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ON THE STRUCTURE OF GENERAL STOCHASTIC AUTOMATA (*)

by Hans DADUNA ([1])

Communicated by W. BRAUER

Abstract. — *We describe the inner structure of stochastic automata over measurable spaces by their lattice of congruences, where a congruence is given by a diminished set of possible events and by a factorization of the underlying set. Using the knowledge of this (almost "countably dual-algebraic") lattice structure we construct the least quotient describing a given event and show when we are able to infer properties of a (hidden) event by measuring events in quotients which do not include this event.*

Résumé — *Nous décrivons la structure interne d'un automate stochastique sur un espace mesurable par son treillis de congruences, où une congruence est donnée par un ensemble diminué d'événements possibles et par une factorisation de l'ensemble associé. En utilisant la connaissance de cette structure de treillis (qui est presque un treillis dénombrablement dual-algébrique) nous construisons le quotient minimal qui contient un événement donné et nous démontrons dans quelles conditions il est possible de trouver les propriétés d'un événement caché en mesurant des événements dans des quotients qui ne contiennent pas cet événement.*

1. INTRODUCTION

Stochastic automata with general measurable state space were implicitly used as models in many areas (*see* for example: non-stationary Markovian programming [9]; learning theory [10], statistical search [11]), while the usual structure theory of stochastic automata only dealt with discrete state spaces. During the last years some articles appeared, which used explicitly the notion of a general stochastic automaton over measurable spaces (e. g. [1-7]). In this note we will study the class of state automata and describe the inner structure of such systems.

The auxiliary instrument to get knowledge about this structure is the lattice of congruences which we defined in [2]. This investigation enables us to speak of a least quotient automaton which describes a given event in an experiment. Looking for criteria which say when we are able to infer properties of a (hidden) event B from the measurement of events in smaller automata, which don't include B itself, we prove decomposition theorems in the manner of Hartmanis and Bacon.

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II. STOCHASTIC AUTOMATA

1 DEFINITION: A stochastic automaton $\underline{A}=((\Omega, \mathfrak{A}), Q; F)$ is defined by a measurable space $(\Omega, \mathfrak{A})$, an initial distribution Q on $(\Omega, \mathfrak{A})$ and a set F of transition kernels; i e., $\lambda \in F$ implies that $\lambda : \Omega \times \mathfrak{A} \to [0, 1]$ is a stochastic kernel. (In the following let $\underline{A}$ be fixed.)

2. REMARKS: *a*. The automata we consider in this note are "initial Medvedev automata" with state space Ω, a set $\mathfrak{A} \subseteqq \mathscr{P}(\Omega)$ of possible events, an "input alphabet" F and an initial distribution Q on $(\Omega, \mathfrak{A})$.

The time development of the automaton is given in the following way: at $t=0$ the automaton is in state Q; if at time $t \in \mathbb{N}$ the automaton is in state P [a probability measure on $(\Omega, \mathfrak{A})$], and if at time t there is an input $\lambda \in F$, then the automaton is in state:

$$\int_{\Omega} P(dx)\, \lambda(x, .) =: (\lambda P)(.) \qquad \text{at time} \quad t+1$$

So for every sequence $(\lambda_t : t \in \mathbb{N})$ of input symbols the behavior of the automaton is given by an inhomogeneous Markov process.

b. Formally one should distinguish between an input symbol λ and the associated transition kernel the application of which is caused by an input of λ into the system. (Different inputs my cause the same transition.) But there will raise no difficulties if we identify input set and set of associated kernels: if λ and μ are different input symbols with the same associated kernel k, then we have k_{μ} and k_{λ} in F.

This description enables us to talk about stochastic automata of the "same (input)-type" although the sets of associated transition kernels are defined on different measure spaces (*see* e. g. lemma 7).

c. The transcription of the following investigations to the case of output automata is possible. (*See* [3], chapt. 10.)

d. For some practical purposes (stochastic dynamic optimization, e. g) one needs a measurable structure on the set F of input symbols. Since in our investigations we consider automata as sequential machines it would be sufficient to introduce the σ-algrebra generated by the countable subsets of F.

3. EXAMPLES: *a*. The first time, a continuous automaton was constructed explicitly seems to be in the note [11]: The automaton constructed there was an

automaton with output. It had state set and output set $(\mathbb{R}^n, \mathfrak{B}^n)$. But the transition kernels from state space $(\mathbb{R}^n, \mathfrak{B}^n)$ to the cartesian product of state and output space $(\mathbb{R}^n, \mathfrak{B}^n)^2$ are in their second component concentrated on the diagonal diag $\mathbb{R}^{2n}$. So we have essentially an automaton $\underline{S}=((\mathbb{R}^n, \mathfrak{B}^n), Q; \{\mu_0, \mu_1\}$ with kernels $\mu_i : \mathbb{R}^n \times \mathfrak{B}^n \to [0, 1]$, $i=0, 1$, and $\mu_1(a, .)$ is the point mass in $(-a)$, $\mu_0(a, .)$ is uniformly distributed on a ball with center a and fixed radius r. ($\underline{S}$ in state $s \in \mathbb{R}^n$ means that the automaton at this time makes a walk given by vector s; μ_1 is used if the last step was not successful: the automaton goes back its last way; μ_0 means: the automaton walks in any direction a way the length of which is at most r).

b. Given any initial topological state automaton $T=((\mathbb{R}^1, \mathcal{O}^1), s_0; F_t)$ with state space $(\mathbb{R}^1, \mathcal{O}^1)$, where $\mathcal{O}^1$ are the open sets and F_t is a set of continuous mappings from $\mathbb{R}^1$ into $\mathbb{R}^1$. Then there exists a stochastic automaton $\underline{T}=((\mathbb{R}^1, \mathfrak{B}^1), Q; F)$, where $\mathfrak{B}^1=\sigma(\mathcal{O}^1)$ and the cernels of F are defined in the following way:

if

$$f: \quad (\mathbb{R}^1, \mathcal{O}^1) \to (\mathbb{R}^1, \mathcal{O}^1) \in F_t,$$

then

$$\lambda_f: \quad \mathbb{R}^1 \times \mathfrak{B}^1 \to [0, 1],$$

$$(x, C) \to \frac{1}{\sqrt{2\pi}\,\sigma} \int_C \lambda^1(dy) \exp\left(-\frac{(y-f(x))^2}{2\sigma^2}\right)$$

is in F. (λ^1 is Lebesgue measure.)

$\underline{T}$ is a perturbated version of T: at any transition [which should be controlled by $f(x)$] the automaton makes some faults which are given normally distributed, centered at the "true" value $f(x)$.

c. Any homogeneous chain, any Markov process with continuous state space and discrete time scale is an initial Medvedev automaton.

III. THE LATTICE OF STOCHASTIC CONGRUENCES

4. DEFINITION: A (stochastic) congruence on $\underline{A}$ is a pair $\theta=(\bar{\theta}, \mathfrak{A}_\theta)$, where $\bar{\theta}$ is an equivalence on Ω and $\mathfrak{A}_\theta$ a sub-σ-algebra of $\mathfrak{A}$, such that the following holds:

(i) $(\mathfrak{A}_\theta \text{ nat } \bar{\theta}) \text{ nat } \bar{\theta}^{-1}=\mathfrak{A}_\theta$, where $\mathfrak{A}_\theta \text{ nat } \bar{\theta}=\{D \subseteq A/\bar{\theta} : D \text{ nat } \bar{\theta}^{-1} \in \mathfrak{A}_\theta\}$;

(ii) for all $\lambda \in F$, $a, b \in \Omega$, $C \in \mathfrak{A}_\theta$: $a\,\bar{\theta}\,b \Rightarrow \lambda(a, C)=\lambda(b, C)$;

(iii) for all $\lambda \in F$, $C \in \mathfrak{A}_\theta$: $\lambda : (., C) : \Omega \to [0, 1]$ is $\mathfrak{A}_\theta - \mathfrak{B}^1 \cap [0, 1]$-measurable.

5. REMARKS: *a*. The equivalence $\bar{\theta}$ tells us which points should not be distinguished further, and the sub-σ-algebra $\mathfrak{A}_0$ of the congruence contains the possible events of our experiment to which we want to reduce our interest. Condition (i) and (iii) represent the compatibility of the measurable structures of the automaton and its quotient, while (ii) reflects an algebraic compatibility: Transition operators and equivalence are interchangeable.

b. It is a little bit surprising that from (i) and (iii) we can deduce (ii):

If $[x]\bar{\theta} := \{ y \in A : x \bar{\theta} y \}$, then from $(\mathfrak{A}_0 \text{ nat } \bar{\theta}) \text{ nat } \bar{\theta}^{-1} = \mathfrak{A}_0$ it follows that $[x]\bar{\theta} \cap B = \emptyset$ or $[x]\bar{\theta} \subseteq B$ for all $B \in \mathfrak{A}_0$. But from the $\mathfrak{A}_\theta - \mathfrak{B}^1 \cap [0, 1]$-measurability of $\lambda(., C)$ for $C \in \mathfrak{A}_0$ we have for $t \in [0, 1] : \{ t \} \lambda(., C)^{-1} \in \mathfrak{A}_\theta$.

There is a bijection between (stochastic) congruences on $\underline{A}$ and state homomorphisms with domain $\underline{A}$ (*see* [2]). The given definition permits the construction of quotient automata.

6. DEFINITION: *a*. Let $\underline{A}$, $\underline{B} = ((\Lambda, \mathfrak{B}), P; F)$ be a pair of stochastic automata [of the same input type — *see* 2 *b*)] A state homomorphism $f : \underline{A} \to \underline{B}$ is given by a function $f : \Omega \to \Lambda$ with the following properties:

(i) f is $\mathfrak{A} - \mathfrak{B}$-measurable;

(ii) $P = Q f$ (the image mesure of Q under f);

(iii) for all $a \in \Omega$ and for all $\lambda \in F$ $\lambda(af, .)$ is the image measure of $\lambda(a, .)$ under f;

(iv) for all $\lambda \in F$ $(\lambda Q) f = \lambda P = \lambda (Q f)$.

b. Isomorphisms are those state homomorphisms $f : \underline{A} \to \underline{B}$ that are given by a bijection $f : \Omega \to \Lambda$ which has a $\mathfrak{B} - \mathfrak{A}$-measurable inverse.

7. LEMMA: *Let* $\theta = (\bar{\theta}, \mathfrak{A}_\theta)$ *be a congruence on* $\underline{A}$. *Then* $\underline{A}/\theta := ((\Omega/\bar{\theta}, \mathfrak{A}/\theta), Q/\theta; F)$ *is a stochastic automaton, where* $\mathfrak{A}/\theta := \mathfrak{A}_0 \text{ nat } \bar{\theta}$. $Q/\theta := Q \text{ nat } \bar{\theta}$ (*image measure under* nat $\bar{\theta}$) *and* $\lambda([a]\bar{\theta}, C) = \lambda(a, C \text{ nat } \bar{\theta}^{-1})$, $C \in \mathfrak{A}/\theta$. ($\underline{A}/\theta$ *called a quotient modulo* θ *of* $\underline{A}$. *Remember that* $\underline{A}$ *and* $\underline{A}/\theta$ *are only of the same input type, the associated kernels are distinct.*)

Proof: From 4 (ii) follows that $\lambda : A/\bar{\theta} \times \mathfrak{A}/\theta \to [0, 1]$ is a mapping. For fixed $D \in \mathfrak{A}/\theta$ $\lambda(., D)$ is $\mathfrak{A}/\theta - \mathfrak{B}^1 \cap [0, 1]$-measurable by 4 (i) and 4 (iii) and for fixed $[x]\bar{\theta} \in A/\bar{\theta}$ $\lambda([x]\bar{\theta}, .)$ is a probability measure on $(A/\bar{\theta}, \mathfrak{A}/\theta)$.

8. EXAMPLE: $\underline{A} = ((\mathbb{R}^1, \mathfrak{B}^1), Q; \{ \mu \})$ with:

$$\mu(x, E) = \frac{1}{\sqrt{2\pi}} \int_E \lambda^1(dy) \exp\left(- \frac{(y - [|x|])^2}{2} \right),$$

where $Q(.)=\int_{(.)} 2e^{-2x} 1_{[0,\infty)}(x)\,\lambda^1(dx)$ is the exponential distribution with parameter 2. ($[|x|]$ is the greatest integer t with $t \leqq |x|$.).

Let us define $\mathfrak{A}_0=\sigma(\{[n,n+1) : n\in\mathbb{Z}\})$ and:

$$x\bar{\theta}y \Leftrightarrow [x]=[y].$$

Then $\theta=(\bar{\theta}, \mathfrak{A}_0)$ is a congruence on $\underline{A}$ and we have:

$$\underline{A}/\theta=(\{[n]\bar{\theta} : n\in\mathbb{Z}\}, \mathscr{P}\{[n]\bar{\theta} : n\in\mathbb{Z}\}, Q/\theta; \{\lambda\}),$$

where $[n]\bar{\theta}$ is the class modulo $\bar{\theta}$ which contains n.

$$Q/\theta([n]\bar{\theta})=(1-e^{-2})\,e^{(-2)n}\,1_{[0,\infty)}(n)$$

is the geometric distribution with parameter $1-e^{-2}$ on $\mathbb{Z}$, and:

$$\mu([n]\bar{\theta}, \{[m]\bar{\theta}\})=\frac{1}{\sqrt{2\pi}}\int_m^{m+1}\lambda^1(dy)\exp\left(-\frac{(y-[|n|])^2}{2}\right).$$

In the following we need some knowledge from the theory of equivalence relations (*see* [8]):

Given a set $A\neq\emptyset$; the set con A of equivalences is partially ordered by:

$$\bar{\theta}, \bar{\varphi}\in \text{con } A, \qquad \text{then} \quad \bar{\theta}\leqq\bar{\varphi} \Leftrightarrow (a\,\bar{\varphi}\,b \Rightarrow a\,\bar{\theta}\,b \text{ for } a, b\in A).$$

con A contains a least element $\bar{\omega}$ and a greatest element $\bar{\iota}$, where $\bar{\omega}$ is equality (in set representation the diagonal) and $\bar{\iota}$ is the all-relation A^2.

The partial ordering induces a complete lattice (con A; $\vee$, $\wedge$); the (arbitrary) meet-operator is given by $\wedge(\bar{\theta}_i : i\in I) := \bigcap(\bar{\theta}_i : i\in I)$, where the right side of the equation must be read as a meet of the set representations of the equivalence relations.

9. THEOREM [2]: *The partial ordering "$\leqq$" defined on the set con $\underline{A}$ of congruences on $\underline{A}$ by:* $\theta\leqq\varphi \Leftrightarrow (\bar{\theta}\leqq\bar{\varphi} \wedge \mathfrak{A}_0\supseteq\mathfrak{A}_\varphi)$, *is a complete lattice.*

Proof: Since $\omega:=(\bar{\omega}, \mathfrak{A})$, resp. $\iota=(\bar{\iota},(\varphi,\Omega))$, are the least resp. greatest congruence, it is sufficient to show that:

$$\sup(\theta_i : i\in I)=(\vee(\bar{\theta}_i : i\in I), \bigcap(\mathfrak{A}_{\theta_i} : i\in I)), \text{ is a congruence.}$$

(i) we have to show: $(\mathfrak{A}_0 \text{ nat } \bar{\theta}) \text{ nat } \bar{\theta}^{-1}\supseteq\mathfrak{A}_0$.

Suppose $B\in\mathfrak{A}_0=\bigcap(\mathfrak{A}_{\theta_i} : i\in I)$. Then for every $i\in I$, B is a disjoint union of equivalence classes of $\bar{\theta}_i$.

Consider x, $y \in \Omega$, $x \in B$ and x nat $\bar{\theta} = y$ nat $\bar{\theta}$. Then there exists a sequence $x = c_0, c_1, \ldots, c_n = y$, $c_i \in \Omega$, $n \in \mathbb{N}$, with $c_i \, \bar{\theta}_{j(i)} \, c_{i+1}$ for $i = 0, \ldots, n-1$, $\bar{\theta}_{j(i)} \in (\bar{\theta}_i : i \in I)$, which implies: $c_i \in B \Rightarrow c_{i+1} \in B$;

(ii) we have to show: $\lambda(., C) : A \to [0, 1]$ is $\mathfrak{A}_\theta - \mathfrak{B}^1 \cap [0, 1]$-measurable for all $C \in \mathfrak{A}_\theta$. But we have for $C \in \quad (\mathfrak{A}_{\theta_i} : i \in I)$ and

$$t \in [0, 1] : \{ c \in A_i : \lambda(c, C) \leqq t \} \in \mathfrak{A}_{\theta_i} \quad \text{for all } i \in I.$$

The infima in (con $\underline{A}$, $\leqq$) of a set $(\theta_i : i \in I)$ are defined by the least sub-σ-algebra $\mathfrak{B}$ of $\mathfrak{A}$ containing all $\mathfrak{A}_{\theta_i}$ such that all $\lambda(., B)$, $B \in \mathfrak{B}$, are $\mathfrak{B} - \mathfrak{B}^1 \cap [0, 1]$-measurable and the greatest equivalence $\bar{\varphi}$ on Ω contained in all $\bar{\theta}_i$ such that $\bar{\varphi}$ is compatible with $\mathfrak{B}$ in the sense of 4(i). There is no real constructive description of the infimum, but for some proofs we need a recursive procedure. So we introduce a generating scheme for congruences.

10. Lemma: *If $\{ B_i : i \in I \} \subseteq \mathfrak{A}$, then define:*

$$\mathfrak{A}_0^- := \bigcup_{i \in I} \bigcup_{\lambda \in F} (\mathfrak{B}^1 \cap [0, 1]) \, \lambda(., B_i)^{-1} \cup \{ B_i : i \in I \};$$

if for $r \leqq n \in \mathbb{N}$, $\mathfrak{A}_r$ is defined, then:

$$\mathfrak{A}_{n+1}^- = \bigcup_{B \in \bigcup_{r \leqq n} \mathfrak{A}_r} \bigcup_{\lambda \in F} (\mathfrak{B}^1 \cap [0, 1]) \, \lambda(., B)^{-1} \cup \{ B_i : i \in I \}$$

and $\mathfrak{A}_{n+1}$ is the closure of $\mathfrak{A}_{n+1}^-$ under finite intersections. Now we set $\mathfrak{A}_{(B_i : i \in I)} := \sigma(\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n)$ and conclude: For all $C \in \mathfrak{A}_{(B_i : i \in I)}$ the function $\lambda(., C) : \Omega \to [0, 1]$ is $\mathfrak{A}_{(B_i : i \in I)} - (\mathfrak{B}^1 \cap [0, 1])$-measurable.
$a \to \lambda(a, C)$

Proof: $\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$ is closed under finite intersections: If $B_i \in \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$, $i = 1, \ldots, m$, then there exist natural numbers $j(i)$, $i = 1, \ldots, m$, such that $B_i \in \mathfrak{A}_{j(i)}$. From the definition we have for all $i = 1, \ldots, m$, $B_i \in \mathfrak{A}_{\max(j(i) : i = 1, \ldots, m)}$, which is closed under finite intersections.

Again from the definition of $\mathfrak{A}_n$ the function $\lambda(., C) : \Omega \to [0, 1]$ is $\mathfrak{A}_{(B_i : i \in I)} - (\mathfrak{B}^1 \cap [0, 1)]$-measurable for $C \in \mathfrak{A}_n$, and this proves the desired measurability of all $\lambda(., C)$, $C \in \mathfrak{A}_{(B_i : i \in I)}$.

Given any σ-algebra $\mathfrak{A} = \{ A_i : i \in I \}$ on Ω, then $\mathfrak{A}$ generates a partition on Ω by the blocks $\bigcap (A_i' : i \in I)$, where $A_i' = A_i$ or $A_i' = A_i^c$. These blocks are the classes of the greatest equivalence $\bar{\theta}$ on Ω which fulfils $(\mathfrak{A}$ nat $\bar{\theta})$ nat $\bar{\theta}^{-1} = \mathfrak{A}$. We say that $\bar{\theta}$ is generated by $\mathfrak{A}$. From 10 we get the following corollary:

11. COROLLARY: *a. For* $\{B_i : i \in I\}$ *there exists a greatest element* $\varphi(B_i : i \in I)$ *in the set of congruences* φ *on* $\underline{A}$ *which fulfill* $\mathfrak{A}_\varphi = \mathfrak{A}_{(B_i : i \in I)}$.

b. If θ *is a congruence with* $\mathfrak{A}_\theta \supseteq \{B_i : i \in I\}$, *then* $\mathfrak{A}_{(B_i : i \in I)} \subseteq \mathfrak{A}_\theta$. *[Not necessary* $\theta \leq \varphi(B_i : i \in I)$.]

c. The infimum of $\{\theta_i : i \in I\} \subseteq \text{con}\,\underline{A}$ *is:*

$$\overline{(\overline{\varphi(\bigcup(\mathfrak{A}_{\theta_i} : i \in I))}} \cap \bigcap(\bar{\theta}_i : i \in I), \mathfrak{A}_{\bigcup(\mathfrak{A}_{\theta_i} : i \in I)}).$$

Proof: a. $\varphi(B_i : i \in I)$ is defined by the σ-algebra $\mathfrak{A}_{(B_i : i \in I)}$ and $\overline{\varphi(B_i : i \in I)}$, the equivalence generated by $\mathfrak{A}_{(B_i : i \in I)}$.

c. The choice of $\mathfrak{A}_{\bigcup(\mathfrak{A}_{\theta_i} : i \in I)}$ gives the measurability condition 4(iii) and the extremal property with respect to the σ-algebra, while the choice of an equivalence smaller than $\bigcap(\bar{\theta}_i : i \in I)$ gives the extremal property with respect to the equivalence. Property 4(i) holds because for every σ-algebra $\mathfrak{C}$, $\bar{\theta} \geq \bar{\varphi}$ and $(\mathfrak{C}\ \text{nat}\ \bar{\theta})\ \text{nat}\ \bar{\theta}^{-1} = \mathfrak{C}$ imply $(\mathfrak{C}\ \text{nat}\ \bar{\varphi})\ \text{nat}\ \bar{\varphi}^{-1} = \mathfrak{C}$ and because $\overline{\varphi(\bigcup(\mathfrak{A}_{\theta_i} : i \in I))}$ is generated by $\mathfrak{A}_{\bigcup(\mathfrak{A}_{\theta_i} : i \in I)}$.

12. EXAMPLE: Given the stochastic automaton of 8. We want to find $\varphi([-1, 1)$.

Let for $x \in \mathbb{R}^1$ be $\mu(x, [-1, 1)) = t_0$. Then we have:

$$\mu(., [-1, 1))^{-1}(t_0) = [x]\,\bar{\theta} \cup [-x]\,\bar{\theta}$$

(where $\bar{\theta}$ is defined as in 8) which means $\mathfrak{A}_{[-1,1)} \supseteq \{[x]\,\bar{\theta} \cup [-x]\,\bar{\theta} : x \in \mathbb{R}^1\}$. From $\sigma\{[x]\,\bar{\theta} \cup [-x]\,\bar{\theta} : x \in \mathbb{R}^1\} - \mathfrak{B}^1 \cap [0, 1]$-measurability of the functions $\mu(., C)$ we get:

$$\mathfrak{A}_{[-1,1)} = \sigma(\{[x]\,\bar{\theta} \cup [-x]\,\bar{\theta} : x \in \mathbb{R}^1\}).$$

In analogy to algebraic lattices we define:

13. DEFINITION: If $(V, \leq)$ is a complete lattice, then:

(i) $v \in V$ is countably dual-compact (c. d. c.)

$$\Leftrightarrow [v \geq \wedge (v_i : i \in I) \subseteq I) \Rightarrow v \geq \wedge (v_i : i \in I_0 \subseteq I)$$

for some countable I_0];

(ii) $(V, \leq)$ is countably dual-algebraic (c. d. a.) $\Leftrightarrow$ every $v \in V$ is infimum of a set of countably dual-compact elements.

Now we can prove:

14. LEMMA: *Let* $\underline{A}$ *be a stochastic automaton:*

(a) for $\{B_n : n \in \mathbb{N}\} \subseteq \mathfrak{A}$, $\varphi(B_n : n \in \mathbb{N})$ *is c. d. c.;*

(b) if $\theta = (\bar{\theta}, \mathfrak{A}_0)$ *is c. d. c., then* $\theta = (\bar{\theta}, \mathfrak{A}_{(B_n : n \in \mathbb{N})})$, *for a countable set* $\{B_n : n \in \mathbb{N}\} \subseteq \mathfrak{A}$;

(c) (con $\underline{A}$, $\leqq$) *contains a c.d.a. lattice* (V, $\lesssim$); *the embedding respects the partial orderings;*

(*d*) *the set of c.d.c. congruences on $\underline{A}$ is a σ-inf-semilattice under the restriction of "$\leqq$". (There are examples, showing that the embedding must not be a suphomomorphism.)*

Proof: The crucial fact is (*a*). By an inverse induction on the construction of 10 we show that for the generation of every B_n we only need a countable set of σ-algebras $\mathfrak{A}_{\theta_i}$, $i \in I_0 \subseteqq I$:

Let $(\theta_i : i \in I) \subseteq \text{con } A$ and $\varphi(B_n : n \in \mathbb{N}) \geqq \wedge(\theta_i : i \in I) =: \theta$. This implies $\mathfrak{A}_{(B_n, n \in \mathbb{N})} \subseteqq \mathfrak{A}_\theta$ and from lemma 10, we get $\mathfrak{A}_\theta = \mathfrak{A}_{\cup(\mathfrak{A}_{\theta_i} : i \in I)}$ and (with notation of 10) a generating set $\bigcup(\mathfrak{A}_n : n \in \mathbb{N})$ of $\mathfrak{A}_\theta$.

For every B_n, $n \in \mathbb{N}$, we have a countable set:

$$\{ C_{n_i} : i \in \mathbb{N} \} \subseteqq \bigcup(\mathfrak{A}_n : n \in \mathbb{N}) \qquad \text{with} \quad B_n \in \sigma\{ C_{n_i} : i \in \mathbb{N} \}.$$

Given C_{n_i}, $n, i \in \mathbb{N}$, there exists a least element $i_0 \in \mathbb{N}$ of the set of numbers $m \in \mathbb{N}$ which fulfill $C_{n_i} \in \mathfrak{A}_m$.

So there is a finite set $\{ C^{i_0}_{n_{ij}} : j = 0, \ldots, n_{i_0} \} \subseteqq \mathfrak{A}^-_{i_0}$ such that $C_{n_i} = \bigcap_{j=0}^{n_{i_0}} C^{i_0}_{n_{ij}}$. By definition of $\mathfrak{A}^-_{i_0}$ there is for every $C^{i_0}_{n_{ij}}$ a $T^{i_0}_{n_{ij}} \in \mathfrak{B}^1 \cap [0, 1]$, a $\lambda \in F$, and a $C^{i_0^*}_{n_{ij}} \in \bigcup(\mathfrak{A}_r : r \leqq i_0 - 1)$, such that $C^{i_0}_{n_{ij}} = (T^{i_0}_{n_{ij}})\lambda(., C^{i_0^*}_{n_{ij}})^{-1}$.

Starting again, for every $C^{i_0^*}_{\mu_{ij}}$ there exists a least element $i_{01} \in \mathbb{N}$ of the set of numbers $m \in \mathbb{N}$ which fulfill $C^{i_0^*}_{n_{ij}} \in \mathfrak{A}_m$, and we have $i_{01} < i_0$.

Continuation of this construction stops after a finite set of steps. At each step we need only a countable set of elements for generation and thus we get a countable set $\{ C_n : n \in \mathbb{N} \}$ of generators for the B_n, $n \in \mathbb{N}$.

Taking for every C_n, $n \in \mathbb{N}$, some $\theta_n \in \{ \theta_i : i \in I \}$ with:

$$C_n \in \mathfrak{A}_{\theta_n} \in \bigcup(\mathfrak{A}_{\theta_i} : i \in I)$$

we get:

$$\mathfrak{A}_{(B_n, n \in \mathbb{N})} \subseteqq \mathfrak{A}_{(\wedge(\theta_n : n \in N))}.$$

This inclusion and 11 *c*, give:

$$\overline{\varphi(B_n : n \in \mathbb{N})} \geqq \overline{\wedge(\theta_n : n \in \mathbb{N})}.$$

b. For all $B \in \mathfrak{A}_\theta$ we have $(\bar{\theta}, \mathfrak{A}_B) \in \text{con } \underline{A}$; so $\theta = \wedge((\bar{\theta}, \mathfrak{A}_B) : B \in \mathfrak{A}_\theta)$. From the c.d.c. property we have $\theta \geqq \wedge((\bar{\theta}, \mathfrak{A}_{B_n}) : n \in \mathbb{N})$ for suitable $B_n \in \mathfrak{A}_\theta$.

From $\theta \leqq (\bar{\theta}, \mathfrak{A}_{B_n})$ for all $n \in \mathbb{N}$, we conclude the proposition.

c. Define $V=\{\theta \in \text{con } \underline{A} : \bar{\theta}$ is generated by $\mathfrak{A}_\theta\}$ and $\theta \lesssim \varphi \Leftrightarrow \mathfrak{A}_\theta \supseteqq \mathfrak{A}_\varphi$.

d Let θ_i, $i \in \mathbb{N}$, be c.d.c. with $\wedge(\theta_i : i \in I) \geqq \wedge(\varphi_j : j \in J)$. We have $\theta_i \geqq \wedge(\varphi_j : j \in J_i)$ with countable $J_i \subseteqq J$ for all $i \in \mathbb{N}$.

15. REMARKS: *a*. The isotone embedding of 14*c*, is generally not a σ-sup-morphism: Take an automaton $\underline{A}$ with underlying set $A :=(\{a\} \times X) \cup (\{b\} \times X)$ (X not countable, $a \neq b$, $a, b \notin X$) and σ-algebra $\sigma(\mathfrak{A}_\theta \cup \mathfrak{A}_\varphi)=\mathfrak{A}$ $(\mathfrak{A}_\theta :=\sigma(\{t\} : t \in \{a\} \times X)$, $\mathfrak{A}_\varphi :=\sigma(\{s\} : s \in \{b\} \times X))$, which has congruences θ and φ with $\mathfrak{A}_\theta$, $\mathfrak{A}_\varphi$ as defined.

From $\mathfrak{A}_\theta \cap \mathfrak{A}_\varphi = \{\varphi, A\}$ we have in $(V, <)$ $\theta \vee \varphi = \iota$; but generally in (con $\underline{A}$, $\leqq$) we have $\theta \vee \varphi \neq \iota$: For if $\bar{\theta} \vee \bar{\varphi} = \bar{\iota}$ holds, for $(a, x), (b, x) \in A$ there must exist a finite sequence $(a, x)=c_0, c_1, \ldots, c_{n-1}, c_n=(b, x)$, which fulfills $c_i \bar{\theta} c_{i+1}$ or $c_i \bar{\varphi} c_{i+1}$, $i=0, \ldots, n-1$ So at least one $i \in \{0, \ldots, n-1\}$ must exist with $(a, x_i)=c_i \bar{\theta} c_{i+1}=(b, x_{i1})$ or $(a, x_i)=c_i \bar{\varphi} c_{i+1}=(b, x_{i+1})$. But this is impossible, since such points are seperated by $\mathfrak{A}_\theta$ and $\mathfrak{A}_\varphi$.

b. Given a physical experiment described by the automaton $\underline{A}$ with $\mathfrak{A}$ as set of possible events. If $B \in \mathfrak{A}$ is a special interesting event, then $\underline{A}/\varphi(B)$ is the smallest quotient automata for a direct investigation of B.

IV. SUBTENSORPRODUCT DECOMPOSITION OF STOCHASTIC AUTOMATA

Quotient automata are "smaller" than $\underline{A}$, so they are easier to handle with. But they only reflect parts of the structure of $\underline{A}$ – so we ask for a method, to put all the information contained in $\underline{A}$ in a suitable set of quotients and for an inversion of this splitting to get back the whole $\underline{A}$.

14. DEFINITION: Let $\underline{A}_m=((\Omega_m, \mathfrak{A}_m), Q_m; F)$; $m \in M$, be a family of stochastic automata with the same input set. The tensorproduct $\otimes(\underline{A}_m : m \in M)= : \underline{A}$ of the $\underline{A}_m$ is defined by:

$$\Omega = X(\Omega_m : m \in M); \qquad \mathfrak{A} = \otimes(\mathfrak{A}_m : m \in M),$$
$$Q = \otimes(Q_m : m \in M);$$

for all $\lambda \in F$, $\lambda : \Omega \times \mathfrak{A} \to [0, 1]$ is defined by:

$$\lambda((a_m : m \in M), X(C_j : j \in J)\,\text{pr}_J^{M-1}) = \prod(\lambda(a_j, C_j) : j \in J),$$

where $\text{pr}_J^M : X(\Omega_m : m \in M) \to X(\Omega_j : j \in J \subseteqq M)$ is the measurable projection and J is finite, $C_j \in \mathfrak{A}_j$. (Tensorproduct is the usual shunt connection and defines a monoidal automata category.)

b. If $\otimes(\underline{A}_m : m \in M)$ is given as defined in *a*, then the projection:

$$\mathrm{pr}_J^M : \quad X(\Omega_m : m \in M) \to X(\Omega_m : m \in J),$$
$$(a_m : m \in M) \to (a_m : m \in J),$$

defines for $J \subseteq M$, $J \neq \emptyset$, an unique state homomorphism called tensor projection.

The connection between quotient decomposition and tensor composition gives the following idea: Try to find a representation of $\underline{A}$ as tensorproduct of some of its quotients in a way that all those quotients that are used in the representation are irreducible, i. e., they are not decomposable in the sketched way. Then you can investigate the whole system $\underline{A}$ *via* the simpler quotients—and you have found the "best" decomposition to do this.

But from deterministic automata theory it is known that such a decomposition is possible only rarely. The succesful concept there is a representation as a subautomaton of a product of irreducible quotients (*see* definition 19 *a*). The irreducibility as usual means that a representation by a product of quotients is only possible if one of the factors is isomorphic to the automaton to be represented (*see* definition 19 *b*).

17. Definition: $\underline{B} = ((\chi, \mathfrak{B}), P; F)$ is a subautomaton of $\underline{A}$ iff:

$$\chi \subseteq \Omega, \mathfrak{B}\mathfrak{A} \cap \chi, Q|_{\mathfrak{B} \cap \chi} = P,$$

and for all $\lambda \in F$, $b \in \chi$:

$$\lambda(b, .)|_{\mathfrak{A} \cap \chi} = \lambda(b, .)|_{\mathfrak{B}}.$$

The connection between quotient decomposition and tensorcomposition gives the following idea:

18. Remark: The automaton $\underline{B}$ as in 17, is a subautomaton of $\underline{A}$ if and only if the set inclusion inc: $\chi \to \Omega$ is bimeasurable as mapping onto χ inc and induces a state homomorphism which we call an embedding of $\underline{B}$ into $\underline{A}$. Any state homomorphism $j : \underline{B} \to \underline{A}$ which is defined by an injective mapping $j : \chi \to \Omega$ which is bimeasurable as mapping onto χj is called an embedding and defines an isomorphism between $\underline{B}$ and the subautomaton $\underline{B}\, j$ of $\underline{A}$.

19. Definition: *a*. $\underline{A}$ is subtensorproduct of $\underline{A}_m$, $m \in M \Leftrightarrow \underline{A}$ is a subautomaton of $\otimes(\underline{A}_m : m \in M)$ and the restrictions of the tensorprojections are surjective.

b. $\underline{A}$ is subtensorially irreducible $\Leftrightarrow$ for every set $K=\{\theta_i : i\in I\}\subseteqq \operatorname{con} \underline{A}$, such that $\underline{A}$ is isomorphic to a subtensorproduct of the $\underline{A}/\theta_i$, $i\in I$, there exists a $\theta\in K$ with $\theta=\omega$.

20. THEOREM: *$\underline{A}$ is isomorphic to a subtensorproduct of $\underline{A}/\theta_i$, $i\in I$, $\theta_i=(\bar{\theta}_i, \mathfrak{A}_{\theta_i})\in \operatorname{con} \underline{A}$, iff the following holds:*

Z1. $\bigcap(\bar{\theta}_i : i\in I)=\bar{\omega}$;

Z2. $\mathfrak{A}=\sigma(\bigcup(\mathfrak{A}_{\theta_i} : i\in I))$;

Let $\mathscr{H}$ be the set of finite nonempty subsets of I, inc *the inclusion of Ω into $X(\Omega/\bar{\theta}_i : i\in I)$:*

Z3. *for all $J\in\mathscr{H}$, $C_j\in\mathfrak{A}/\theta_j$:*

$$Q((\underset{j\in J}{X} C_j)\operatorname{pr}_J^{I-1} \operatorname{inc}^{-1})=\prod_{j\in J} Q/\theta_j(C_j));$$

Z4. *for all $J\in\mathscr{H}$, $C_j\in\mathfrak{A}/\theta_j$, $a\in\Omega$, $\lambda\in F$:*

$$\lambda(a,(\underset{j\in J}{X} C_j)\operatorname{pr}_J^{I-1} \operatorname{inc}^{-1})=\prod_{j\in J}\lambda([a]\theta_j, C_j).$$

Proof: We have to show that an embedding of $\underline{A}$ into $\otimes(\underline{A}/\theta_i : i\in I)$ exists, which respects the automaton transformations. A set inclusion

$$\operatorname{inc}: \Omega\to X(\Omega/\bar{\theta}_i : i\in I),\ a\to([a]\theta_i : i\in I),$$

is given by Z1:

$$a \operatorname{inc}=b \operatorname{inc}\Leftrightarrow[a]\bar{\theta}_i=[b]\bar{\theta}_i$$

for all

$$i\in I\Leftrightarrow(a, b)\in\bigcap(\bar{\theta}_i : i\in I)\Leftrightarrow a\,\omega\, b\Leftrightarrow a=b.$$

We have to show that $\otimes(Q/\theta_i : i\in I)$ is the image measure of Q under inc, i.e., for all $C\in\otimes(\mathfrak{A}/\theta_i : i\in I)$ we have $\otimes(Q/\theta_i : i\in I)(C)=Q(C \operatorname{inc}^{-1})$. For sets $C=X(C_j : j\in J)\operatorname{pr}_J^{I-1}$, $J\in\mathscr{H}$, $C_j\in\mathfrak{A}/\theta_j, j\in J$, this equation holds by Z3. But the collection of these sets is stable under finite intersections and generates $\otimes(\mathfrak{A}/\theta_i : i\in I)$.

In the same way we prove condition 6 (iii) from Z4.

Finally we must show that inc is (restricted on its image) bimeasurable. Let C be as above. Then $C \operatorname{inc}^{-1}=\bigcap(C_j \operatorname{nat} \bar{\theta}_j^{-1} : j\in J)$. So inc is measurable Let

$$g : \Omega\operatorname{inc}=X(\Omega/\bar{\theta}_i : i\in I)\cap\Omega\operatorname{inc}\to X(\Omega/\bar{\theta}_i : i\in I)$$

be the $(\otimes(\mathfrak{A}/\theta_i : i\in I)\cap\Omega\operatorname{inc}-\otimes(\mathfrak{A}/\theta_i : i\in I)$-measurable injection, $h : \Omega \operatorname{inc}\to\Omega$ the inverse of inc. Then for all $i\in I$ we have $h\circ\operatorname{nat}\bar{\theta}_i=g\circ\operatorname{pr}_{\{i\}}^I$ and the mapping on the right side of the equation is

$$\otimes(\mathfrak{A}/\theta_i)\cap\Omega\operatorname{inc}-\mathfrak{A}/\theta_i\text{- measurable.}$$

So from the $(\mathfrak{A}_{\theta_i}-\mathfrak{A}/\theta_i)$ measurability of nat θ_i and Z2, we conclude the desired measurability of h.

21. EXAMPLE: Given $\underline{B}=((\mathbb{R}^1, \mathfrak{B}^1), R; \{\mu\})$, where μ is as in example 8, and $R=(3/4)D\,0+(1/4)D\,3$. (Dx is the one-point distribution in x.) The congruence $\theta=(\bar{\theta}, \mathfrak{A}_\theta)$ defined in 8, is a congruence on $\underline{B}$ too.

We show that $\underline{B}/\theta$ is subtensorially irreducible.

Proof: Suppose there exists a set of congruences $(\theta_i : i \in I) \subseteqq \text{con}\,\underline{B}/\theta$, such that $\underline{B}/\theta$ is isomorphic to a subtensorproduct of the $(\underline{B}/\theta)/\theta_i$, $i \in I$. From associativity of tensorproduct, the definition of the infimum in $(\text{con}\,\underline{B}/\theta; \leqq)$ and theorem 20, we conclude that $\underline{B}/\theta$ is isomorphic to a subtensorproduct of two quotients $(\underline{B}/\theta)/\varphi_i$, $i=1, 2$.

Now one can prove (in an analoguous manner as in 12), that the following holds: $(\mathfrak{A}/\theta)_{\varphi_i} \supseteqq \sigma\{[x]\bar{\theta} \cup [-x]\bar{\theta} : x \in \mathbb{R}^1\}$, $i=1, 2$. So for $a, b \in \mathbb{N}$, $a \neq b$, the points $[a]\bar{\theta}$ and $[b]\bar{\theta}$ of $\mathbb{R}^1/\bar{\theta}$ are separated by $(\mathfrak{A}/\theta)_{\varphi_i}$, $i=1, 2$. From this we see that no injection from $\underline{B}/\theta$ into $(\underline{B}/\theta)/\varphi_1 \otimes (\underline{B}/\theta)/\varphi_2$ can exist.

22. REMARK: The decomposition scheme of 19 a, is the one we want to work with. If we add to this definition: "$\underline{A}=\otimes(\underline{A}_m : m \in M)$", we get the decomposition of $\underline{A}$ into a tensorproduct of quotient automata in 20. There we have to add some conditions to get an analogous theorem. The possibility of such a decomposition is of some practical importance:

If we have to calculate the state of a (finite) stochastic automaton $\underline{B}$ at time $t=n$, given its state distribution at time $t=0$, and if we have the decomposition of $\underline{B}$ into a tensorproduct $\underline{B}_1 \otimes \underline{B}_2$ of quotients, it is easy to see that we must carry out fewer multiplications in calculating the n-th state of the quotients first, and then the common distribution of $\underline{B}$, than in calculating the distribution of $\underline{B}$ directly. The difference of expenses increase with n. So (for large n) it would be worthwhile to work even with a subtensorproduct decomposition.

The aim of a decomposition theory is to get a decomposition of every object into irreducible factors, in our case: A stochastic automaton is isomorphic to a subtensorproduct of subtensorially irreducible quotients.

First we characterize the irreducible components:

23. LEMMA: *$\underline{A}$ is subtensorially irreducible iff $|\Omega|=1$ or for every set $\{\theta_i : i \in I\} \subseteqq \text{con}\,\underline{A}$ (with $\theta_i > \omega$ for all $i \in I$) which fulfill* Z1, Z3, Z4, *there is a $\varphi > \omega$ in* con $\underline{A}$, *such that $\varphi \leqq \theta_i$ holds for all $i \in I$.*

Proof: We show that for a set $\{\theta_i : i \in I\}$ under condition Z1, Z3, Z4 the infimum is representable as:

$$\wedge(\theta_i : i \in I)=(\bar{\omega}, \sigma(\bigcup(\mathfrak{A}_{\theta_i} : i \in I))).$$

For this it is sufficient to show that for any set E contained in the closure of $\bigcup(\mathfrak{A}_{\theta_i} : i \in I)$ under finite intersections the mapping $\lambda(., E) : \Omega \to [0, 1]$ is $\sigma(\bigcup(\mathfrak{A}_{\theta_i} : i \in I)) - \mathfrak{B}^1 \cap [0, 1]$-measurable.

Now $E = \bigcap(C_j : j \in J)$ for some $J \in \mathscr{H}$, $C_j \in \mathfrak{A}_{\theta_j}$, and there exist $D_j \in \mathfrak{A}/\theta_j$ such that $D_j = C_j$ nat $\bar{\theta}_j$, $j \in J$.

So:

$$x \in E \iff [x]\bar{\theta}_j \in D_j \quad \text{for:} \quad j \in J$$

$$\iff x \in (X(D_j : j \in J)\,\mathrm{pr}_J^{I-1})\,\mathrm{inc}^{-1}.$$

For $t \in [0, 1]$ we have:

$$\begin{aligned}\{a \in \Omega : \lambda(a, E) \leqq t\} &= \{a \in \Omega : \lambda(a, X(D_j : j \in J)\,\mathrm{pr}_J^{I-1}\,\mathrm{inc}^{-1}) \leqq t\} \\ &= \{a \in \Omega : \textstyle\prod(\lambda([a]\bar{\theta}_j, D_j) : j \in J) \leqq t\} \\ &= \{a \in \Omega : \textstyle\prod(\lambda(a, C_j) : j \in J) \leqq t\} \subseteqq \sigma(\bigcup_{\theta_i} : i \in I)),\end{aligned}$$

because every $\lambda(., C_j)$ is $\mathfrak{A}_{\theta_j} - \mathfrak{B}^1 \cap [0, 1]$-measurable, hence $\sigma(\bigcup(\mathfrak{A}_{\theta_i} : i \in I)) - \mathfrak{B}^1 \cap [0, 1]$-measurable.

Now sufficiency and necessity are proved by way of contradiction:

(i) suppose $(\theta_i : i \in I) \subseteqq \mathrm{con}\ \underline{A}$ has properties Z1, Z3, Z4 and $\underline{A}$ is subtensorially irreducible. Then $\wedge(\theta_i : i \in I) = (\omega, \overline{\sigma(\bigcup(\mathfrak{A}_{\theta_i} : i \in I))}) > \omega$ because from $\wedge(\theta_i : i \in I) = \omega$ we would have Z2 and a possible decomposition of $\underline{A}$;

(ii) suppose for every set $(\theta_i : i \in I) \subseteqq \mathrm{con}\ \underline{A}$ which has properties Z1, Z3, Z4, there exists some $\varphi \in \mathrm{con}\ \underline{A}$ with $\omega < \varphi \leqq \theta_i$ for all $i \in I$ and $\underline{A}$ is not subtensorially irreducible. Then there is some set $(\theta_k : k \in K) \subseteqq \mathrm{con}\ \underline{A}$ such that $\underline{A}$ is isomorphic to the subtensorproduct of the $\underline{A}/\theta_k$, $k \in K$.

From theorem 20 $(\theta_k : k \in K)$ fulfils Z1, Z3, Z4, and therefore there exists a $\varphi \in \mathrm{con}\ \underline{A}$, with $\omega < \varphi \leqq \theta_k$, $\mathfrak{A}_\varphi \neq \mathfrak{A}$. So Z2 does not hold.

In 15, we characterized the quotients $\underline{A}/\varphi(B)$, defined by the c.d c congruences $\varphi(B)$, $B \in \mathfrak{A}$, as those automata, which are the smallest one to investigate B directly.

While for deterministic automata the greatest congruence containing a given pair (a, b) of states generates a subdirectly irreducible quotient, this is not true for our automata in general.

24. EXAMPLE: Given a stochastic automaton $\underline{A} = ((\mathbb{R}^1, \mathfrak{B}^1), \exp\lambda; \{\lambda_0, \lambda_1\})$, where $\exp\lambda$ is the exponential distribution with parameter λ, and:

$$\lambda_0: \quad \mathbb{R}^1 \times \mathfrak{B}^1 \to [0, 1],$$
$$(x, C) \to D0(C),$$
$$\lambda_1: \quad \mathbb{R}^1 \times \mathfrak{B}^1 \to [0, 1],$$
$$(x, C) \to \begin{cases} D0(C), & x > 0, \\ \int_C \lambda^1 \, dt \, \lambda e^{-\lambda t} 1_{[0, \infty)}(t), & x \leqq 0, \end{cases}$$

Computing the c.d.c. congruence $\varphi(\{0\}) = (\overline{\varphi(\{0\})}, \mathfrak{A}_{\{0\}})$, we get $\mathfrak{A}_{\{0\}} = \sigma([0, \infty), \{0\})$.

The quotient automaton:

$$\underline{A}/\varphi(\{0\}) = \{(A/\overline{\varphi(\{0\})}, \mathfrak{A}/\varphi(\{0\})), \quad (\exp\lambda) \text{ nat } \overline{\varphi(\{0\})}; \{\lambda_0, \lambda_1\}\},$$

is isomorphic to a subtensorproduct of two quotients:

$$(\underline{A}/\varphi(\{0\}))/\theta_i, \qquad i = 1, 2,$$

where:

$$(\mathfrak{A}/\varphi(\{0\}))_{\theta_1} = \sigma([(-\infty, 0)] \overline{\varphi(\{0\})})$$

and:

$$(\mathfrak{A}/\varphi(\{0\}))_{\theta_2} = \sigma([(0, \infty)] \overline{\varphi(\{0\})}),$$

and $\overline{\theta}_i$ is generated by $(\mathfrak{A}/(\{\varphi\}))/\theta_i$, $i = 1, 2$ (the proof is an easy checking of the $((\mathfrak{A}/\varphi(\{0\}))_{\theta_i} - \mathfrak{B}^1 \cap [0, 1])$ measurability of $\lambda_j(., D)$, $i, j = 1, 2$, and of Z1, Z2, Z3, Z4.) We see that none of the two quotients of $\underline{A}/\varphi(\{0\})$ contain $[\{0\}]\varphi(\{0\})$ in $(\mathfrak{A}/\varphi(\{0\}))_{\theta_i}$.

This "irregularity" of stochastic automata shown in 24, enables us to decompose the automaton $\underline{A}/\varphi(B)$ into smaller components. Now the question arises whether we can decompose every $\underline{A}/\varphi(B)$ in such a way as example 24, looks like:

The quotient as a single automaton does not reflect properties of B, only their common structure gives us information about B. The important fact is this: We get information on B without measuring B itself, only working on smaller quotients. Saying it in other words the next theorem will prove the following:

If in some physical system described by a stochastic automaton a hidden event B should be investigated and if the smallest quotient system which includes B is not irreducible then we are able to measure B indirectly. We factorize the smallest quotient including B such that none of the factors includes B itself — and from the knowledge of the factors we get the whole information about B.

22. THEOREM: *If for* $B \in \mathfrak{A}$ *the quotient* $\underline{A}/\varphi(B)$ *is not subtensoriel irreducible, and if* $\underline{A}/\varphi(B)$ *is a subtensorproduct of the* $(A/\varphi(B))/\theta_i$, $i \in I$, $(\theta_i > \omega)$, *then* B nat $\overline{\varphi(B)} \notin (\mathfrak{A}/\varphi(B))_{\theta_i}$ *for all* $i \in I$.

Proof: By a corollary to the II. Isomorphism theorem [9, 8] there exists a lattice isomorphism:

$$m: \quad (\text{con } \underline{A}/\varphi(B), \leqq) \to ([\varphi(B)), \leqq),$$
$$\varphi \quad \to \quad \varphi',$$

where $[\varphi(B)) = \{\theta \in \text{con } \underline{A} : \theta \geqq \varphi(B)\}$ is the dual ideal generated by $\varphi(\underline{B})$ in $(\text{con } \underline{A}, \leqq)$ [*see* (11)] and for $\varphi \in \text{con } \underline{A}/\varphi(B)$ $\varphi' \in \text{con } \underline{A}$ is defined by:
$a, b \in \Omega$, then

$$a\,\varphi', b \;\Leftrightarrow\; [a]\overline{\varphi(B)}\,\overline{\varphi}\,[b]\overline{\varphi(B)} \qquad \text{and} \qquad \mathfrak{A}_{\varphi'} = (\mathfrak{A}/\varphi(B))_{\varphi} \text{ nat } \varphi(B)^{-1}.$$

[If for example $\omega \in \text{con } \underline{A}/\varphi(B)$ is the least congruence on $\underline{A}/\varphi(B)$ then $m(\omega) = \omega' = \varphi(B)$.]

Now from $\theta_i > \omega$ in the lattice $(\text{con } \underline{A}/\varphi(\underline{B}), \leqq)$ we derive in $([\varphi(B), \leqq)$ [which is a sublattice of $(\text{con } \underline{A}, \leqq)$,] the following relation:

$$(\star) \qquad \theta_i' = (\bar{\theta}_i', \mathfrak{A}_{\theta_i'}) = (\bar{\theta}_i', \mathfrak{A}/\varphi(B))_{\theta_i} \text{ nat } \overline{\varphi(B)}^{-1}) > (\overline{\varphi(B)}, \mathfrak{A}_B).$$

If B nat $\overline{\varphi(B)} \in (\mathfrak{A}/\varphi(B))_{\theta_i}$, we get from 9 b, and $(\star)$ $\mathfrak{A}_B = \mathfrak{A}_{\theta_i}'$. If $\bar{\theta}_i' = \overline{\varphi(B)}$, we have a contradiction to $\omega < \theta_i$, and if $\bar{\theta}_i' > \overline{\varphi(B)}$, we have $(\bar{\theta}_i', \mathfrak{A}_B) > (\overline{\varphi(B)}, \mathfrak{A}_B)$, a contradiction to the definition of $\varphi(B)$.

The general problem of decomposition theory: "Is every stochastic automaton isomorphic to some subtensorproduct of a set of subtensorially irreducible quotients ", sems to be open up to now.

In connection with the problem of theorem 25, it should be said that there are stochastic automata which are decomposable into factors which are of the form $\varphi(B)$ for every event B of the factor σ-algebra, such that each factor is subtensorially irreducible.

A partial answer for the decomposition question was given in [2] for the case of stochastic automata having Noetherian lattice of congruences, i.e.: every ascending chain in the lattice terminates.

26. THEOREM: *If* (con $\underline{A}$, $\leqq$) *is a Noetherian lattice, then* $\underline{A}$ *is isomorphic to a subtensorproduct of subtensorial irreducible quotient automata.*

REFERENCES

1. J. ASENDORPF, *Reduktion nichtdiskreter stochastischer Automaten*, Diplomarbeit, Kiel, 1973.
2. H DADUNA, *Stochastic Algebras and Stochastic Automata over General Measurable Spaces: Algebraic Theory and a Decomposition Theorem;* in: *Fundamentals of Computation Theory*, Lect. Notes Comp. Sc., Vol. 56, 1977, pp 72-77, Berlin, Heidelberg, New York.
3. H. DADUNA, *Stochastische Algebra, Theorie und Anwendungsbeispiele*, Dissertation, Hamburg, 1979
4. E.-E. DOBERKAT, *Gute Strategien für bewertete Zustands-automaten über metrischen Räumen*, to appear in: Math.-Phys, Semesterberichte, Preprint, 1978.
5. E.-E. DOBERKAT, *Stochastic behavior of Initial Nondeterministic Automata*, to appear in: Proc. Fourth European Meeting on Cybernetics and Systems Research, Linz, 1978, Preprint, 1978.
6. E.-E. DOBERKAT, *Convergence Theorems for Stochastic automata and Learning Systems*, Preprint, 1978.
7. E.-E. DOBERKAT, *On a representation of Measurable Automaton-Transformations by Stochastic Automata*, To appear in: J. Math. Anal. Appl., Preprint, 1978.
8. G. GRATZER, *Universal Algebra*, Princeton, N.J., 1968.
9. K. HINDERER, *Foundations of Non-Stationary Dynamic Programming*, Berlin, Heidelberg, New York, 1970.
10. M. IOSIFESCU and R. THEODORESCU, *Random Processes and Learning*, Berlin, Heidelberg, New York, 1969.
11. L. A. RASTRIGIN and K. K. RIPA, *Statistical Search as a Probability Automaton*, Avtomatika i Vychislitel'naya Tekhnika, Vol. 5, No. 1, 1971, pp. 50-55.