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ON THE STRUCTURE OF IDEMPOTENT SEMIGROUPS

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ABSTRACT. An idempotent semigroup (band) is a semigroup in which every element is an idempotent. We describe the structure of idempotent semigroups in terms of semilattices Ω, partial chains Ω of left zero semigroups, and partial chains Ω of right zero semigroups. We also describe bands of maximal left zero semigroups in terms of partial chains Ω of left zero semigroups and semilattices Ω of right zero semigroups.

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Unless otherwise specified we employ the definitions and notation of [2]. The following theorem is a starting point in the proof of both of our structure theorems.

THEOREM 1 (CLIFFORD [1], MCLEAN [3]). *Let E be an idempotent semigroup. Then, E is a semilattice Ω of rectangular bands $(E_\delta : \delta \in \Omega)$.*

We begin by introducing the following concepts.

Let W be a partial groupoid which is a union of a collection of pairwise disjoint subsemigroups $(T_\delta : \delta \in \Lambda)$ where Λ is a semilattice. If $x \in T_\nu$, $y \in T_\delta$, and $\delta \leqq \nu$ (in Λ) imply xy is defined (in W) and $xy \in T_\delta$ and if $\xi \leqq \delta$ and $z \in T_\xi$ imply $(xy)z = x(yz)$, W is termed a (lower) partial chain Λ of the semigroups $(T_\delta : \delta \in \Lambda)$. If $x \in T_\nu$, $y \in T_\delta$, and $\nu \leqq \delta$ imply xy is defined (in W) and $xy \in T_\nu$, and $\xi \geqq \delta$ and $z \in T_\xi$ imply $(xy)z = x(yz)$, W is termed an (upper) partial chain of the semigroups $(T_\delta : \delta \in \Lambda)$.

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We are now in a position to give the first theorem.

Let Ω be a semilattice, let I be a (lower) partial chain Ω of left zero semigroups $(I_\delta:\delta\in\Lambda)$, and let J be an (upper) partial chain Ω of right zero semigroups $(J_\delta:\delta\in\Lambda)$. Let α be a mapping of $J\times I$ into I and let β be a mapping of $J\times I$ into J subject to the conditions:

I. If $r, s\in\Omega$, $(J_r\times I_s)\alpha\subseteq I_{rs}$ and $(J_r\times I_s)\beta\subseteq J_{rs}$.

II. If $j\in J_s$, $p\in I_t$, $q\in J_t$, and $m\in I_g$,

$$(j,p)\alpha((j,p)\beta q,m)\alpha=(j,p((q,m)\alpha))\alpha \quad \text{and}$$

$$(j,p((q,m)\alpha))\beta(q,m)\beta=((j,p)\beta q,m)\beta.$$

Let $(\Omega,I,J,\alpha,\beta)$ denote $\bigcup(I_s\times J_s:s\in\Omega)$ under the multiplication $(i,j)(p,q)=(i((j,p)\alpha),(j,p)\beta q)$.

THEOREM 2.[1] *E is an idempotent semigroup if and only if $E\cong(\Omega,I,J,\alpha,\beta)$ for some collection Ω, I, J, α, β.*

PROOF. Let E be an idempotent semigroup. Select and fix an $\mathscr{L}$-class I_δ of E_δ and select and fix an $\mathscr{R}$-class J_δ of E_δ ($\mathscr{L}$ and $\mathscr{R}$ are Green's relations [**2**]). Thus every element of E may be expressed uniquely in the form $x=ij$ where $i\in I_\delta$ and $j\in J_\delta$ for some $\delta\in\Omega$. If $e\in I_\delta$, $f\in I_\nu$ and $\nu\leqq\delta$, $(ef,f)\in\mathscr{L}$ $(\in E_\nu)$ and, hence, $ef\in I_\nu$. Let $I=\bigcup(I_\delta:\delta\in\Omega)$ and, if $a,b\in I$, define $a\circ b=ab$ (product in E) if $ab\in I$ while $a\circ b$ is undefined if $ab\,\bar{\in}\, I$. Hence, the partial groupoid $(I,\circ)$ is a (lower) partial chain Ω of left zero semigroups $(I_\delta:\delta\in\Omega)$ (since no confusion will arise, we replace "$\circ$" by juxtaposition). Similarly, $J=\bigcup(J_\delta:\delta\in\Omega)$ is an (upper) partial chain Ω of right zero semigroups $(J_\delta:\delta\in\Omega)$. We may define a mapping α of $J\times I$ into I and a mapping β of $J\times I$ into J satisfying I by the expression $ji=(j,i)\alpha(j,i)\beta$ where $j\in J_r$ and $i\in I_s$, say. If $j\in J_s$, $p\in I_t$, $q\in J_t$, and $m\in I_g$,

$$((jp)q)m=(j,p)\alpha((j,p)\beta qm)=(j,p)\alpha((j,p)\beta q,m)\alpha((j,p)\beta q,m)\beta$$

and

$$j(p(qm))=j(p((q,m)\alpha(q,m)\beta))=(jp((q,m)\alpha)).$$

[1] The referee informs me that a result similar to Theorem 2 has also been obtained by Petrich (unpublished).

$(q, m)\beta = (j, p((q, m)\alpha))\alpha(j, p((q, m)\alpha))\beta(q, m)\beta$ and II follows. Since

$$(ij)(pq) = i(jp)q = (i((j, p)\alpha))((j, p)\beta q)$$

for $i \in I_s, j \in J_s, p \in I_t$, and $q \in J_t$, say, $(ij)\varphi = (i, j)$ defines an isomorphism of S onto $(\Omega, I, J, \alpha, \beta)$. We next show that $T = (\Omega, I, J, \alpha, \beta)$ is a band. We utilize I to establish closure and II to establish associativity while $(i, j) \in I_s \times J_s$ implies $(i, j)^2 = (i, j)$ by a routine calculation.

We will need the following definition.

A partial transformation λ of a partial groupoid W is termed an inner left translation of W determined by $e \in W$ if the domain D of λ is the set of $s \in W$ such that es is defined and $s\lambda = es$, $\forall\, s \in D$. We write $\lambda = \lambda_e$.

We next give a structure theorem for bands of maximal left zero semigroups.

Let X be a semilattice Ω of right zero semigroups $(X_\delta : \delta \in \Omega)$. For each $\delta \in \Omega$, select $e_\delta \in X_\delta$ and let $B = (e_\delta : \delta \in \Omega)$. Under the order, $e_\beta \leqq e_\delta$ if $e_\delta e_\beta = e_\beta$, B is a semilattice order isomorphic to Ω. Let W be a (lower) partial chain B of left zero semigroups $(T_{e_\delta} : e_\delta \in B)$. For each $s \in X_\delta$, let $s' = e_\delta$. Let $r \to \alpha_r$ be a mapping of X into $\mathscr{T}_W$, the full transformation semigroup on W, subject to the conditions:

I(a) $T_{e_\delta}\alpha_r \subseteq T_{(re_\delta)'}$;

(b) $(g_{e_\delta} h_{e_\beta})\alpha_r = (g_{e_\delta}\alpha_r)(h_{e_\beta}\alpha_{re_\delta})$ for $g_{e_\delta} \in T_{e_\delta}$, $h_{e_\beta} \in T_{e_\beta}$, and $e_\beta \leqq e_\delta$.

II. $\alpha_{st}\lambda_e = \alpha_t\alpha_s$ for all $e \in T_{(st)'}$ where λ_e is the inner left translation of W determined by e.

Let (X, W, α) denote $\{(g_{s'}, s) : s \in X, g_{s'} \in T_{s'}\}$ under the multiplication $(g_{s'}, s)(h_{t'}, t) = (g_{s'}(h_{t'}\alpha_s), st)$.

THEOREM 3. *E is a band of maximal left zero semigroups if and only if $E \cong (X, W, \alpha)$ for some collection X, W, α.*

PROOF. Let E be a band b of maximal left zero semigroups. Hence, $b = \mathscr{L}$ and $E/\mathscr{L} = X$ is a semilattice Ω of right zero semigroups $(X_\delta : \delta \in \Omega)$ where $X_\delta = E_\delta\mathscr{L}$. If $T_s = s\mathscr{L}^{-1}$ $(s \in X)$, $(T_s : s \in X)$ is the collection of $\mathscr{L}$-classes of E with $T_sT_t \subseteq T_{st}$. Let u_s be a representative element for T_s. For each $\delta \in \Omega$, select $e_\delta \in X_\delta$, and, if $s \in X_\delta$, let $s' = e_\delta$. Hence, every element of E may be uniquely expressed in the form $x = g_{s'}u_s$ where $g_{s'} \in T_{s'}$. If we let $B = (e_\delta : \delta \in \Omega)$, then, under the order $e_\beta \leqq e_\delta$ if $e_\delta e_\beta = e_\beta$, B is a semilattice order isomorphic to Ω. As above, $W = \bigcup\, (T_{e_\delta} : e_\delta \in B)$ is a (lower) partial chain B of left zero semigroups $(T_{e_\delta} : e_\delta \in B)$. For each $r \in X$, the expression $u_r g_{e_\delta} = (g_{e_\delta}\alpha_r)u_{re_\delta}$ defines a unique $\alpha_r \in \mathscr{T}_{W}$ satisfying I(a). We obtain I(b) from the expression

$$(g_{e_\delta}h_{e_\beta})\alpha_r u_{re_\delta e_\beta} = u_r(g_{e_\delta}h_{e_\beta}) = (u_r g_{e_\delta})h_{e_\beta}$$
$$=(g_{e_\delta}\alpha_r)(u_{re_\delta}h_{e_\beta}) = (g_{e_\delta}\alpha_r)(h_{e_\beta}\alpha_{re_\delta})u_{re_\delta e_\beta}$$

where $e_\beta \leqq e_\delta$. We may write $u_s u_t = f_{s,t}u_{st}$ where $f_{s,t} \in T_{(st)'}$. Hence, we obtain II from the expression

$$f_{s,t}(g_{z'}\alpha_{st})u_{stz'} = f_{s,t}(u_{st}g_{z'}) = (f_{s,t}u_{st})g_{z'} = u_s(u_t g_{z'}) = u_s(g_{z'}\alpha_t u_{tz'})$$
$$= g_{z'}\alpha_t\alpha_s u_{s(tz')'}u_{tz'} = g_{z'}\alpha_t\alpha_s f_{s(tz')',tz'}u_{stz'} = g_{z'}\alpha_t\alpha_s u_{stz'}.$$

The last equality follows since $g_{z'}\alpha_t\alpha_s$ and $f_{s(tz')',tz'}$ are both contained in the same $\mathscr{L}$-class of E. We have

$$(g_{s'}u_s)(h_{t'}u_t) = g_{s'}(u_s h_{t'})u_t = g_{s'}(h_{t'}\alpha_s)u_{st'}u_t = g_{s'}(h_{t'}\alpha_s)f_{st',t}u_{st}$$
$$= g_{s'}(h_{t'}\alpha_s)u_{st}.$$

The last equality follows since $h_{t'}\alpha_s$ and $f_{st',t}$ are contained in the same $\mathscr{L}$-class of E. Hence, $(g_{s'}u_s)\varphi = (g_{s'}, s)$ defines an isomorphism of S onto (X, W, α). Next, we show that (X, W, α) is a band of maximal left zero semigroups. We utilize I(a) to establish a closure and I(b) and II to establish associativity. If we let $L_s = ((g_{s'}, s): g_{s'} \in T_{s'})$, E is the band X of maximal left zero semigroups $(L_s : s \in X)$.

REMARK. Using the previous proof, E is a band of maximal left zero semigroups if and only if E is a band and $\mathscr{L}$ is a congruence on E.

REMARK (ADDED IN PROOF). We may show that "left zero semigroups" may be replaced by "left groups" in Theorem 3 provided we make the following modifications: In the definition of W, replace "left zero semigroups" by "left groups". Let $(r, s) \rightarrow f_{r,s}$ be a mapping of X^2 into W. Replace II by the condition II$'$ $f_{s,t}(g_{z'}\alpha_{st}) = g_{z'}\alpha_t\alpha_s f_{s(tz')',tz'}$, where $g_{z'} \in T_{z'}$. Add the conditions: I(c) $f_{k,r} \in T_{(kr)'}$; I(d) $f_{s',s} \in E(T_{s'})$, the set of idempotents of $T_{s'}$; I(e) if $s \in X$, there exists $g_{s'} \in E(T_{s'})$ such that $g_{s'}\alpha_s \in E(T_{s'})$; III $f_{s,t}f_{st,z} = f_{t,z}\alpha_s f_{s(tz)',tz}$. The multiplication becomes $(g_{s'},s)(h_{t'}, t) = (g_{s'}(h_{t'}\alpha_s)f_{st',t}, st)$. A proof is given in [4]. A semigroup E is a band of maximal left groups if and only if E is a union of groups and $\mathscr{L}$ is a congruence on E.

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