Remarks. — 1) By Nakayama's lemma, v induces an isomorphism

$$R \otimes_{\mathbf{C}} \overline{M} \xrightarrow{\sim} M(\text{as } R\text{-modules}). \tag{3.3.1}$$

2) In general, we have $(v^*V)^\alpha \subset (\mathrm{pr}_*V)^\alpha$, because $\mathrm{pr} \circ v = \mathrm{id}$.

3.4. Proposition. — *Let the notations and the assumptions be as above, and U an opposite filtration of F on $M^\alpha (\alpha \in \mathbf{Q})$. Then there exists a unique good section v of* pr *satisfying*

$$\mathrm{Im}\, v = M_0 \cap P = \oplus_\alpha M_0 \cap P^\alpha, \tag{3.4.1}$$

where $P = \sum_\alpha U^0 M^\alpha$ *and* $P^\alpha = G^\alpha + \sum_{\beta > \alpha} U^1 M^\beta$, *cf.* (3.1.6).

Proof. — We define a finite decreasing filtration $U = \{U^\alpha\}_{\alpha \in \mathbf{Q}}$ of M by

$$U^\alpha M = \mathrm{Im}(\oplus_{\beta \geq \alpha} K \otimes_{\mathbf{C}} G^\beta \hookrightarrow M)$$

and put $U^\alpha M_0 = M_0 \cap U^\alpha M$. Then we have

$$U^\alpha M_0 \subset V^\alpha M, \tag{3.4.2}$$

because $\mathrm{Gr}_V^\beta U^\alpha M_0 \subset \mathrm{Gr}_V^\beta M_0 \cap \mathrm{Gr}_V^\beta U^\alpha M \subset F_0 \mathrm{Gr}_V^\beta M \cap U^1 \mathrm{Gr}_V^\beta M = 0$ for $\beta < \alpha$, cf. (3.1.7).

We take $v_{\alpha,i} \in V^\alpha M_0$ such that

$$[v_{\alpha,i}] := v_{\alpha,i} (\mathrm{mod.} V^{>\alpha} M_0) \in \mathrm{Gr}_V^\alpha M_0 \cap \mathrm{Gr}_V^\alpha U^\alpha M \xleftarrow{\sim} G^\alpha$$

and $\{[v_{\alpha,i}]\}_i$ is a $\mathbf{C}$-basis of G^α. Because $V^\alpha M_0$ are R-submodules and R is a discreet valuation ring (cf. 1.8), we can change $v_{\alpha,i}$ (without changing $[v_{\alpha,i}]$) by induction on α so that

$$v_{\alpha,i} \in U^\alpha M_0. \tag{3.4.3}$$

Note that (3.4.3) implies

$$\{v_{\alpha,i}\}_i \text{ is an } R\text{-basis of } \mathrm{Gr}_U^\alpha M_0 = \mathrm{Gr}_U^\alpha V^\alpha M (\subset \mathrm{Gr}_U^\alpha M), \tag{3.4.4}$$

because the inclusion $\mathrm{Gr}_U^\alpha M_0 \subset \mathrm{Gr}_U^\alpha V^\alpha M$ is clear by (3.4.2) and $\{v_{\alpha,i}\}_i$ is an R-basis of $\mathrm{Gr}_U^\alpha V^\alpha M$ by $\mathrm{Gr}_U^\alpha \mathrm{Gr}_V^\alpha M \xleftarrow{\sim} G^\alpha$. Then if $v_{\beta,i}$ satisfies

(3.4.3) for $\beta < \alpha$ and $v_{\alpha,i} \in U^{\gamma}M_0$ for some $\gamma < \alpha$, we can apply (3.4.4) to γ, and change $v_{\alpha,i}$ by adding $\sum_j g_{\gamma,j}v_{\gamma,j}$ with $g_{\gamma,j} \in R$, so that $v_{\alpha,i} \in U^{>\gamma}M$ for any $\gamma < \alpha$ by induction on γ, where $g_{\gamma,j}v_{\gamma,j} \in V^{>\alpha}M$, because $\mathrm{Gr}_U^{\gamma}v_{\alpha,j} = \sum_j g_{\gamma,j}v_{\gamma,j}$ in $V^{>\alpha}\,\mathrm{Gr}_U^{\gamma}M$ so that $g_{\gamma,j} \in \partial_t^{-k}R$ with $k > \alpha - \gamma$. Using the base change matrix of the two R-bases $\{v_{\alpha,i}\}_i, \{[v_{\alpha,i}]\}_i$ of the R-module $\mathrm{Gr}_U^{\alpha}V^{\alpha}M = R \otimes_{\mathbf{C}} G^{\alpha}$, we can further change $v_{\alpha,i}$ so that

$$v_{\alpha,i} \in G^{\alpha} + U^{>\alpha}M, \text{ and then } v_{\alpha,i} \in P^{\alpha}$$

by adding $\sum_j g_{\beta,j}v_{\beta,j}$ for $g_{\beta,j} \in R(\beta > \alpha)$ by increasing induction on β.

Let

$$p_{\alpha} : M_0 \cap P^{\alpha} \to G^{\alpha}$$

be the natural projection induced by the decomposition (1.5.1). Then p_{α} is surjective by the above construction of $v_{\alpha,i}$. We can also check the injectivity using the definition of P^{α}, because $M_0 \cap \sum U^1M^{\alpha} = 0$. Therefore we get a unique section v of pr satisfying (3.4.1), because $M_0 \cap P \supset \oplus M_0 \cap P^{\alpha}$ is clear and we have the inclusion $\subset$ by

$$\mathrm{Gr}_U^{\alpha}(M_0 \cap P) \subset \mathrm{Gr}_U^{\alpha}V^{\alpha}M \cap \mathrm{Gr}_U^{\alpha}P \xleftarrow{\sim} G^{\alpha}.$$

Then v is a good section by definition. In fact, $(v^*V)^{\alpha} \subset (\mathrm{pr}_*V)^{\alpha}$ is clear and we have the isomorphisms

$$M_0 \cap P^{\alpha} \xrightarrow{\sim} \mathrm{Gr}_0^F\mathrm{Gr}_V^{\alpha}M \simeq \mathrm{Gr}_V^{\alpha}\mathrm{Gr}_0^FM \simeq \mathrm{Gr}_{\mathrm{pr}_*V}^{\alpha}\overline{M}_0$$

which implies the inclusion $\supset$.

3.5. PROPOSITION. — *Let the notations and the assumptions be as in 3.1, and v a good section of* pr *in 3.3. Let $U^{\alpha}M$ be the free K-submodule generated by $v(V^{\alpha}\overline{M}_0)$ (i.e. $K \otimes_{\mathbf{C}} V^{\alpha}\overline{M}_0 \simeq U^{\alpha}M$ as K-modules). Assume $U^{\alpha}M$ are E-submodules of M (i.e. stable by t). Then the filtration U on M^{α} defined by*

(3.5.1) $$U^pM^{\alpha} = M^{\alpha} \cap U^{\alpha+p}M$$

is an opposite (A)-filtration (cf. 3.1).

Proof. — Set $V = \mathrm{pr}_* V = v^* V$ on $\overline{M}_0$, and define

$$G^\alpha = \mathrm{Im}\,(v\ :\ \mathrm{Gr}_V^\alpha \overline{M}_0 \to \mathrm{Gr}_V^\alpha M \xleftarrow{\ \sim\ } M^\alpha).$$

Then the assumption $\mathrm{pr}_* V = v^* V$ implies that the composition

$$\mathrm{Gr}_V^\alpha \overline{M}_0 \xrightarrow{\ v\ } \mathrm{Gr}_V^\alpha M_0 \xrightarrow{\ \mathrm{pr}\ } \mathrm{Gr}_V^\alpha \overline{M}_0$$

is well-defined and the identity. Therefore we get a splitting of F on $\mathrm{Gr}_V^\alpha M$ by the direct sum decomposition

(3.5.2) $$\mathrm{Gr}_V^\alpha M = \oplus \partial_t^i G^{\alpha+i},$$

because $\mathrm{Gr}_0^F\ \mathrm{Gr}_V^\alpha M = \mathrm{Gr}_V^\alpha \mathrm{Gr}_0^F M = \mathrm{Gr}_V^\alpha \overline{M}_0$. On the other hand we have a natural isomorphism

$$U^i M^\alpha \cong (U^{\alpha+i} M)^\alpha \xrightarrow{\ \sim\ } \mathrm{Gr}_V^\alpha U^{\alpha+i} M,$$

because the $\mathcal{D}$-linear morphism $U^{\alpha+i} M \to M$ is compatible with (1.5.1) and strictly compatible with V. In particular, we get $U^i M^\alpha \supset \partial_t^i G^{\alpha+i}$, because $U^{\alpha+i} M \supset \partial_t^i v(V^{\alpha+i}\overline{M_0})$. This implies

(3.5.3) $$U^\alpha M = \mathrm{Im}\,(\oplus_{\beta \geq \alpha} K \otimes_{\mathbb{C}} G^\beta \hookrightarrow M),$$

because the injectivity of the morphism in the right hand side is clear by (3.5.2) and the both sides of (3.5.3) have the same dimension over K by $U^\alpha M = K \otimes_{\mathbb{C}} V^\alpha \overline{M}_0$. Therefore (3.5.2) gives also the splitting of U on each M^α, which implies the assertion because (3.1.2) is clear.

3.6. THEOREM. — *Let the notations and the assumptions be as in 3.1 and 3.3. Then the constructions in 3.4 and 3.5 give a one-to-one correspondence between the opposite (A)-filtrations U and the good sections v of* pr *satisfying*

(3.6.1) $$tv = vA_0 + \partial_t^{-1} v A_1 \text{ for } A_0, A_1 \in \mathrm{End}_{\mathbb{C}}(\overline{M}_0).$$

Here A_0, A_1 are uniquely determined by v, and $V^\alpha \overline{M}_0$ is the direct sum of the eigenspaces of the semisimple part of A_1 with eigenvalue $\geq \alpha$. Moreover U is an opposite (B)-filtration, iff its corresponding A_1 is semisimple.

Proof. — Let U be an opposite (A)-filtration, and P, v as in 3.4. We have $\partial_t tP \subset P$ by (3.1.5), and $\partial_t^{-1}P \subset P$ by (3.1.2). Then

$$t(M_0 \cap P) \subset M_0 \cap \partial_t^{-1}P = M_0 \cap P + \partial_t^{-1}(M_0 \cap P),$$

because $\quad M_0 \cap P \xrightarrow{\sim} M_0/\partial_t^{-1}M_0 \xleftarrow{\sim} M_0 \cap \partial_t^{-1}P/\partial_t^{-1}(M_0 \cap P)$. Therefore we get (3.6.1), and the uniqueness of A_0 and A_1 are clear. Moreover the opposite filtration constructed in (3.5.1) coincides with the original one by the proof of 3.4-5 (i.e. the two G^α coincide).

Now let v be a good section of pr satisfying (3.6.1), and put $V = \mathrm{pr}_* V = v^* V$ on $\overline{M}_0$. Then

$$A_0 V^\alpha \subset V^{\alpha+1},\ A_1 V^\alpha \subset V^\alpha, \tag{3.6.2}$$

because $tV^\alpha M_0 \subset V^{\alpha+1}M_0$, $\partial_t^{-1}V^a M = V^{\alpha+1}M$. In particular, $U^a M$ in 3.5 are $\mathcal{E}$-submodules, and we get the corresponding opposite filtration U on each M^α by (3.5.1). Let G^α, P^α be as in 3.4. We have to show

$$\mathrm{Im}\ v \subset \oplus M_0 \cap P^\alpha, \tag{3.6.3}$$

which establishes the one-to-one correspondence. Put

$$\overline{M}_0^\alpha = v^{-1}(G^\alpha + \Pi_{\beta>\alpha}(F_{-1}M^\beta + U^1 M^\beta)),$$

cf. (1.5.1), where the two G^α in 3.4 and 3.5 coincide by the proof of 3.5. Then by decreasing induction on α we check the surjectivity of the projection (induced by (1.5.1)) :

$$v(\overline{M}_0^\alpha) \to G^\alpha, \tag{3.6.4}$$

because Im $(v\ :\ V^\alpha \overline{M}_0 \to \mathrm{Gr}_V^\alpha M) = G^\alpha$ by definition. We have also the injectivity of (3.6.4) by

$$\mathrm{Im}\ v \cap \Pi_\beta(F_{-1}M^\beta + U^1 M^\beta) = 0, \tag{3.6.5}$$

which follows from the definition of G^α (taking Gr_V^α of (3.6.5)). Thus we get

$$V^\alpha \overline{M}_0 = \oplus_{\beta \geq \alpha} \overline{M}_0^\beta,$$

i.e. $\oplus \overline{M}_0^\alpha$ gives a splitting of V on $\overline{M}_0$. Let

$$c_{\beta\alpha}\ :\ G^\alpha \to M^\beta$$

be $\mathbb{C}$-linear maps such that Im $\left(\sum_\beta c_{\beta\alpha}\right) = v(\overline{M}_0^\alpha)$ and $c_{\alpha\alpha}$ is the natural inclusion. By decreasing induction on α we shall show Im $c_{\beta\alpha} \subset U^1M^\beta(\alpha < \beta)$, which is equivalent to (3.6.3). Here $c_{\beta\alpha}$ are uniquely determined by the bijectivity of (3.6.4). We denote by π_α the isomorphism

$$v : \overline{M}_0^\alpha \xrightarrow{\sim} G^\alpha(\subset \mathrm{Gr}_V^\alpha M)(\text{cf. } (3.6.4)).$$

Then we have

$$\sum_\beta c_{\beta\alpha}\pi_\alpha = v|_{\overline{M}_0^\alpha} : \overline{M}_0^\alpha \to M_0.$$

By (3.6.1-2) and by inductive hypothesis we get

$$\partial_t tv|\overline{M}_0^\alpha \equiv vA_1|\overline{M}_0^\alpha \equiv v\mathrm{Gr}_V^\alpha A_1(\mathrm{mod.} \sum_{\beta>\alpha} U^0M^\beta),$$

where $\mathrm{Gr}_V^\alpha A_1$ is identified with an endomorphism of $\overline{M}_0^\alpha$ by the isomorphism $\overline{M}_0^\alpha = \mathrm{Gr}_V^\alpha \overline{M}_0$. This implies

$$(\beta - N)c_{\beta\alpha}\pi_\alpha = c_{\beta\alpha}\pi_\alpha \ \mathrm{Gr}_V^\alpha A_1(\mathrm{mod.}\ U^0M^\beta) \text{ for } \beta > \alpha,$$

where $N = -(\partial_t t - a)$ on M^α. Therefore the assertion is reduced to

(3.6.6) the eigenvalues of $\mathrm{Gr}_V^\alpha A_1$ are α,

or equivalently

(3.6.7) $$\mathrm{Gr}_U^\alpha M_0 = V^\alpha\ \mathrm{Gr}_U^\alpha M,$$

because $U^\alpha M_0 = R \otimes_{\mathbf{C}} v(V^\alpha \overline{M}_0)$. But (3.6.7) follows from the proof of 3.5, because $\mathrm{Gr}_U^\alpha M_0 \subset V^\alpha\ \mathrm{Gr}_U^\alpha M$ and $G^\alpha \xrightarrow{\sim} \mathrm{Gr}_V^\alpha\ \mathrm{Gr}_U^\alpha M$.

To show that $\overline{M}_0^\alpha$ is the α-eigenspace of A_1, it is enough to show its stability by A_1 (cf. (3.6.6)). We have the inclusion $\partial_t tP^\alpha \subset G^\alpha + \partial_t P$, which implies

$$\begin{aligned}\partial_t t(M_0 \cap P^\alpha) &\subset \partial_t M_0 \cap (G^\alpha + \partial_t P)\\ &= M_0 \cap P^\alpha + \partial_t(M_0 \cap P),\end{aligned}$$

because $\partial_t M_0 \cap (G^\alpha + \partial_t P)/M_0 \cap P^\alpha \xrightarrow{\sim} \partial_t M_0/M_0$. Thus $\overline{M}_0^\alpha$ is stable by A_1.

Let $A_{\beta\alpha} : \overline{M}_0^\alpha \to \overline{M}_0^\beta$ be the $\mathbf{C}$-linear maps such that $\sum_\beta A_{\beta\alpha} = A_0|\overline{M}_0^\alpha$. Put $c'_{\beta\alpha} = c_{\beta\alpha}\pi_\alpha$. Then

(3.6.8) $$(\beta - N)c'_{\beta\alpha} = \sum_\gamma \partial_t c'_{\beta+1,\gamma}A_{\gamma\alpha} + c'_{\beta\alpha}(A_1|\overline{M}_0^\alpha)$$

follows from (3.6.1) for any $\beta \geq \alpha$. In particular

$$(3.6.9) \qquad -N\pi_\alpha = \partial_t \pi_{\alpha+1} A_{\alpha+1,\alpha} + \pi_\alpha (A_1 - \alpha)|\overline{M_0^\alpha}$$

for $\alpha = \beta$. This shows the last assertion (cf. the remark below) and completes the proof of 3.6.

Remark. — Let the notations be as in the proof of 3.6. Then

$$-N : G^\alpha \to G^\alpha \oplus \partial_t G^{\alpha+1}$$

and

$$(\mathrm{Gr}_V^\alpha A_1 - \alpha) \oplus \mathrm{Gr}_V^\alpha A_0 : \overline{M_0^\alpha} \to \overline{M_0^\alpha} \oplus \overline{M_0^{\alpha+1}}$$

are identified by π_α and $\partial_t \pi_{\alpha+1}$, where $\mathrm{Gr}_V^\alpha A_0 = A_{\alpha+1,\alpha}$ and $\mathrm{Gr}_V^\alpha A_1 = A_1|\overline{M_0^\alpha}$.

3.7. Proposition. — *Let M and M_0 be as in 3.1. Let W be the monodromy filtration of M (i.e. $NW_i \subset W_{i-2}$ and $\mathrm{Gr}^W N^j : \mathrm{Gr}_j^W \xrightarrow{\sim} \mathrm{Gr}_{-j}^W$ for $j > 0$). Then the following conditions are equivalent :*

a) M_0 is a (B)-lattice,
b) $N^j(F_p \mathrm{Gr}_j^W M^\alpha) = F_{p+j} \mathrm{Gr}_{-j}^W M^\alpha$ for any $j > 0, p, \alpha$,
c) $N^j : (M^\alpha, F) \to (M^\alpha, F[-j])$ are strict morphisms for any $j > 0, \alpha$,
d) (M^α, F, N) are isomorphic to direct sums of the copies of $(\mathbb{C}[N]/\mathbb{C}[N]N^m, F[p], N)$ for $p \in \mathbb{Z}, m \in \mathbb{N}$, where $F_p \mathbb{C}[N] = \oplus_{i \leq p} \mathbb{C}N^i$ and $F[p]_j = F_{j-p}$.

Proof. — The implications d) $\Longrightarrow$a) $\Longrightarrow$ c) and d) $\Longrightarrow$ b) are clear. It is enough to show b) $\Longrightarrow$ *d)* and c) $\Longrightarrow$ *d)*. Put $H = M^\alpha$.

Assume first b). Let $P\mathrm{Gr}_j^W H = \mathrm{Ker}\,(N^{j+1} : \mathrm{Gr}_j^W H \to \mathrm{Gr}_{-j-2}^W H)$ be the primitive part of $\mathrm{Gr}_j^W H$ for $j \geq 0$. By the condition b) the primitive decomposition

$$(3.7.1) \qquad \mathrm{Gr}_i^W H = \oplus_{j-2m=i} N^m P\mathrm{Gr}_j^W H$$

is compatible with F (applying N^{j-m} to (3.7.1) and using b) by decreasing induction on j). Let

$$s_j : (P\mathrm{Gr}_j^W H, F) \to (W_j H, F)$$

be a filtered $\mathbb{C}$-linear map whose composition with the canonical projection $W_j H \to \mathrm{Gr}_j^W H$ is the natural inclusion. (Such a map exists using a basis of $P\mathrm{Gr}_j^W H$ compatible with F.) By assumption $\mathrm{Im} N^{j+1}s_j \subset W_{-j-3}H$ and we can modify s_j so that $\mathrm{Im}\, N^{j+1}s_j \subset W_{-j-a}H$ for any $a \geq 4$ by induction on a. In fact we use the condition b) for $j+a-1$ and the factorization $N^{j+a-1} = N^{j+1}N^{a-2}$, where $N^{a-2}W_{j+a-1} \subset W_{j-1}$ follows from $a \geq 4$. Thus we get s_j such that $N^{j+1}s_j = 0$, and these s_j give a lifting of (3.7.1) to H, which implies d).

Assume now c). Put $K_j = \mathrm{Ker}\, N^j$ and $I^i = \mathrm{Im}\, N^i$.Then we have isomorphisms

$$N : \mathrm{Gr}_I^i \mathrm{Gr}_j^K H \xrightarrow{\sim} \mathrm{Gr}_I^{i+1} \mathrm{Gr}_{j-1}^K H \text{ for } j \geq 2,$$

because d) holds always if we forget F. In particular we get the isomorphisms

$$N^j : \mathrm{Gr}_I^0 \mathrm{Gr}_{j+1}^K \xrightarrow{\sim} \mathrm{Gr}_I^j K_1 H \text{ for } j \geq 0. \tag{3.7.2}$$

By the condition c), (3.7.2) induces the isomorphisms

$$N^j : F_{p-j}\, \mathrm{Gr}_I^0 \mathrm{Gr}_{j+1}^K H \xrightarrow{\sim} F_p \mathrm{Gr}_I^j K_1 H,$$

because $(N^j)^{-1}(K_1 H) = K_{j+1}H$ and $I^0 H = H$. Then we have the isomorphisms

$$F_{p-j}\mathrm{Gr}_I^0 \mathrm{Gr}_{j+1}^K H \xrightarrow[\sim]{N^{j-i}} F_{p-i}\mathrm{Gr}_I^{j-i}\mathrm{Gr}_{i+1}^K H \xrightarrow[\sim]{N^i} F_p \mathrm{Gr}_I^i K_1 H \tag{3.7.3}$$

for $0 \leq i \leq j$, because these maps are injective (forgetting F). Let

$$s_j : (\mathrm{Gr}_I^j K_1 H, F) \to (K_{j+1}H, F[j])$$

be a filtered $\mathbf{C}$-linear map whose composition with the natural projection $K_{j+1}H \to \mathrm{Gr}_I^0 \mathrm{Gr}_{j+1}^K H$ is the inverse of (3.7.2). Then these s_j induce the filtered isomorphisms for $i \geq 0$:

$$\sum_{j \geq i} N^{j-i}s_j : \oplus_{j\geq i}(\mathrm{Gr}_I^j K_1 H, F[-i]) \xrightarrow{\sim} (\mathrm{Gr}_{i+1}^K H, F)$$

by (3.7.3). Therefore we get

$$\sum_{j\geq i\geq 0} N^{j-i}s_j : \oplus_{j\geq i\geq 0}(\mathrm{Gr}_I^j K_1 H, F[-i]) \xrightarrow{\sim} (H, F)$$

which implies d).

3.8 *Remark.* — In the case of Brieskorn lattice (cf. 2.6), the conditions b) and c) are satisfied by (2.6.3). By the proof of 3.6, $\mathrm{Gr}_V A_0 = \oplus A_{\alpha+1,\alpha}$ is identified with $-N$ by the isomorphisms π_α if we take a good section corresponding to an opposite (B)-filtration, cf. the remark after the proof of 3.6.

The differential equation (3.6.1) is uniquely determined by the eigenspace decomposition by A_1, because A_0 represents the action of f on $\Omega_f := \Omega^{n+1}_{X,0}/df \wedge \Omega^n_{X,0} (= M_0/\partial_t^{-1}M_0)$.

Problem. — Find the characterization of the direct sum decomposition of Ω_f corresponding to an opposite (A)-or (B)-filtration.

3.9. *Remark.* — If M is not quasi-unipotent, we have to choose a total order $\succ$ of $\mathbf{C}$ satisfying $a+1 \succ a$, $a \succ b$ iff $a+1 \succ b+1$, and there exists an integer m such that $a+m \succ b$, for any $a, b \in \mathbf{C}$. It is equivalent to a choice of $\Lambda \subset \mathbf{C}$ and a total order of Λ such that $\Lambda \to \mathbf{C} \to \mathbf{C}/\mathbf{Z}$ is bijective, $\Lambda \ni 1$ and $\sup\Lambda = 1$. (Put $\Lambda = \{a \in \mathbf{C} | 0 \underset{\prec}{\neq} a \prec 1\}$.) Then the same argument as in 3.4-6 applies. Here the restriction of the order to $\mathbf{R}$ may be different from the usual one. This will be used in 4.4 for the construction of a pathological example.

3.10. Let G be the splitting of the Hodge filtration F on the Milnor cohomology constructed in 2.8. Then it gives an opposite (B)-filtration U by the isomorphism (1.6.1). Let $G^\alpha \subset M^\alpha, c_{\beta\alpha}$ and $\overline{M_0^\alpha}$ be as in 3.4-6 (i.e. $v(\overline{M_0^\alpha}) = \mathrm{Im} \sum_\beta c_{\beta\alpha}$). Then by the identification of the polarization $\mathbf{S}$ in 2.7, we have

$$(3.10.1)\qquad \begin{aligned} &\mathbf{S}(G^\alpha, G^\beta) = 0 \quad \text{for } \alpha+\beta \neq n+1, \\ &\mathbf{S} : G^\alpha \otimes G^{n+1-\alpha} \to \mathbf{C} \otimes \partial_t^{-n-1} \text{ is a perfect pairing .} \end{aligned}$$

Since $M = \oplus_\alpha K \otimes_{\mathbf{C}} G^\alpha$ as K-modules and $\mathrm{Im}\, c_{\beta\alpha} \subset \sum_\gamma \mathbf{C}[\partial_t^{-1}]\partial_t^{-1}G^\gamma$ for $\beta > \alpha$, we have

$$(3.10.2)\qquad \mathbf{S}\left(\sum_\gamma c_{\gamma\alpha}(u), \sum_\delta c_{\delta\beta}(v)\right) = \mathbf{S}(u,v) \ \text{ for } u \in G^\alpha, v \in G^\beta,$$

because $\mathbf{S}(M_0, M_0) \subset F_{-1-n}K = \mathbf{C}\{\{\partial_t^{-1}\}\}\partial_t^{-n-1}$ by (2.7.10). (This gives some relation between $c_{\beta\alpha}$'s, cf. 4.1 for a special case.) In particular the direct sum decomposition $\mathrm{Im}\ v = \oplus v(\overline{M_0^\alpha})$ is orthogonal for $\mathbf{S}$, and $\mathbf{S}(\mathrm{Im}\ v,\ \mathrm{Im}\ v) \in \mathbf{C}\partial_t^{-n-1}$.

Remark. — Assume n is even (i.e.the duality $\mathbf{S}$ on $H^n(X_\infty, \mathbf{Q})_1$ is anti-symmetric), and $H^n(X_\infty, \mathbf{Q})_1$ contains $H' \oplus H''$ as a direct factor of mixed Hodge structure compatible with $\mathbf{S}$ and N, where $H' = \oplus_{0\le i\le 3} H'_i, H'' = \oplus_{1\le i\le 2} H''_i, H'_i = H''_i = \mathbf{Q}(-i-(n-2)/2), NH'_i = H'_{i-1} (i > 0), NH''_2 = H''_1, NH'_0 = NH''_1 = 0$, and $\mathbf{S}(H', H'') = 0$. Then $\mathbf{S}(H'_i, H'_j) \ne 0$ iff $i+j = 3$ (same for H''), because $H'_i = F^p W_{2p} H'$ for $p = i + (n-2)/2$.

In this case we have an (A)-(but not (B)-) opposite filtration U of F on $H' \oplus H''$ (hence on $H^n(X_\infty, \mathbf{Q})$) compatible with $\mathbf{S}$. For example the splitting $F^p U^{-p}$ for $p = i + (n-2)/2$ is generated by $e_3 (i = 3), e_2$ and $f_2 + e_3 (i = 2), e_1$ and $f_1 (i = 1)$ and $e_0 - f_1 (i = 0)$, where e_i and f_i are a generator of H'_i and H''_i such that $\mathbf{S}(e_0, e_3) = \mathbf{S}(f_1, f_2)$. By the same argument as above, the corresponding section v satisfies $\mathbf{S}(\mathrm{Im}\ v,\ \mathrm{Im}\ v) \subset \mathbf{C}\partial_t^{-n-1}$.

It is very probable that this case actually occurs (e.g. $f = x^{10} + y^{10} + z^{10} + w^{10} + (xyzw)^2 + v^2$).

4. Applications.

4.1. *b-function.* Let $M, M_0 = F_{-n}M, \widetilde{M_0}$ and $b(s)$ be as in the introduction. We have $t\widetilde{M_0} = \partial_t^{-1}\widetilde{M_0}$, because $\widetilde{M_0} \subset V^{>0}M$ by [K1][M1] and $\partial_t t$ is bijective on $\widetilde{M_0}$. Let U be an opposite (B)-filtration to the Hodge filtration F (cf. 3.1 and 3.8) and v the corresponding good section in 3.4 (or 3.6). Let G^α and $c_{\beta\alpha} : G^\alpha \to M^\beta$ be as in the proof of 3.6, i.e. $\oplus_\alpha \mathrm{Im} \sum_\beta c_{\beta\alpha} = \mathrm{Im}\ v$. Here $G^\alpha = 0$ except for $0 < \alpha < n+1$ by (2.6.3).

Put $\widetilde{M_0^\alpha} = \widetilde{M_0} \cap M^\alpha$. Then

(4.1.1) $\widetilde{M_0} = \widehat{\oplus}\widetilde{M_0^\alpha}$, i.e. $\widetilde{M_0} \subset M$ is compatible with the decomposition (1.5.1), and $\widetilde{M_0^\beta} = \sum_{i,j\ge 0, \alpha\le\beta-j} \mathrm{Im}\ N^i \partial_t^{-j} c_{\beta-j,\alpha}$,

because $\widetilde{M_0}$ is stable by $\partial_t t$ and generated by $\mathrm{Im} \sum_\beta c_{\beta\alpha}$ over $\mathbf{C}\{\{\partial_t^{-1}\}\}$ (use the theory of Jordan decomposition). Let $\beta_1, \ldots, \beta_\mu$ be the eigenvalues of $\partial_t t$ on $\widetilde{M}_0/\partial_t^{-1}\widetilde{M}_0$. Then (4.1.1) implies

$$\#\{i|\beta_i = \beta\} = \dim \widetilde{M}_0^\beta/\partial_t^{-1}\widetilde{M}_0^{\beta-1}, \tag{4.1.2}$$

$$\min \alpha_i = \min \beta_i > 0, \max \beta_i \leq \max \alpha_i < n+1, \tag{4.1.3}$$

where $\alpha_1, \ldots, \alpha_\mu$ are the exponents of f, i.e. $\#\{i|\alpha_i = \alpha\} = \dim G^\alpha = \dim \Omega_f^\alpha$. Let F be the filtration on M^α as in 3.1, i.e. $F_pM^\alpha = \oplus_{i\leq p}\partial_t^i G^{\alpha+i}$. We define F' on M^α by

$$F'_pM^\alpha = \partial_t^p \widetilde{M}_0^{\alpha+p}.$$

Then $F_p \subset F'_p$ by (4.1.1) and $\dim \mathrm{Gr}_p^F \mathrm{Gr}_q^{F'} M^\alpha$ is the number of i such that $\alpha_i = \alpha + p$ and $\beta_i = \alpha + q$, if we index $\{\alpha_i\}$ and $\{\beta_i\}$ properly. We have also $\dim_{\mathbb{C}} \widetilde{M}_0/M_0 = \sum \beta_i - \sum \alpha_i$.

From now on (in 4.1) we *assume* $n = 1$. (A similar argument holds if $\min \alpha_i > (n-1)/2$.) Then we have by (4.1.3) :

$$\mathrm{Im}\ c_{\beta\alpha} \subset \partial_t G^{\beta+1} \text{ for } \beta > \alpha.$$

In particular $c_{\beta\alpha} = 0$ for $\beta \geq 1$ and $\beta > \alpha$. This implies

$$\widetilde{M}_0^\beta = \begin{cases} G^\beta + NG^\beta + \sum_{\alpha<\beta} \mathrm{Im}\ c_{\beta\alpha} & \text{for } 0 < \beta < 1 \\ M^\beta & \text{for } 1 \leq \beta < 2. \end{cases} \tag{4.1.3}$$

Therefore $\widetilde{M}_0^\beta$ is compatible with the decomposition $M^\beta = G^\beta \oplus \partial_t G^{\beta+1}$ for $0 < \beta < 1$. Now we assume that the splitting of Hodge filtration is *orthogonal* with respect to the duality (cf. 2.8). Then by (3.10.2) we have

(4.1.4) $\partial_t^{-1}c_{\beta-1,\alpha} : G^\alpha \to G^\beta$ and $\partial_t^{-1}c_{1-\alpha,2-\beta} : G^{2-\beta} \to G^{2-\alpha}$ are dual of each other by the duality $\mathbf{S}$ for $\beta > \alpha + 1$.

We have the same for $\partial_t^{-1}N : G^\alpha \to G^{\alpha+1}$ and $\partial_t^{-1}N : G^{1-\alpha} \to G^{2-\alpha}$, and we conclude that

$$\begin{aligned} \dim(\widetilde{M}_0^{\beta-1} \cap \partial_t G^\beta) &= \dim \mathrm{Im}((N, c_{\beta-1,\alpha}) : \oplus_{\alpha\leq\beta-1}G^\alpha \to \partial_t G^\beta) \\ &= \dim \mathrm{Im}((N, c_{1-\alpha,2-\beta}) : G^{2-\beta} \to \oplus_{\alpha\leq\beta-1}\partial_t G^{2-\alpha}). \end{aligned} \tag{4.1.5}$$

On the other hand we have

$$\mathrm{Ker}\ f \cap \Omega_f^\alpha = \mathrm{Ker}\ (N, c_{\beta\alpha}) : G^\alpha \to \oplus_{\beta\geq\alpha}\partial_t G^{\beta+1}. \tag{4.1.6}$$

In fact the multiplication of f on Ω_f is identified with A_0, i.e. the action of t on $\overline{M}_0$ (cf. 3.6), and

$$(\partial_t t - \alpha)v(u) = \partial_t v(A_0 u) \quad \text{for } u \in \Omega_f^\alpha (= \overline{M}_0^\alpha) \tag{4.1.7}$$

by (3.6.1). Since Im $\sum_\beta c_{\beta\alpha} = v(\Omega_f^\alpha)$, we have

$$\text{Ker } f \cap \Omega_f^\alpha = \text{Ker } (\partial_t t - \alpha \ : \ \text{Im } \Big(\sum_\beta c_{\beta\alpha}\Big) \to M).$$

Then (4.1.6) follows from $\partial_t t - \beta = 0$ on Im $c_{\beta\alpha}$ for $\beta > \alpha$ (in fact, Im $Nc_{\beta\alpha} \subset \partial_t^2 G^{\beta+2} = 0$), and it implies

(4.1.8)

$$\text{Ker } f \cap \Omega_f^\alpha = \Omega_f^\alpha \qquad \text{for } 1 \le \alpha < 2,$$

$$\dim (\widetilde{M}_0^{\beta-1} \cap \partial_t G^\beta) = \dim \Omega_f^{2-\beta}/\text{Ker } f \cap \Omega_f^{2-\beta} \qquad \text{for } 1 < \beta < 2.$$

Here the first assertion follows also from (3.6.2). Therefore we get the assertion in the introduction, because $b(s)(s+1)^{-1}\Pi_{\alpha\in\Delta}(s+\alpha)^{-1}$ is the product of $(s+\alpha)$ such that $\widetilde{M}_0^\alpha/\partial_t^{-1}\widetilde{M}_0^{\alpha-1} \ne 0$, where Δ is as in the introduction. (If $n > 1$ and min $\alpha_i > (n-1)/2$, we replace the condition $0 < \alpha < 1$ by $(n-1)/2 < \alpha < (n+1)/2$ in the definition of Δ.)

Remark. — Assume $n = 1$, the monodromy is semisimple, and $\alpha_i \ne \alpha_j (\text{mod}\mathbf{Z})$ for $i \ne j$, i.e. dim $M^\alpha = 1$ or 0. The last condition implies that v is unique, because $G^\alpha = M^\alpha$ or 0. Let u_i be a generator of G^{α_i} for $1 \le i \le \mu$, where we assume $\alpha_i < \alpha_j$ for $i < j$ and $\mathbf{S}(u_i, u_{\mu+1-i}) = \partial_t^{-2}$. Put

$$v_i = u_i + \sum_j c_{\alpha_j-1,\alpha_i} u_i = u_i + \sum_j c_{ji}\partial_t u_j = v(u_i),$$

where $c_{ji} = 0$ if $\alpha_j \le \alpha_i + 1$, and $v_i = u_i$ if $\alpha_i \ge 1$. Therefore we get

$$\text{(4.1.9)} \qquad \text{pr } (tv_i) = \sum_j (\alpha_j - \alpha_i - 1)c_{ji}\text{pr } (v_j).$$

On the other hand, $\beta_j = \alpha_j - 1$ iff $\widetilde{c}_{ji} := (\alpha_j - \alpha_i - 1)c_{ji} \ne 0$ for some i, where β_j's are uniquely indexed by the condition $\beta_j = \alpha_j (\text{mod } \mathbf{Z})$. Since u_i's are orthogonal with respect to the duality, i.e. $\mathbf{S}(u_i, u_j) \ne 0$ iff $i + j = \mu + 1$, we have the same for v_i's, and we get by (4.1.4) :

(4.1.10)

$$\beta_j = \alpha_j - 1 \text{ iff } \widetilde{c}_{\mu+1-i,\mu+1-j} \ne 0 \text{ for some } i, \text{ i.e. pr}(v_{\mu+1-j}) \notin \text{Ker } f.$$

This is equivalent to the assertion in the introduction in this case. In general, $A_0 (= (\widetilde{c}_{ji})$ in this case) is symmetric with respect to the duality $\mathbf{S}$ (i.e. $\widetilde{c}_{ji} = \widetilde{c}_{\mu+1-i,\mu+1-j}$ in this case), because $\mathbf{S}$ induces Grothendieck's residue pairing on $\Omega_f = \text{Gr}_{-n}^F M$ by (2.7.10), cf. [O][Sk].

4.2. *b-function and μ-constant deformation*. — Let $\{f_{t'}\}_{t' \in T}$ be a μ-constant deformation of a holomorphic function with isolated singularity, and (M, F) the associated Gauss-Manin system on $S \times T$, where S and T are as in 2.9. Then the condition in 2.9 : M is trivial along T, is satisfied, i.e. the Gauss-Manin systems for $f_{t'}$ are canonically identified by the parallel translation in 2.9 and the Brieskorn lattices $M_0(t')$ vary holomorphically. Let U be an opposite filtration for f_0. Then it remains to be an opposite filtration for $t' \in T$ sufficiently near 0 by the above triviality, because (2.9.1) is satisfied (cf. 2.10). We can also check that G^α and $c_{\beta\alpha}$ vary holomorphically with respect to t'. In particular we get a holomorphic stratification of T (by restricting T if necessary) such that the b-function and $\beta_{j'}$s are constant on each stratum, because these invariants are determined by the rank of the morphisms of vector spaces $\sum N^i \partial_t^{-j} c_{\beta-j,\alpha}$ in (4.1.1) which depend on the parameter holomorphically. Using a desingularization we get a similar statification of the μ-constant stratum of the versal deformation by decreasing induction on the dimension of strata, because these invariants are constant on a Zariski open subset. If the μ-constant stratum is *irreductible*, the b-function on its open stratum is called the b-function of the generic μ-constant deformation of $f \ := f_0$. Note that on the open stratum dim $\widetilde{M}_0^\alpha$ and dim $\widetilde{M}_0/M_0$ become maximal. We shall determine later the b-function of generic μ-constant deformation of a quasi-homogeneous polynomial of two variables. First we shall review the case of quasi-homogeneous polynomial, and show how to calculate b-function, etc. of its μ-constant deformation.

Let f be a quasi-homogeneous polynomial of $n+1$ variables with weight $(w_0, \dots, w_n)$, i.e. f is a linear combination of monomials x^ν for $\nu \in \mathbf{N}^{n+1}$ such that $\sum_i w_i \nu_i = 1$, where w_i's are rational numbers. We assume f has an isolate singularity at 0. For $g(x)dx \in M_0$ such that $g(x)$ is a monomial $x^{\nu-1} = \prod_i x_i^{\nu_i - 1}$ we have by Brieskorn (cf. also (4.2.4)) :

$$\left(\partial_t t - \sum_i w_i \nu_i\right) g(x)dx = 0 \quad \text{in } M, \tag{4.2.1}$$

i.e. $g(x)dx \in M^{\alpha(\nu)}$ with $\alpha(\nu) = \sum_i w_i \nu_i$. Let $g_i(x) (1 \le i \le \mu)$ be monomials such that $\{g_i(x)dx\}$ is a $\mathbf{C}$-basis of Ω_f. Then $\{g_i(x)dx\}$ is a basis of M_0 over $\mathbb{C}\{\{\partial_t^{-1}\}\}$ and over $\mathbb{C}\{t\}$. In particular we get

$$\sum_i T^{\alpha_i} = \prod_j (T^{w_j} - T)(1 - T^{w_j})^{-1} \tag{4.2.2}$$

using the morphism $\partial f \ : \ \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, as is well-known. By [V4] the μ-constant stratum is smooth, and any (sufficiently small) μ-constant deformation is given by

$$f' = f + \sum_{\alpha_i \geq \alpha_1 + 1} t'_i g_i(x), \tag{4.2.3}$$

where $\alpha_i = \alpha(\nu_i)$ for $g_i(x) = x^{\nu_i - 1}$ and $\alpha_1 = \sum w_i$, i.e. $g_1(x) = 1$. From now on we assume $t'_i = 0$ for $\alpha_i = \alpha_1 + 1$ by replacing f by $f + \sum_{\alpha_i = \alpha_1 + 1} t'_i g_i(x)$. For $\nu \in \mathbb{N}^{n+1}$, put $u^\nu(\text{resp.} v^\nu) = x^{\nu - 1} dx$ in M the Gauss-Manin system for $f(\text{resp.} f')$ for $\nu \in \mathbb{Z}_+^{n+1}$, where $1 = (1, \dots, 1) \in \mathbb{N}^{n+1}$. Here we identify the two Gauss-Manin systems for f and f' by the parallel translation in 2.9, and u^ν also denotes its parallel translate. Then we have $u^\nu, v^\nu \in V^{\alpha(\nu)}M$ and $\mathrm{Gr}_V^{\alpha(\nu)} v^\nu = u^\nu$ in $M^{\alpha(\nu)}$, because $f = f'$ modulo $V^{>1}\mathcal{O}_X$, cf. [S7]. To calculate $c_{\beta\alpha}$ (hence $\beta'_i s$ and $b(s)$) for f', it is enough to express v^ν's by a linear combination of $\partial_t^{-j} u^\nu$'s modulo $V^n M$. Here the monodromy is semi-simple (i.e. $N = 0$) and the choice of g_i's gives the section v such that G^α in 3.4-6 is generated by $\mathrm{Gr}_V^\alpha(g_i(x)dx)$ with $\alpha_i = \alpha$. We first check

$$\begin{aligned} (4.2.4) \qquad \alpha(\nu)\partial_t^{-1} v^\nu &= \sum w_i x_i (\partial f'/\partial x_i) x^{\nu - 1} dx \\ &= (f + \sum (\alpha_i - \alpha_1) t'_i g_i(x)) x^{\nu - 1} dx, \\ \text{i.e.} \qquad (\partial_t t - \alpha(\nu)) v^\nu &= \sum \partial_t (1 + \alpha_1 - \alpha_i) t'_i g_i(x) x^{\nu - 1} dx) \end{aligned}$$

in M. Therefore $v^\nu - u^\nu$ is determined by $\partial_t v^{\nu'}$ such that $\alpha(\nu') > \alpha(\nu) + 1$, and we can calculate $\partial_t^j v^\nu$ modulo $V^n M$ by induction on $\alpha(\nu) - j$. (For the determination of $c_{\beta\alpha}$, we have to repeat the argument in the proof of 3.4.)

From now on (in 4.2) we assume $n = 1$. Then we have $\widetilde{M}_0^\alpha \supset G^\alpha$ and $\widetilde{M}_0^\alpha = G^\alpha \oplus \widetilde{M}_0^\alpha \cap \partial_t G^{\alpha+1} (0 < \alpha < 1)$ and M^α (otherwise), cf. 4.1 for the notation. The claim is that for f' a generic μ-constant deformation of a quasi-homogeneous polynomial f and for $\alpha < 1$, we have

$$\text{(4.2.5)} \qquad \dim \widetilde{M}_0^\alpha \cap \partial_t G^{\alpha+1} = \min (\dim \Omega_f^{\alpha+1}, \sum_{\beta<\alpha} \dim \Omega_f^\beta),$$

where $G^\alpha = \Omega_f^\alpha (= \overline{M}_0^\alpha)$ by 3.4-6. As a corollary we get

(4.2.6) $(s+1)^{-1}b(s)$ is the product of $(s+\alpha)$ such that $\alpha = \alpha_i - 1$ with $\alpha_i > \alpha_1 + 1$ or $\alpha = \alpha_i$ with $\#\{j : \alpha_j < \alpha_i - 1\} < \#\{j : \alpha_j = \alpha_i\}$,

where $b(s)$ is the b-function of generic deformation of f, and $\alpha_1, \dots, \alpha_\mu$ are the exponents of f such that α_1 is the minimal one. (Here $(s+1)^{-1}b(s)$ is reduced, because $N = 0$.)

For the proof of (4.2.5) we need some construction. By the condition of isolated singularity, f must contain a monomial x^m or $x^m y$, and y^n or xy^n. Therefore the weight of f is either $(m^{-1}, n^{-1}), (m^{-1}, (m-1)m^{-1}n^{-1})$ or $((n-1)(mn-1)^{-1}, (m-1)(mn-1)^{-1})$, i.e. f contains $x^m + y^n, x^m + xy^n$ and $x^m y + xy^n$ respectively. In these three cases, we define $\Gamma \subset \mathbb{C}[x,y]$ to be the subspace generated by $x^i y^j$ such that $0 \le i \le m-2$, $0 \le j \le n-2$; $0 \le i \le m-1$, $0 \le j \le n-2$ or $(i,j) = (0, n-1)$; $0 \le i \le m-1$, $0 \le j \le n-1$ respectively. We denote by Γ^α the subspace of Γ generated by $x^i y^j$ such that$(i+1)w_0 + (j+1)x_1 = \alpha$. Then for f generic the natural map $\Gamma \to \Omega_f$ defined by $g \to g dx \wedge dy$ is an isomorphism, and hence induces $\Gamma^\alpha \xrightarrow{\sim} \Omega_f^\alpha$ (i.e. we have chosen g_i's).

We prove the assertion (4.2.5) in the case $\sum_{\beta<\alpha} \dim \Gamma^\beta \le \dim \Gamma^{\alpha+1}$. The argument is same in the other case if we restrict to a good subspace of $\oplus_{\beta<\alpha}\Gamma^\beta$ whose dimension is same as $\Gamma^{\alpha+1}$. We take monomials h_γ for $0 < \gamma \le \alpha - \alpha_1$ such that $h_\gamma \Gamma^{\alpha-\gamma} \subset \Gamma^{\alpha+1}$ (in particular $\alpha(h_\gamma) - \alpha_1 = \alpha + 1 - \alpha + \gamma = \gamma + 1 > 1$) and $\oplus h_\gamma \Gamma^{\alpha-\gamma} \to \Gamma^{\alpha+1}$ is injective. Put

$$f' = f + \sum_{0<\gamma\le\alpha-\alpha_1} t_\gamma h_\gamma \quad \text{for } t_\gamma \in \mathbb{C} \text{ generic.}$$

Then it is enough to show the equality (4.2.5) for this f', because the inequality $\le$ is clear and $\dim \widetilde{M}_0^\alpha \cap \partial_t G^{\alpha+1}$ is maximal on the open stratum. Here we may assume that f is generic and the morphism $\Gamma \to \Omega_f$ is bijective. For $g \in \Gamma^\beta (\beta < \alpha)$, we have

$$(\partial_t t - \beta)(g dx \wedge dy) = \sum_{0<\gamma\le\alpha-\alpha_1} -\partial_t \gamma t_\gamma (h_\gamma g ds \wedge dy)$$

in M by (4.2.4), where the right hand side modulo $V^{>\alpha}M$ is independent of t_γ for $\gamma > \alpha - \beta$ and $\partial_t(h_\gamma g dx \wedge dy)$ modulo $V^{>\alpha}M$ for $0 < \gamma \leq \alpha - \beta$ is independent of $t_{\alpha-\beta}$. In fact with the notation of (4.2.4) we can check that v^ν modulo $V^{>\alpha}M$ is independent of t_γ for any α, γ, ν such that $\alpha(\nu) + \gamma >$ α by induction on $\alpha - \alpha(\nu)$, where the case $\alpha = \alpha(\nu)$, i.e. $\mathrm{Gr}_V^{\alpha(\nu)} v^\nu = u^\nu$ is independent of any t_γ, was shown before (4.2.4). Therefore by the proof of 3.4, we get a decomposition $c_{\alpha,\alpha-\gamma} = c'_{\alpha,\alpha-\gamma} + c''_{\alpha,\alpha-\gamma}$ such that $c'_{\alpha,\alpha-\gamma}$ depends at most on t_δ for $0 < \delta < \gamma$ and $c''_{\alpha,\alpha-\gamma}$ is identified with $t_\gamma h_\gamma : \Gamma^{\alpha-\gamma} \to \Gamma^{\alpha+1}$ by the isomorphism $\Gamma^\alpha = \Omega_f^\alpha = G^\alpha$ up to a non-zero constant multiple. We choose t_γ inductively so that the composition

$$\oplus c_{\alpha,\alpha-\gamma} \; : \; \oplus_{\gamma \leq \gamma'} G^{\alpha-\gamma} \to \partial_t G^{\alpha+1} = \Gamma^{\alpha+1} \twoheadrightarrow \Gamma^{\alpha+1} / \oplus_{\delta > \gamma'} h_\delta \Gamma^{\alpha-\delta}$$

is injective. Then this completes the proof of (4.2.5).

Remark. — In the case $\alpha_i \neq \alpha_j (i \neq j)$, e.g. $f = x^a + y^b$ with $(a, b) = 1$, the assertions (and the proof of) (4.2.5) and (4.2.6) become very simple (because $\dim \Omega_f^\alpha = 1$ or 0), and they are proved by Cassou-Noguès in the case $f = x^a + y^b$ with $(a, b) = 1$ using another method cf. [CN1].

4.3. *Primitive forms.* With the notations and the assumptions of 2.5, put $(M, F) = \left(\int_p \mathcal{C}_{X|Z}, F \right)_\eta$, i.e. (M, F) is the germ of the filtered microlocal Gauss-Manin system associated to a deformation of a holomorphic function with an isolated singularity. First we assume $Y = S$, i.e. the base space is one dimensional. By 3.6 and 3.8, we have a basis $\{v_j\}$ of the Brieskorn lattice $F_{-n}M$ over $\mathbb{C}\{\{\partial_t^{-1}\}\}$ satisfying

(4.3.1) $$tv = A_0 v + A_1 \partial_t^{-1} v,$$

where $v = {}^t(v_1, \ldots, v_\mu)$ and A_0, A_1 are $\mu \times \mu$ matrices. Here we may assume furthermore that A_1 is *semi-simple*, i.e.

(4.3.2) $$A_1 \text{ is a diagonal matrix } (\alpha_j),$$

by changing v_j's. We can also show that $\{v_j\}$ is a basis over $\mathbb{C}\{t\}$, because $\mathrm{Gr}_U^\alpha M_0 = V^\alpha \mathrm{Gr}_U^\alpha M$, cf. (3.4.3). By 3.10 we may assume furthermore :

(4.3.3) $$\mathbf{S}(v_i, v_j) \in \mathbb{C}\partial_t^{-n-1}.$$

Here note that the eigenvalues $\alpha_1, \ldots, \alpha_\mu$ of A_1 coincide with the usual exponents, i.e. $\#\{i \; : \; \alpha_i = \alpha\} = \dim \mathrm{Gr}_V^\alpha \Omega_f$.

But in some cases, there exists a basis $\{v_j\}$ of M_0 over $\mathbf{C}\{\{\partial_t^{-1}\}\}$ and over $\mathbf{C}\{t\}$, which satisfies (4.3.1-3), but whose eigenvalues α_i's are different from the usual exponents, cf. the example below.

Corresponding to a basis satisfying (4.3.1-3), we can show the existence of a primitive form in the sense of K. Saito [Sk][O] using a result of Malgrange.

We now assume that (M, F) is associated to a versal deformation, i.e. $\dim Y = \mu$. Let $(M(0), F)$ be its restriction to $S \times \{0\} \subset Y = S \times T$, i.e. $F_pM(0) = \mathcal{O}_S \otimes_{\mathcal{O}_Y} F_pM$. Let $\{v_j\}$ be a basis of $F_{-n}M(0)$ over $\mathbf{C}\{\{\partial_t^{-1}\}\}$ satisfying (4.3.1-3). According to Malgrange, the theorem (5.9) in [M3] is true without the hypothesis on the monodromy $(\mathbf{Z}^*)$, cf. [M4]. Therefore a basis satisfying (4.3.1) is uniquely extended to a basis of $F_{-n}M$ over $R = \mathcal{O}_T\{\{\partial_t^{-1}\}\}$ (cf. (2.7.1)), denoted also by $\{v_j\}$, such that

$$(4.3.4)\qquad \begin{aligned} t_1 v &= A_0(t')v + A_1\partial_1^{-1}v,\\ \partial_1^{-1}\partial_i v &= B_i(t')v \end{aligned}$$

where $A_0(t'), B_i(t')$ are $\mu \times \mu$ matrices whose coefficients are holomorphic functions of t' such that $A_0(0) = A_0$ (and A_1 is same as in (4.3.1)). Here $t = t_1$ is the coordinate of S, $t' = (t_2, \dots, t_\mu)$ is the coordinate system of T, and $\partial_i = \partial/\partial t_i$. Then we can easily check (4.3.3) for the extended v_j's using (4.3.3) for the restricted v_j's and (4.3.4), (2.7.8-10). In fact we have

$$\partial_k \mathbf{S}(v_i, v_j) = -\partial_1\mathbf{S}(\sum b_{i\ell}^k v_\ell, v_j) + \partial_1\mathbf{S}(v_i, \sum b_{jm}^k v_m)$$
$$(\partial_2^{\nu_2}\dots\partial_\mu^{\nu_\mu}\mathbf{S}(v_i, v_j))|_{t'=0} \in \mathbf{C}[\partial_1]\partial_1^{-n} \text{ for } \nu \neq 0,$$

where $B_k(t') = (b_{ij}^k)$, and the assertion follows from

$$\mathbf{S}(F_{-n}M, F_{-n}M) \subset F_{-n-1}K = R\partial_1^{-n-1}, \text{ cf. (2.7.10).}$$

By (4.3.2) there exists some j such that pr (v_j) generates $\mathrm{Gr}_{-n}^F M = (\Omega_{X/T}^{n+1}/df' \wedge \Omega_{X/T}^n)_0$ over $\mathcal{O}_{X,0}$. (By Nakayama's lemma it is enough to check this condition for the restriction to $t' = 0$.) Then $\zeta = \partial_1^{-1}v_j$ satisfies the conditions of primitive form. In fact it remains to check the condition on the Euler operator E, but this follows immediately from $A_1v_j = \alpha_j v_j$, i.e. v_j is an eigenvector of A_1. We get $E\zeta = \alpha_j\zeta$, i.e. $Ev_j = (\alpha_j - 1)v_j$, because E is uniquely characterized by the conditions $t_1\partial_1 - E \in \sum_{i\geq 1} \mathcal{O}_{T,0}\partial_i$ and $E(F_{-n}M) \subset F_{-n}M$ (cf. [loc. cit]), and we use (4.3.4).

4.4. *Example.* — $f = x^6 + y^6$, $f' = x^6 + y^6 + x^4y^4$.

Let M be the Gauss-Manin system of f and f', and u^ν, v^ν as in 4.2. Then $\{u^\nu : 0 < \nu_0, \nu_1 < 6\}$ is a basis of M_0 for f, and a basis $\{u'^\nu : 0 < \nu_0, \nu_1 < 6\}$ of M_0 for f' is given by

$$u'^\nu \text{ for } \nu \neq (1,1)$$
$$u'^{(1,1)} = u^{(1,1)} + a\partial_t u^{(5,5)} \text{ for } a \in \mathbf{C}^*.$$

But it is also possible that

$$\begin{aligned} u'^\nu &= u^\nu \text{ for } \nu \neq (1,1),(5,5) \\ u'^{(1,1)} &= \partial_t^{-1} u^{(1,1)} \\ u'^{(5,5)} &= u^{(1,1)} + a\partial_t u^{(5,5)}, \end{aligned}$$

where we can also check $\mathbf{S}(u'^\nu, u'^\mu) \in \mathbf{C}\partial_t^{-2}$, i.e. the corresponding section v satisfies (4.3.1-3). For the second basis the eigenvalues of A_1 associated to the corresponding section are

$$8/6, 4/6, \text{ and } (i+j)/6 \text{ for } 0 < i,j < 6, 3 \le i+j \le 9,$$

where the minimal is 3/6 and its multiplicity is 2. In fact this is the case where we choose an order $\succ$ of $\mathbf{C}$ in 3.9 such that $1/3 \succ 2/3$, and apply the same argument.

4.5. *Remark.* — In 4.3 we used the condition (4.3.2) essentially for the existence of an eigenvector generating $\mathrm{Gr}^F_{-n}M$. If A_1 is not semi-simple, we have to prove something non-trivial :

(4.5.1)

the natural morphism $\mathrm{Ker}\,((A_1)_u - \mathrm{id}) \to \Omega_f \otimes \mathcal{O}_{X,0}/m_{X,0}$ is nonzero,

where $m_{X,0}$ is the maximal ideal of $\mathcal{O}_{X,0}$ and we restrict to $t' = 0$, i.e. $\mathrm{Gr}^F_{-n}M = \Omega_f$. This point is completely missing in the proof of (4.2) in [Sk] (where the definition of good section is different from that in this paper, i.e. K. Saito calls a section *good*, if its image is generated over $\mathcal{O}_T$ by a basis satisfying (4.3.3-4)). Probably there is a basis satisfying (4.3.3-4) but A_1 is not semi-simple, cf. the remark in 3.10 (and 3.6). I don't know whether (4.5.1) is true in general. Here note also that in (4.4) of [Sk], the matrix $S(t', \delta_1^{-1})$ is not uniquely determined by the condition (4.4.3) in [loc. cit], because there still remains the ambiguity of $M_\mu(\mathcal{O}_T)$. In (4.3.4) (in this paper) the expansion of the action of ∂_i is given by only one term; a coordinate system satisfying $\sum \mathbf{C}\partial_i\zeta = \sum \mathbf{C}v_i$ should be the flat coordinate in the sense of K. Saito [loc. cit].

As is shown by the above example 4.4, there is a primitive form in the sense of [loc. cit] whose exponents (i.e. the eigenvalues of A_1) are different from the usual exponents, and such that the eigenvalue α_j of the generating vector in 4.2 cannot be the minimal eigenvalue and the multiplicity of the minimal is not one. For the primitive form corresponding to good sections in 3.6 and 3.10, its exponents are the usual ones and α_j in 4.2 is the minimal with multiplicity one. This shows that we cannot control everything by using only the higher residue pairing, and we have to add some axiom to the condition for primitive form, but it is rather difficult to find a good one which can be effectively combined with the other conditions. In fact the condition on the filtration V is restricted only to $t' = 0$, and it is not easy to use this condition on the base space of the versal deformation.

Originally the definition of primitive form was obtained to axiomize the arguments used in the case of rational double and simple elliptic singularity. But in those cases there was another important ingredient : the existence of weight or degree on the variables (i.e. the $\mathbf{C}^*$-action on the base space), and the non-negativity of the degree (with the stability of quasi-homogeneity by μ-constant deformation) was crucial in the argument (for example, the natural simultaneous compactification of fibers was used). I don't know whether the second ingredient can be generalized or well replaced. It might be clever to restrict to the case of quasi-homogeneous polynomial to avoid the above pathology. But the problem is still difficult even in the case of 14 exceptional singularity (e.g. $f = x^7 + y^3 + z^2$) where we cannot compactify all the fibers simultaneously in the natural way (i.e. have to restrict to the non-negative degree part of the base space.)

Note also that the definition of primitive form was *local*(i.e. as a germ), because the base space of the versal deformation is defined only as a germ. But I don't know whether we can expect a good theory of period mapping of (a partial compactification of) the complement of the discriminant of the base space (as a germ). It is rather interesting whether the primitive form can be extended globally on the base space in the case of quasi-homogeneous polynomial (especially, the 14 exceptional singularities). As to the naturality of the period mapping associated to the primitive form, we'll have to find a more intrinsic definition of $\mathcal{M}^{(n/2)}$

(e.g. for which primitive form (the monodromy group of) Sol($\mathcal{M}^{(n/2)}$) is naturally defined over $\mathbf{Q}$ or $\mathbf{R}$).

BIBLIOGRAPHY

[B] E. BRIESKORN, Die Monodromie der isolierten Singularitäten von Hyperflächen, Manuscripta Math., 2 (1970), 103-161

[BGM] J. BRIANCON, M. GRANGER and Ph. MAISONOBE, Sur le polynôme de Bernstein des singularités semi quasihomogènes, Prépublication de l'Université de Nice, n°138, novembre 1986.

[BM] J. BRIANCON and Ph. MAISONOBE, Idéaux de germes d'opérateurs différentiels à une variable, Enseign. Math., 30 (1984), 7-38.

[Bo] L BOUTET DE MONVEL, $\mathcal{D}$-modules holonômes réguliers en une variable, in Mathématique et Physique, Prog. in Math., Birkhäuser, 37, (1983), 281-288.

[CN1] P. CASSOU-NOGUÈS, Étude du comportement du polynôme de Bernstein lors d'une déformation à μ constant de $X^a + Y^b$ avec $(a, b) = 1$, Compositio Math., 63 (1987), 291-313.

[CN2] P. CASSOU-NOGUÈS, Calculs explicites sur les singularités isolées semi quasihomogènes. II, preprint.

[D1] P. DELIGNE, Équations différentielles à points singuliers réguliers, Lect. Notes in Math., 163, Springer, (1970).

[D2] P. DELIGNE, Théorie de Hodge II, Publ. Math. IHES, 40, (1971), 5-58.

[K1] M. KASHIWARA, B-function and holonomic systems, Invent. Math., 38 (1976), 33-53.

[K2] M. KASHIWARA, Introduction to Microlocal Analysis, Enseign. Math., 32 (1986), 227-259.

[K3] M. KASHIWARA, Vanishing cycle sheaves and holonomic systems of differential equations, in Algebric Geometry, Lect. Notes in Math., Springer, 1016, (1983), 134-142.

[KK] M. KASHIWARA and T. KAWAI, On the holomic systems of microdifferential equations III, Systems with regular singularites, Publ. RIMS, Kyoto Univ., 17 (1981), 813-979.

[Ka] N.M. KATZ, The regularity theorem in Algebraic Geometry, Act. Congrès Intern. Math., (1970), 437-443.

[Lo] E. LOOIJENGA, On the semi-universal deformation of a simple-elliptic hypersurface singularity, part II : The discrimiant, Topology, 17 (1978), 17-32.

[M1] B. MALGRANGE, Intégrales asymptotiques et monodromie, Ann. Sci. École Norm. Sup. Paris (4), 7 (1974), 405-430.

[M2] B. MALGRANGE, Le polynôme de Bersnstein d'une singularité isolée, in Lect. Notes in Math., Springer, 459, (1975), 98-119.

[M3] B. MALGRANGE, Déformations de systèmes différentiels et microdifférentiels, in Mathématique et Physique, Prog. in Math., Birkhäuser, 37, (1983), 353-379.

[M4] B. MALGRANGE, Deformations of differentiel systems. II, Journal of the Ramanujan Math. Soc., 1 (1986), 3-15.

[O] T. ODA, K. Saito's period map for holomorphic functions with isolated critical points, Advanced Studies in Pure Math., 10 (1987), 591-648.

[Ph] F. PHAM, Singularités des Systèmes Différentiels de Gauss-Manin, Prog. in Math., Birkhäuser, 2, (1979).

[S1] M. SAITO, Gauss-Manin system and mixed Hodge structure, Proc. Japan Acad., 58 (A) (1982), 29-32; Supplement, in Astérisque, 101/102 (1983), 320-331.

[S2] M. SAITO, Hodge filtrations on Gauss-Manin systems I, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math., 30 (1984), 489-498; II, Proc. Japan Acad., 59 (A) (1983), 37-40.

[S3] M. SAITO, On the structure of Brieskorn lattices (preprint at Grenoble Sept. 1983).

[S4] M. SAITO, Hodge structure via filtered $\mathcal{D}$-Modules, Astérisque, 130 (1985), 342-351.

[S5] M. SAITO, Modules de Hodge polarisables, to appear in Publ. RIMS.

[S6] M. SAITO, Mixed Hodge Modules, preprint RIMS-585, July 1987.

[S7] M. SAITO, Exponents and Newton polyhedra for isolated hypersurface singularities, to appear in Math. Ann.

[Sk] K. SAITO, Period mapping associated to a primitive form, Publ. RIMS, Kyoto Univ., 19 (1983), 1231-1264.

[SKK] M. SATO, T. KAWAI and M. KASHIWARA, Microfunctions and pseudo-differential equations, Lect. Notes in Math., Springer, 287, (1973), 264-529.

[SS] J. SCHERK and J. STEENBRINK, On the mixed Hodge structure on the cohomology of the Milnor fiber, Math. Ann., 271 (1985), 641-665. (This is the corrected version of the preprint quoted in [S1][S2].)

[Se] J.P. SERRE, Algèbre locale, multiplicités, Lect. Notes in Math., Springer 11, (1975).

[St] J. STEENBRINK, Mixed Hodge structure on the vanishing cohomology, in Real and Complex Singularities, Sijthoff & Noordhoff, Alphen aan den Rijn, (1977), 525-563.

[V1] A. VARCHENKO, The asymptotics of holomorphic forms determine a mixed Hodge structure, Soviet Math. Dokl., 22 (1980), 772-775.

[V2] A. VARCHENKO, On the monodromy operator in vanishing cohomology and the operator of multiplication by f in the local ring, Soviet Math. Dokl., 24 (1981), 248-252.

[V3] A. VARCHENKO, The complex exponent of a singularity does not change along strata μ = const, Func. Anal. Appl., 16 (1982), 1-9.

[V4] A. VARCHENKO, A lower bound for the codimension of the strata $\mu =$ const in terms of the mixed Hodge structure, Moscow Univ. Math. Bull., 37-6 (1982), 30-33.

Manuscrit reçu le 1er février 1988,
révisé le 1er juin 1988.

Morihiko SAITO,
Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606 (Japan).