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ON THE STRUCTURE OF JORDAN *-DERIVATIONS

BY

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1. Introduction. Let R be a *-ring, i.e., a ring with involution *. An additive mapping E from R to R is called a *Jordan* **-derivation* if

$$E(x^2) = E(x)x^* + xE(x) \quad \text{for all } x \in R\,.$$

Note that the mapping $x \to ax^* - xa$, where a is a fixed element in R, is a Jordan *-derivation; such Jordan *-derivations are said to be *inner*.

The study of Jordan *-derivations has been motivated by the problem of the representativity of quadratic forms by bilinear forms (for the results concerning this problem we refer to [8, 12, 14–16]). It turns out that the question whether each quadratic form can be represented by some bilinear form is intimately connected with the question whether every Jordan *-derivation is inner, as shown by Šemrl [14].

In [4] Brešar and Vukman studied some algebraic properties of Jordan *-derivations. As a special case of [4; Theorem 1] we have that every Jordan *-derivation of a complex algebra A with unit element is inner. Clearly, the requirement that A must contain the unit element cannot be omitted—for example, if A is a self-adjoint ideal in an algebra B, then the mapping $x \to bx^* - xb$, where $b \in B$, is a Jordan *-derivation of A which is not necessarily inner. In this paper we prove that Jordan *-derivations of a rather wide class of complex *-algebras (in general without unit) can be represented by double centralizers (Theorem 2.1). As an application we obtain a result on automatic continuity of Jordan *-derivations (Corollary 2.3). As another application we determine the structure of Jordan *-derivations on the algebra of all compact linear operators on a complex Hilbert space (Corollary 2.4).

Roughly speaking, it is much more difficult to study Jordan *-derivations on real algebras than on complex algebras. Nevertheless, in [13] Šemrl showed that every Jordan *-derivation of $B(H)$, the algebra of all bounded linear operators on a real Hilbert space H ($\dim H > 1$), is inner. In the present paper, using a completely different approach, we give a new proof of this result. Our proof is based on two well-known results. The first is from algebra (due to Martindale, concerning Jordan derivations of the symmetric elements of a *-ring), while the second is from analysis (due to Chernoff, stating that all derivations on $B(H)$ are inner). In fact, throughout this paper we combine algebraic and analytic methods.

2. Jordan *-derivations of complex *-algebras.

Let A be an algebra (resp. a ring). A linear (resp. additive) mapping T from A to A is called a *left centralizer* of A if $T(xy) = T(x)y$ for all $x, y \in A$. Analogously, a linear (resp. additive) mapping S from A to A satisfying $S(xy) = xS(y)$ for all $x, y \in A$ is called a *right centralizer* of A. For T a left centralizer of A and S a right centralizer of A, the pair (S,T) is called a *double centralizer* of A if $xT(y) = S(x)y$ for all $x, y \in A$.

Let A be a *-ring. Note that every double centralizer (S,T) of A induces a Jordan *-derivation E, defined by $E(x) = T(x^*) - S(x)$. In the following theorem we show that in certain complex *-algebras all Jordan *-derivations are induced in such a way.

THEOREM 2.1. *Let A be a complex *-algebra such that $Aa = 0$ or $aA = 0$ (where $a \in A$) implies $a = 0$. If E is a Jordan *-derivation of A then there exists a unique double centralizer (T,S) such that $E(x) = T(x^*) - S(x)$ for all $x \in A$.*

Obviously, as a special case of Theorem 2.1 we obtain the known result stating that all Jordan *-derivations of a complex *-algebra with unit are inner.

Proof of Theorem 2.1. Define an additive mapping S_1 of A by $S_1(x) = 2E(ix) + 2iE(x)$. We have

$$\begin{aligned}
&S_1(x^2) - xS_1(x)\\
&\quad = 2E(ix^2) + 2iE(x^2) - 2xE(ix) - 2ixE(x)\\
&\quad = E((1+i)^2x^2) + 2iE(x)x^* + 2ixE(x) - 2xE(ix) - 2ixE(x)\\
&\quad = E(x+ix)(x^* - ix^*) + (x+ix)E(x+ix) + 2iE(x)x^* - 2xE(ix)\\
&\quad = \{E(x)x^* + xE(x) - iE(ix)x^* + ixE(ix)\}\\
&\qquad + i\{E(x)x^* + xE(x) - iE(ix)x^* + ixE(ix)\}
\end{aligned}$$

Expanding the identity $E(x^2) = -E((ix)^2)$ we obtain

$$E(x)x^* + xE(x) = iE(ix)x^* - ixE(ix)\,,$$

and therefore $S_1(x^2) = xS_1(x)$. In a similar fashion we see that the mapping T_1 of R, defined by $T_1(x) = 2iE(x^*) - 2E(ix^*)$, satisfies $T_1(x^2) = T_1(x)x$. Now, define $T = -\frac{1}{4}iT_1$ and $S = \frac{1}{4}iS_1$. Clearly, $E(x) = T(x^*) - S(x)$ for every x in A. We claim that (T,S) is a double centralizer of A. Let us first verify that $xT(x^*) = S(x)x^*$ for all $x \in A$. We have

$$\begin{aligned} xT(x^*) - S(x)x^* &= xE(x) + xS(x) - T(x^{*2}) + T(x^*)x^* - S(x)x^* \\ &= xE(x) - (-S(x^2) + T(x^{*2})) + (T(x^*) - S(x))x^* \\ &= xE(x) - E(x^2) + E(x)x^* = 0\,. \end{aligned}$$

Linearizing $xT(x^*) = S(x)x^*$ (i.e., replacing x by $x + y$) we get

$$xT(y^*) + yT(x^*) = S(x)y^* + S(y)x^*\,. \tag{1}$$

From the definition of S and T we see that $S(ix) = iS(x)$ and $T(ix) = iT(x)$ for all $x \in A$. Therefore, replacing y by iy in (1) we obtain

$$-ixT(y^*) + iyT(x^*) = -iS(x)y^* + iS(y)x^*\,.$$

Comparing this identity with (1) we see that $xT(y) = S(x)y$ for all $x, y \in A$. Consequently,

$$xT(yz) = S(x)yz = xT(y)z\,,$$

that is, $A(T(yz) - T(y)z) = 0$. By hypothesis, this implies that $T(yz) = T(y)z$. Similarly we see that T is linear; namely, $xT(\lambda y) = S(x)\lambda y = x\lambda T(y)$. Thus T is a left centralizer of A. Analogously one shows that S is a right centralizer of A. Thus the pair (T, S) is a double centralizer of A.

In order to prove that T and S are uniquely determined we assume that $L(x^*) = R(x)$ where L is a left and R is a right centralizer of A. Then

$$L(y^*)x^* = L(y^*x^*) = L((xy)^*) = R(xy)$$

for all $x, y \in R$. Replacing y by iy yields $-iL(y^*)x^* = iR(xy)$. But then, comparing the last two relations we obtain $L(y^*)x^* = 0$ for all $x, y \in A$, that is, $L(A)A = 0$, which yields $L = 0$, and, therefore, $R = 0$. This completes the proof of the theorem.

As an immediate consequence of Theorem 2.1 we obtain the following result which can be compared with [4; Corollary 1].

COROLLARY 2.2. *Let A be a complex *-algebra such that $Aa = 0$ or $aA = 0$ implies $a = 0$. Then every Jordan *-derivation of A is real linear.*

COROLLARY 2.3. *Let A be a complex Banach *-algebra such that $Aa = 0$ or $aA = 0$ implies $a = 0$. If the involution is continuous then every Jordan *-derivation of A is continuous.*

We remark that every semisimple Banach *-algebra satisfies the requirements of Corollary 2.3 (see [1; p. 191]).

Proof of Corollary 2.3. By Theorem 2.1, it suffices to show that every one-sided centralizer of A is continuous. Let T be a left centralizer. Suppose that $x_n, y \in A$ with $\lim_{n\to\infty} x_n = 0$, $\lim_{n\to\infty} T(x_n) = y$. By the closed graph theorem, it is enough to prove that $y = 0$. Given any $a \in A$, we have $ay = \lim_{n\to\infty} aT(x_n) = \lim_{n\to\infty} S(a)x_n = 0$. Hence $y = 0$. In a similar fashion one shows that every right centralizer of A is continuous.

Combining Theorem 2.1 with [6; Theorem 3.9] we obtain

COROLLARY 2.4. *Let A be the algebra of all compact linear operators on a complex Hilbert space H. Then every Jordan *-derivation of A is of the form $x \to ax^* - xa$ for some bounded linear operator a on H.*

It is an open question whether Corollary 2.4 remains true in the real case.

In the proof of Theorem 2.1, there occur additive mappings S, T satisfying $S(x^2) = xS(x)$, $T(x^2) = T(x)x$. The question arises whether S (resp. T) is then necessarily a right (resp. left) centralizer. Using a similar approach to [2, 3, 5], where some Jordan mappings are considered, we now prove

PROPOSITION 2.5. *Let R be a prime ring of characteristic not 2. If an additive mapping $T : R \to R$ satisfies $T(x^2) = T(x)x$ for all $x \in R$, then T is a left centralizer of R. Similarly, if an additive mapping $S : R \to R$ satisfies $S(x^2) = xS(x)$ for all $x \in R$, then S is a right centralizer of R.*

Recall that a ring R is said to be *prime* if $aRb = 0$ implies $a = 0$ or $b = 0$.

Proof of Proposition 2.5. Linearizing $T(x^2) = T(x)x$ we get

$$T(xy + yx) = T(x)y + T(y)x \quad \text{for all } x, y \in R. \tag{2}$$

In particular,

$$T(x(xy + yx) + (xy + yx)x) = T(x)(xy + yx) + (T(x)y + T(y)x)x.$$

But on the other hand,

$$\begin{aligned} T(x(xy + yx) + (xy + yx)x) &= T(x^2y + yx^2) + 2T(xyx) \\ &= T(x^2)y + T(y)x^2 + 2T(xyx) = T(x)xy + T(y)x^2 + 2T(xyx). \end{aligned}$$

Comparing the last two relations we arrive at $2T(xyx) = 2T(x)yx$. Since the characteristic of R is not 2, it follows that

$$T(xyx) = T(x)yx \quad \text{for all } x, y \in R. \tag{3}$$

A linearization of (3) gives

$$T(xyz + zyx) = T(x)yz + T(z)yx \quad \text{for all } x, y, z \in R. \tag{4}$$

Now, analogously to [2; Theorem 3], [3; Lemma 2.1] and [5; Proposition 3] we consider $W = T(xyzyx + yxzxy)$. According to (3) we have

$$W = T(x(yzy)x) + T(y(xzx)y) = T(x)yzyx + T(y)xzxy.$$

On the other hand, we see from (4) that

$$W = T((xy)z(yx) + (yx)z(xy)) = T(xy)zyx + T(yx)zxy\,.$$

Comparing the two expressions for W and applying (2), we then get

$$(T(xy) - T(x)y)z(xy - yx) = 0 \qquad \text{for all } x, y, z \in R\,. \tag{5}$$

Since R is prime, for any $x, y \in R$ we have either $T(xy) = T(x)y$ or $xy = yx$. In other words, given $x \in R$, R is the union of its subsets $G_x = \{y \in R \mid T(xy) = T(x)y\}$ and $H_x = \{y \in R \mid xy = yx\}$. Clearly G_x and H_x are additive subgroups of R. However, a group cannot be the union of two proper subgroups, therefore either $G_x = R$ or $H_x = R$. Thus we have proved that R is the union of its subsets $G = \{x \in R \mid T(xy) = T(x)y$ for all $y \in R\}$ and $H = \{x \in R \mid xy = yx$ for all $y \in R\}$. Of course, G and H are also additive subgroups of R. Hence either $G = R$, i.e., T is a left centralizer, or $H = R$, i.e., R is commutative.

Thus, we may assume that R is commutative. Then, of course, R is a domain. Following the proofs of [9; Lemma 2.2] and [5; Theorem 2] we consider $V = 2T(x^2y)$. By (3) we have $V = 2T(xyx) = 2T(x)yx$. However, from (2) we see that $V = T(x^2y+yx^2) = T(x)xy+T(y)x^2$. Comparing both expressions we obtain $(T(x)y - T(y)x)x = 0$. Since R is a domain it follows that $x = 0$ or $T(x)y = T(y)x$; in any case $T(x)y = T(y)x$. Therefore, (2) yields $2T(xy) = 2T(x)y$. Since the characteristic of R is not 2, this means that T is a left centralizer.

Similarly one proves that S is a right centralizer.

3. Jordan *-derivations of $B(H)$. Throughout this section, H will be a Hilbert space such that $\dim H > 1$. We denote by $B(H)$ the algebra of all bounded linear operators on H, and by $S(H)$ the set of all self-adjoint operators in $B(H)$. Our main purpose in this section is to give a new proof of the following theorem of Šemrl [13].

THEOREM 3.1. *If H is a real Hilbert space then every Jordan *-derivation of $B(H)$ is inner.*

Recall that an additive mapping D of a ring R into itself is called a *derivation* if it satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. If R is an algebra and D is a derivation of R which is not necessarily homogeneous, then D will be called an *additive derivation*; otherwise we call D a *linear derivation.*

Outline of the proof of Theorem 3.1. Let E be a Jordan *-derivation of $B(H)$. Using the theorem of Martindale quoted below, we show that there exists an additive derivation D of $B(H)$ such that the restrictions of D and E to $S(H)$ coincide. It turns out that D is in fact linear, therefore, by the well-known theorem of Chernoff [7], D is inner, i.e., $D(A) = TA - AT$ for some $T \in B(H)$. Finally, we show that there is a real number μ such that $E(A) = (T + \mu I)A^* - A(T + \mu I)$ for all $A \in B(H)$.

Let R be a *-ring, and let S denote the set of all symmetric elements of R. A *Jordan derivation* d of S into R is an additive mapping of S into R such that $d(s^2) = d(s)s + sd(s)$ for all $s \in S$ (we will only deal with 2-torsion free rings, i.e., ones where $2a = 0$ implies $a = 0$; in such rings our definition of Jordan derivations coincides with the definition in [10]). Our proof of Theorem 3.1 is based on the fact that the restriction of a Jordan *-derivation to the set of symmetric elements is a Jordan derivation.

In [10; Corollary 3, Theorem 4] Martindale proved

THEOREM M. *Let R be a 2-torsion free *-ring with unit element 1. Suppose that either*

(i) *R contains nonzero orthogonal symmetric idempotents e_1, e_2 and e_3 such that $e_1 + e_2 + e_3 = 1$ and $Re_iR = R$ for $i = 1, 2, 3$, or*

(ii) *R is simple and it contains nonzero orthogonal idempotents e_1 and e_2 such that $e_1 + e_2 = 1$.*

Then every Jordan derivation of R into S can be uniquely extended to a derivation of R.

Remark 3.2. Let us show that the algebra $B(H)$ (H real or complex) satisfies the requirements of Theorem M. First, if H is finite-dimensional, then $B(H)$ satisfies (ii). Now suppose H is infinite-dimensional. Then there exists an orthonormal basis in H of the form $\{e_\alpha, f_\alpha, g_\alpha;\ \alpha \in A\}$. Let H_1 be the subspace generated by $\{e_\alpha;\ \alpha \in A\}$, and let E_1 be the orthogonal projection with range H_1. Analogously we define the subspaces H_2, H_3, and projections E_2, E_3. Of course, $E_1 + E_2 + E_3 = I$, the identity on H. We claim that $B(H)E_iB(H) = B(H)$, $i = 1, 2, 3$. Indeed, there exists a one-to-one bounded linear operator B on H with range contained in H_i. Note that there is $A \in B(H)$ such that $AE_iB = AB = I$. But then $B(H)E_iB(H) = B(H)$.

In order to determine the structure of Jordan derivations of $S(H)$ into $B(H)$ we also need the following simple lemma.

LEMMA 3.3. *If $A, B \in B(H)$ are such that $ASB = 0$ for all $S \in S(H)$ then either $A = 0$ or $B = 0$.*

Proof. It suffices to prove that if a, b are nonzero vectors in H, then there exists $S \in S(H)$ such that $Sb = \lambda a$ for some nonzero scalar λ. If a and b are not orthogonal then this condition is satisfied by the operator $a \otimes a$ (we denote by $u \otimes v$ the operator $(u \otimes v)x = \langle x, v\rangle u$ where $\langle \cdot, \cdot \rangle$ is the inner product); otherwise take $S = a \otimes b + b \otimes a$.

We are now in a position to prove

THEOREM 3.4. *Let H be a* (*real or complex*) *Hilbert space. If a Jordan derivation d of $S(H)$ into $B(H)$ is real linear then there exists $T \in B(H)$ such that $d(S) = TS - ST$ for all $S \in S(H)$.*

Proof. By Theorem M (and Remark 3.2) there is an additive derivation D of $B(H)$ such that $D|S(H) = d$. Since every linear derivation of $B(H)$ is inner [7], the theorem will be proved by showing that D is linear.

Let us first show that D is real linear. For $A \in B(H)$ we may write $A = W + K$ where $W^* = W$ and $K^* = -K$. By assumption, $D(\lambda W) = \lambda D(W)$ for every real λ, therefore it suffices to show that $D(\lambda K) = \lambda D(K)$. Given any $S \in S(H)$, we have $KSK \in S(H)$. Therefore,

$$\begin{aligned} D(\lambda KSK) &= d(\lambda KSK) = \lambda d(KSK) = \lambda D(KSK) \\ &= \lambda D(K)SK + \lambda KD(S)K + \lambda KSD(K)\,; \end{aligned}$$

on the other hand,

$$D(\lambda KSK) = D((\lambda K)SK) = D(\lambda K)SK + \lambda KD(S)K + \lambda KSD(K)\,.$$

Comparing the above expressions for $D(\lambda KSK)$, we arrive at $(D(\lambda K) - \lambda D(K))SK = 0$ for all $S \in S(H)$. By Lemma 3.3 we conclude that $D(\lambda K) = \lambda D(K)$.

Now suppose H is a complex space. Since D is real linear it suffices to show that $D(iA) = iD(A)$ for every $A \in B(H)$. We have $D(I) = 0$. Hence

$$0 = D((iI)^2) = D(iI)iI + iID(iI) = 2iD(iI)\,.$$

Thus $D(iI) = 0$. But then for any $A \in B(H)$ we have

$$D(iA) = D((iI)A) = D(iI)A + iID(A) = iD(A)\,,$$

which completes the proof.

For the proof of Theorem 3.1 we also need the following lemma which is similar to [11; Theorem 1].

LEMMA 3.5. *If $A, B \in B(H)$ are such that $ABS = BSA$ for all $S \in S(H)$, and if $B \neq 0$, then $A = \lambda B$ for some scalar λ.*

Proof. For all $x, y \in H$ we have $A(y \otimes y)Bx = B(y \otimes y)Ax$; that is, $\langle Bx, y\rangle Ay = \langle Ax, y\rangle By$. Consequently,

$$\begin{aligned} \langle Bx, y\rangle\langle By, z\rangle Az &= \langle Bx, y\rangle\langle Ay, z\rangle Bz = \langle\langle Bx, y\rangle Ay, z\rangle Bz \\ &= \langle\langle Ax, y\rangle By, z\rangle Bz = \langle Ax, y\rangle\langle By, z\rangle Bz\,. \end{aligned}$$

Thus $\langle By, z\rangle\{\langle Bx, y\rangle Az - \langle Ax, y\rangle Bz\} = 0$ for all $x, y, z \in H$. Hence for any $y, z \in H$ we have either $\langle By, z\rangle = 0$ or $\langle Bx, y\rangle Az = \langle Ax, y\rangle Bz$ for all $x \in H$. Using the fact that a group cannot be the union of two proper subgroups (cf. the proof of Proposition 2.5) one can easily show that either $\langle By, z\rangle = 0$

for all $y, z \in H$ or $\langle Bx, y\rangle Az = \langle Ax, y\rangle Bz$ for all $x, y, z \in H$. Since we have assumed that $B \neq 0$ it follows at once that $A = \lambda B$ for some λ.

Remark 3.6. It is easy to see [4; Lemma 2] that every Jordan *-derivation E satisfies $E(xyx) = E(x)y^*x^* + xE(y)x^* + xyE(x)$.

Proof of Theorem 3.1. Let E be a Jordan *-derivation of $B(H)$. By [4; Corollary 1], E is linear. Since the restriction of E to $S(H)$ is a Jordan derivation of $S(H)$ to $B(H)$, it follows from Theorem 3.4 that there exists $T \in B(H)$ such that

$$E(S) = TS - ST \quad \text{for all } S \in S(H)\,. \tag{1}$$

Pick $K \in B(H)$ such that $K^* = -K$. For every $S \in S(H)$ we have $KSK \in S(H)$. Therefore,

$$E(KSK) = TKSK - KSKT\,.$$

On the other hand, using Remark 3.6 we obtain

$$\begin{aligned} E(KSK) &= -E(K)SK - KE(S)K + KSE(K) \\ &= -E(K)SK - K(TS - ST)K + KSE(K) \end{aligned}$$

Comparing both expressions we get

$$(E(K) + KT + TK)SK = KS(E(K) + KT + TK)$$

for all $S \in S(H)$. Now Lemma 3.5 yields

$$E(K) + KT + TK = \lambda(K)K \tag{2}$$

for some real $\lambda(K)$. We claim that $\lambda(K)$ is a constant. Pick $K_1, K_2 \in B(H)$ with $K_1^* = -K_1$, $K_2^* = -K_2$. We claim that $\lambda(K_1) = \lambda(K_2)$. First assume that K_1 and K_2 are linearly independent. In view of (2) we have

$$E(K_1 + K_2) = \lambda(K_1 + K_2)(K_1 + K_2) - T(K_1 + K_2) - (K_1 + K_2)T\,.$$

On the other hand,

$$\begin{aligned} E(K_1 + K_2) &= E(K_1) + E(K_2) \\ &= \lambda(K_1)K_1 - TK_1 - K_1T + \lambda(K_2)K_2 - TK_2 - K_2T\,. \end{aligned}$$

Comparing we get

$$(\lambda(K_1 + K_2) - \lambda(K_1))K_1 + (\lambda(K_1 + K_2) - \lambda(K_2))K_2 = 0\,.$$

Since K_1 and K_2 are linearly independent we obtain $\lambda(K_1) = \lambda(K_1 + K_2) = \lambda(K_2)$.

If K_1 and K_2 are linearly dependent, then for any $K \in B(H)$ with $K^* = -K$ which is linearly independent from both K_1 and K_2, we have $\lambda(K_1) = \lambda(K)$ and $\lambda(K_2) = \lambda(K)$. Thus $\lambda(K_1)$ and $\lambda(K_2)$ are also equal in this case. This means that $\lambda(K)$ is a constant λ, so that

$$E(K) = \lambda K - KT - TK \tag{3}$$

for every $K \in B(H)$ with $K^* = -K$.

Take $A \in B(H)$. We have $A = S + K$, where $S^* = S$, $K^* = -K$. Using (1) and (3) we then get

$$\begin{aligned} E(A) &= E(S) + E(K) = TS - ST + \lambda K - KT - TK \\ &= (T - \tfrac{1}{2}\lambda I)(S - K) - (S + K)(T - \tfrac{1}{2}\lambda I) \\ &= (T - \tfrac{1}{2}\lambda I)A^* - A(T - \tfrac{1}{2}\lambda I)\,. \end{aligned}$$

Thus $E(A) = T_1 A^* - AT_1$ for all $A \in B(H)$, where $T_1 = T - \frac{1}{2}\lambda I$. This proves the theorem.

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