On the structure of solutions of ergodic type Bellman equation related to risk-sensitive control

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1 Introduction

We consider the following nonlinear partial differential equation:

$$\frac{1}{2}D_i(a^{ij}D_jW)+\frac{1}{2}\hat{a}^{ij}D_iWD_jW+b\cdot\nabla W+V=\Lambda \text{ in } \mathbb{R}^N \tag{1.1}$$

or equivalently

$$\frac{1}{2}a^{ij}D_{ij}W+\frac{1}{2}\hat{a}^{ij}D_iWD_jW+\tilde{b}\cdot\nabla W+V=\Lambda,\quad \tilde{b}^i(x)\equiv b^i(x)+\frac{1}{2}D_ja^{ij}(x), \tag{1.2}$$

where $a(x)=[a^{ij}(x)]$, $\hat{a}(x)=[\hat{a}^{ij}(x)]$ are symmetric matrices, $b(x)=(b^1(x),\cdots,b^N(x))$ is mapping of $\mathbb{R}^N$ into $\mathbb{R}^N$, $V(x)$ is function on $\mathbb{R}^N$. Here we utilize the notations $D_{ij}=\partial^2/\partial x_i\partial x_j$, $D_i=\partial/\partial x_i$ and summation convention for multiple indexes. We think of a pair (W,Λ) of function $W(x)$ and constant Λ as a solution of (1.1). (1.1) is called ergodic type Bellman equation. This kind of equations is treated in ergodic control problems (cf. [1]). In the ergodic control problems, $\hat{a}$ is *negative-definite* and more general forms of (1.1) have been studied under rather general conditions (cf. [2]). On the other hand, (1.1) also appears in risk-sensitive control problems in infinite time horizon and has been studied under certain conditions (cf. [5], [8], [9], [13]). One of the main features of (1.1) in risk-sensitive control is that $\hat{a}$ might be *positive-definite*. Recently, it is also known that this case happens in some investment problems in mathematical finance (cf. [3], [4], [6], [7], [14]). We shall study the solutions of (1.1) in the case that $\hat{a}$ is *positive-definite*.

The studies of solutions for Bellman equations from analytical point of view are considered to be fundamental to determine an optimal control for control problems (see the explanation later in this section). Note that solutions of (1.1) have ambiguity of additive constant, *i.e.*, if (W,Λ) is a solution of (1.1), $W(x)+c$ still satisfies (1.1) for each constant c. As some examples show, it is known that (1.1) has multiple solutions even if we identify the solutions up to additive constants. So, it is important to study how we pick up a particular solution of (1.1) which gives an optimal control for the problems at hand. A common way to obtain a particular solution for ergodic type Bellman equations is to study the discounted type equations. The discounted type Bellman equation corresponding to (1.1) is as follows:

$$\frac{1}{2}D_i(a^{ij}D_jW_\alpha) + \frac{1}{2}\hat{a}^{ij}D_iW_\alpha D_jW_\alpha + b\cdot\nabla W_\alpha + V = \alpha W_\alpha$$

$\alpha > 0$ is called discount factor. Under certain conditions, it is shown that $W_\alpha(x) - W_\alpha(x_0)$ normalized at some point $x_0 \in \mathbb{R}^N$ and αW_α converge to some function $W(x)$ and some constant Λ respectively. Furthermore (W, Λ) satisfies (1.1) (cf. [5], [8], [9]). Under the conditions including Linear Exponential Quadratic Gaussian (LEQG) control problem, we need to consider the case that $b(x)$ (*resp.* $V(x)$) is at most linearly growing (*resp.* quadratic growing). Under such a kind of settings, W is characterized to meet some growth condition and (W, Λ) obtained by this process is considered to be the right solution (cf. [8], [9]).

In the present paper, we directly tackle (1.1) without the procedure using discounted type equation under the conditions including LEQG case. We shall specify the set of Λ for which (1.1) has a smooth solution. Furthermore we shall characterize the set of Λ by noting the global behavior of diffusion process which is related to some control problem. One of our advantages is that we can treat more general $b(x)$ compared to [8], [9].

To explain how we relate (1.1) to a control problem, we shall give a control interpretation to (1.1). Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a probability space with filtration. Consider the following controlled stochastic differential equation (SDE):

$$dX_t = (\tilde{b}(X_t) + u_t)dt + \sigma(X_t)dB_t,\ X_0 = x \in \mathbb{R}^N, \quad \sigma(x) \equiv a(x)^{1/2}$$

where B_t is N-dimensional $\mathcal{F}_t$-Brownian motion and u_t is $\mathcal{F}_t$-progressively measurable process taking its value in $\mathbb{R}^N$. $\{u_t\}$ is considered as control process. We define the value function as follows:

$$v(t,x) = \sup_{u_\cdot} E_x\left[\int_0^{T-t} V(X_s) - \frac{1}{2}\hat{a}^{-1}_{ij}(X_s)u^i_s u^j_s ds\right],$$

where $\hat{a}^{-1}_{ij}$ is (i,j)-component in inverse of $\hat{a}$. By using Bellman principle, we see that $v(t,x)$ satisfies the following equation formally:

$$\frac{\partial v}{\partial t} + \frac{1}{2}a^{ij}D_{ij}v + \sup_{u\in\ ^N}\left\{(\tilde{b}(x)+u)\cdot\nabla W - \frac{1}{2}\hat{a}^{-1}_{ij}u^iu^j\right\} + V = 0 \text{ in } (0,T)\times\mathbb{R}^N \quad (1.3)$$

$$v(T,x) = 0,\ x \in \mathbb{R}^N. \quad (1.4)$$

Since $\sup_{u\in\ ^N}\{(\tilde{b}+u)\cdot\nabla W - (1/2)\hat{a}^{-1}_{ij}u^iu^j\} = (1/2)\hat{a}^{ij}D_ivD_jv + \tilde{b}\cdot\nabla v$, (1.3) reduces to the following:

$$\frac{\partial v}{\partial t} + \frac{1}{2}a^{ij}D_{ij}v + \frac{1}{2}\hat{a}^{ij}D_ivD_jv + \tilde{b}\cdot\nabla W + V = 0.$$

Note that the supremum is attained at $\bar{u}(x) = \hat{a}\nabla W(x)$. If $(\partial v/\partial t)(0,x)$ converges to some constant Λ and $v(0,x) - v(0,x_0)$ normalized at some point $x_0 \in \mathbb{R}^N$ converges to some function $W(x)$ as $T \to \infty$, we have formally the following equation which we shall discuss in this paper:

$$\frac{1}{2}a^{ij}D_{ij}W + \frac{1}{2}\hat{a}^{ij}D_iWD_jW + \tilde{b}\cdot\nabla W + V = \Lambda.$$

This is considered to characterize the long-time average cost defined as following:

$$\Lambda = \overline{\lim_{T\to\infty}} \sup_{u.} \frac{1}{T}E_x\left[\int_0^T V(X_s) - \frac{1}{2}\hat{a}_{ij}^{-1}(X_s)u_s^i u_s^j ds\right]. \tag{1.5}$$

Following Bellman principle, we can expect that $\bar{u}_t = \hat{a}(X_t)\nabla W(X_t)$ should be an candidate of optimal control for (1.5), where X_t is defined by the controlled SDE with $u_t = \bar{u}_t = \hat{a}(X_t)\nabla W(X_t)$:

$$dX_t = (\tilde{b}(X_t) + \hat{a}\nabla W(X_t))dt + \sigma(X_t)dB_t,\ X_0 = x. \tag{1.6}$$

We shall study the structure of solutions of (1.1) by relating to (1.6) under conditions which include LEQG case, *i.e.*, $b(x)$ (*resp.* $V(x)$) has at most linear growth (*resp.* quadratic growth).

The paper is organized as follows.

In §2, we shall specify the set of Λ for which (1.1) has a solution under rather general conditions on $b(x)$ and $V(x)$. Indeed, it is proved that the set of Λ is equal to closed half-line $[\Lambda^*, \infty)$ for some $\Lambda^* \in (-\infty, \infty)$.

In §3, we shall classify Λ according to the global property of the diffusion process defined by (1.6) under certain stability condition for $b(x)$ (see $(A1)''$). We shall prove that for $\Lambda > \Lambda^*$, the diffusion process $\{X_t\}$ in (1.6) corresponding to solution (W, Λ) is transient and for $\Lambda = \Lambda^*$, $\{X_t\}$ is ergodic. Moreover, we shall show that solution $W(x)$ corresponding to Λ^* is unique up to additive constant.

We note that the structure of Λ specified in this paper is considered to be a generalization in the theory of positive harmonic function for linear differential operators (cf. [15]).

2 The set of Λ having a solution

In the present section, we shall consider the set of Λ for which (1.1) has a classical solution W under rather general conditions. In the next section, under certain stability property of $b(x)$, we shall classify Λ by following the global behavior of the diffusion process related to the solution W corresponding to Λ.

We define the following set:

$$\mathcal{A} \equiv \{\Lambda :\ \text{there exists smooth function } W \text{ satisfying (1.1) for } \Lambda\}.$$

Under the assumptions given below, we can prove that $\mathcal{A}$ has the following form for some $\Lambda^* \in (-\infty, \infty)$:

$$\mathcal{A} = [\Lambda^*, \infty)$$

For simplicity, we always assume a^{ij}, $\hat{a}^{ij}$, b, V are sufficiently smooth. We shall give the following assumptions :

(A1) $Da^{ij}(x)$ is bounded and there exist c_1, $c_2 > 0$ and $m \geq 1$ such that

$$|b(x)| \leq c_1(1+|x|^m),\ |Db(x)| \leq c_1(1+|x|^{m-1}),$$
$$|V(x)| \leq c_2(1+|x|^{2m}),\ |DV(x)| \leq c_2(1+|x|^{2m-1}).$$

(A2) There exist $0 < \nu_1 < \nu_2$ such that

$$\nu_1|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \nu_2|\xi|^2 \ \ \forall x,\ \xi \in \mathbb{R}^N.$$

(A3) There exist $0 < \mu_1 < \mu_2$ such that

$$\mu_1|\xi|^2 \leq \hat{a}^{ij}(x)\xi_i\xi_j \leq \mu_2|\xi|^2 \ \ \forall x,\ \xi \in \mathbb{R}^N.$$

(A4) There exists some function $W_0(x)$ such that

$$\frac{1}{2}D_i(a^{ij}D_jW_0) + \frac{1}{2}\hat{a}^{ij}D_iW_0D_jW_0 + b\cdot\nabla W_0 + V \to -\infty \text{ as } |x| \to \infty.$$

Remark 2.1. Note that it follows from (A2), (A3) that there exist c, $\bar{c} > 0$ such that

$$ca(x) \leq \hat{a}(x) \leq \bar{c}a(x),\ x \in \mathbb{R}^N. \tag{2.1}$$

Remark 2.2. For the purpose of discussion in the present section, we can replace (A4) with the existence of a super-solution of (1.1) for some Λ to ensure that $\mathcal{A} \neq \emptyset$. We need (A4) to classify Λ in the next section

As for sub-solutions, under (A1)–(A3), we can show that for arbitrary Λ, there exists $\tilde{W}_0(x)$ such that

$$\frac{1}{2}D_i(a^{ij}D_j\tilde{W}_0) + \frac{1}{2}\hat{a}^{ij}D_i\tilde{W}_0D_j\tilde{W}_0 + b\cdot\nabla\tilde{W}_0 + V \geq \Lambda \text{ in } \mathbb{R}^N.$$

Indeed, we can take $\tilde{W}_0(x) = \alpha|x|^{m+1} + \beta|x|^2$ and choose α, β satisfying the above inequality.

In order to see $\mathcal{A} \neq \emptyset$, consider the following Dirichlet problem :

$$\frac{1}{2}D_i(a^{ij}D_jW_R) + \frac{1}{2}\hat{a}^{ij}D_iW_RD_jW_R + b\cdot\nabla W_R + V = \Lambda \ \text{ in } B_R, \tag{2.2}$$
$$W_R = W_0 \ \text{ on } \partial B_R, \tag{2.3}$$

where B_R is open ball with radius R centered at 0 and W_0 is taken from (A4). Note that (2.2) is equivalent to

$$\frac{1}{2}a^{ij}D_{ij}W_R + \frac{1}{2}\hat{a}^{ij}D_iW_RD_jW_R + \tilde{b}\cdot\nabla W_R + V = \Lambda \ \text{ in } B_R, \tag{2.4}$$

By (A4), W_0 satisfies the following inequality for some Λ :

$$\frac{1}{2}D_i(a^{ij}D_jW_0) + \frac{1}{2}\hat{a}^{ij}D_iW_0D_jW_0 + b\cdot\nabla W_0 + V \leq \Lambda \text{ in } \mathbb{R}^N.$$

Also, from Remark 2.2, we have

$$\frac{1}{2}D_i(a^{ij}D_j\tilde{W}_0)+\frac{1}{2}\hat{a}^{ij}D_i\tilde{W}_0D_j\tilde{W}_0+b\cdot\nabla\tilde{W}_0+V\geq\Lambda \text{ in } \mathbb{R}^N.$$

Then, under (A1)–(A4), there exists $W_R \in C^{2,\alpha}(\bar{B}_R)$ satisfying (2.2), (2.3) (cf. [10], Chapter 4, Theorem 8.4).

We need a uniform bound for ∇W_R in compact sets to obtain a solution W of (1.1) by sending the radius R to ∞. The following gradient estimate is also useful in the later discussions.

Lemma 2.3. *Let W_R be a smooth function satisfying* (2.2)*. Under* (A1)–(A3)*, we have for each $r>0$ and $R>2r$*

$$\sup_{B_r}|\nabla W_R|\leq C(\Lambda)(1+|r|^m), \tag{2.5}$$

where $C(\Lambda)$ is a constant independent of r and R and non-decreasing on Λ.

Proof. (1.1) has the nonlinear term similar to those treated in [8], [9] and we can follow the same arguments to obtain the gradient estimate. However, we shall give a proof to specify the dependence of Λ and growth order on r.

We set $W=W_R$ for simplicity. By differentiating each side of (2.4) on x_k, we have

$$\frac{1}{2}D_ka^{ij}D_{ij}W+\frac{1}{2}a^{ij}D_{ijk}W+\frac{1}{2}D_k\hat{a}^{ij}D_iWD_jW+\hat{a}^{ij}D_iWD_{jk}W \\ +D_k\tilde{b}^iD_iW+\tilde{b}^iD_{ik}W+D_kV=0. \tag{2.6}$$

Let us set $G\equiv(1/2)\sum_k(D_kW)^2$. Then, using (2.6)

$$\begin{aligned} &-\frac{1}{2}a^{ij}D_{ij}G-\hat{a}^{ij}D_iWD_jG-\tilde{b}^iD_iG \\ &=-\frac{1}{2}a^{ij}D_kWD_{ijk}W-\frac{1}{2}a^{ij}D_{ki}WD_{kj}W-\hat{a}^{ij}D_iWD_kWD_{jk}W-\tilde{b}^iD_kWD_{ik}W \\ &=\frac{1}{2}D_ka^{ij}D_kWD_{ij}W+\frac{1}{2}D_k\hat{a}^{ij}D_iWD_jWD_kW \\ &\qquad +D_k\tilde{b}^iD_iWD_kW+D_kVD_kW-\frac{1}{2}a^{ij}D_{kj}WD_{kj}W. \end{aligned} \tag{2.7}$$

We note the second order derivative terms. Then, we have

$$\begin{aligned} \text{RHS of (2.7)} &\leq \frac{1}{4\delta}(\sum_{i,j}|Da^{ij}|^2)|DW|^2+\frac{\delta}{4}|D^2W|^2 \\ &\quad+\frac{1}{2}D_k\hat{a}^{ij}D_iWD_jWD_kW+D_k\tilde{b}^iD_iWD_kW \\ &\qquad+D_kVD_kW-\frac{1}{4}a^{ij}D_{ki}WD_{kj}W-\frac{1}{4}a^{ij}D_{ki}WD_{kj}W \\ &\leq\frac{1}{4\delta}(\sum_{i,j}|Da^{ij}|^2)|DW|^2+\frac{1}{2}D_k\hat{a}^{ij}D_iWD_jWD_kW+D_k\tilde{b}^iD_iWD_kW \\ &\quad+D_kVD_kW-\frac{1}{4}a^{ij}D_{ki}WD_{kj}W, \end{aligned}$$

where $\delta > 0$ is a small constant. Indeed, we can take δ satisfying $\delta < \nu_1$. From matrix inequality $(\mathrm{tr}\, AB)^2 \le N\nu_2(\mathrm{tr}\, AB^2)$ where A, B are $N\times N$-symmetric matrices, A is non-negative definite and ν_2 is the maximum eigenvalue of A, we finally obtain the following inequality by using (A1):

$$
\begin{aligned}
&-\frac{1}{2}a^{ij}D_{ij}G - \hat{a}^{ij}D_iWD_jG - \tilde{b}^iD_iG \\
&\le C(1+|x|^{2m-1})|DW| + C(1+|x|^{m-1})|DW|^2 + C|DW|^3 - \frac{1}{4N\nu_2}(a^{ij}D_{ij}W)^2 \text{ in } B_{2r},
\end{aligned}
\tag{2.8}
$$

Here and in the proof below, we suppose that C is constant independent of r and R.

Fix arbitrary $\xi \in B_r$ and take a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfying the following:

$$
\begin{gathered}
0 \le \varphi \le 1 \text{ in } \mathbb{R}^N, \quad \varphi(\xi)=1, \quad \varphi \equiv 0 \text{ in } B_r(\xi)^c, \\
|\nabla\varphi| \le C\varphi, \quad |D^2\varphi| \le C,
\end{gathered}
\tag{2.9}
$$

where $B_r(\xi)$ is open ball with radius r centered at ξ. Let x_0 be a maximum point of φG in $\bar{B}_r(\xi)$. By maximum principle, we can see

$$
\begin{aligned}
0 &\le -\frac{1}{2}a^{ij}D_{ij}(\varphi G) - \hat{a}^{ij}D_iWD_j(\varphi G) - \tilde{b}^iD_i(\varphi G) \\
&= \varphi\left\{-\frac{1}{2}a^{ij}D_{ij}G - \hat{a}^{ij}D_iWD_jG - \tilde{b}^iD_iG\right\} \\
&\quad - \frac{1}{2}a^{ij}(D_{ij}\varphi)G - a^{ij}D_i\varphi D_jG - \hat{a}^{ij}D_j\varphi(D_iW)G - \tilde{b}^i(D_i\varphi)G \\
&\le \varphi\left\{-\frac{1}{2}a^{ij}D_{ij}G - \hat{a}^{ij}D_iWD_jG - \tilde{b}^iD_iG\right\} + C(1+|x|^m)G + C\varphi^{1/2}G^{3/2} \text{ at } x_0,
\end{aligned}
\tag{2.10}
$$

where we used $0 = D(\varphi G) = GD\varphi + \varphi DG$ and (2.9). From (2.4) and (2.8), it implies

$$
\begin{aligned}
&\text{RHS of (2.10)} \\
&\le \varphi\left\{C(1+|x|^{2m-1})G^{1/2} + C(1+|x|^{m-1})G + CG^{3/2} - \frac{1}{4N\nu_2}(a^{ij}D_{ij}W)^2\right\} \\
&\quad + C(1+|x|^m)G + C\varphi^{1/2}G^{3/2} \\
&= \varphi\Bigg\{C(1+|x|^{2m-1})G^{1/2} + C(1+|x|^{m-1})G + CG^{3/2} \\
&\quad - \frac{1}{N\nu_2}\left(-\frac{1}{2}\hat{a}^{ij}D_iWD_jW - \tilde{b}^iD_iW - V + \Lambda\right)^2\Bigg\} \\
&\quad + C(1+|x|^m)G + C\varphi^{1/2}G^{3/2} \text{ at } x_0.
\end{aligned}
\tag{2.11}
$$

Noting (A1), (A3), then, the following inequalities hold for some positive constants κ depending on μ_1

$$
\begin{aligned}
-\frac{1}{2}\hat{a}^{ij}D_iWD_jW - \tilde{b}^iD_iW - V + \Lambda &\le -\frac{\mu_1}{2}|DW|^2 + C(1+|x|^m)|DW| - V + \Lambda \\
&\le -\kappa|DW|^2 + C(1+|x|^{2m}) - V + \Lambda.
\end{aligned}
\tag{2.12}
$$

In the case that $-\kappa|DW|^2 + C(1+|x|^{2m}) - V + \Lambda \geq 0$ at x_0, we have

$$\begin{aligned}\kappa|DW|^2(x_0) &\leq C(1+|x_0|^{2m}) - V(x_0) + \Lambda \\ &\leq C(1+|x_0|^{2m}) + \Lambda \leq C(1+|r|^{2m}) + \Lambda,\end{aligned}$$

where we used (A1) and $x_0 \in B_{2r}$. Since $(1/2)|DW|^2(\xi) = (1/2)|DW|^2(\xi)\varphi(\xi) \leq G(x_0)\varphi(x_0)$, we obtain the following gradient estimate at ξ:

$$|DW|^2(\xi) \leq C(1+|r|^{2m}) + \Lambda.$$

We next consider the case that $-\kappa|DW|^2 + C(1+|x|^{2m}) - V + \Lambda \leq 0$ at x_0. By (2.12),

$$\begin{aligned}&\text{RHS of (2.11)} \\ &\leq \varphi\Big\{C(1+|x|^{2m-1})G^{1/2} + C(1+|x|^{m-1})G + CG^{3/2} \\ &\quad - \frac{1}{N\nu_2}\left(-\kappa|DW|^2 + C(1+|x|^{2m}) - V + \Lambda\right)^2\Big\} + C(1+|x|^m)G + C\varphi^{1/2}G^{3/2} \\ &\leq \varphi\Big\{C(1+|x|^{2m-1})G^{1/2} + C(1+|x|^{m-1})G + CG^{3/2} \\ &\quad - \frac{4\kappa^2}{N\nu_2}G^2 + \frac{8\kappa}{N\nu_2}G(C(1+|x|^{2m}) - V + \Lambda)\Big\} + C(1+|x|^m)G + C\varphi^{1/2}G^{3/2} \qquad (2.13)\end{aligned}$$

If $C(1+|x_0|^{2m}) - V + \Lambda \geq \kappa G(x_0)/4$ or $C(1+|x_0|^{2m-1}) \geq G(x_0)$ we have the bound $|DW|^2(\xi) \leq C(1+|r|^{2m}) + \Lambda$ in the same way as the above case. We shall consider the case that $C(1+|x_0|^{2m}) - V + \Lambda \leq \kappa G(x_0)/4$ and $C(1+|x_0|^{2m-1}) \leq G$. Then, from (2.13), we have

$$\begin{aligned}0 &\leq \varphi\left\{CG^{3/2} + C(1+|x|^{m-1})G + CG^{3/2} - \frac{4\kappa^2}{N\nu_2}G^2 + \frac{2\kappa^2}{N\nu_2}G^2\right\} \\ &\quad + C(1+|x|^m)G + C\varphi^{1/2}G^{3/2} \\ &\leq -C_1\varphi G^2 + C_2\varphi^{1/2}G^{3/2} + C_3(1+r^m)G \\ &\equiv -C_1\varphi G^2 + C_2\varphi^{1/2}G^{3/2} + \tilde{C}_3 G \text{ at } x_0,\ \tilde{C}_3 \equiv C_3(1+r^m)\end{aligned}$$

where C_1, C_2, C_3 are positive constants independent of r, R and Λ. By setting $X \equiv \varphi^{1/2}G^{1/2}$, we have

$$0 \leq -C_1X^2 + C_2X + C_3.$$

Therefore, we have

$$X^2 = \varphi G(x_0) \leq \frac{C_2^2}{C_1^2} + \frac{2\tilde{C}_3}{C_1} \leq \frac{C_2^2}{C_1^2} + \frac{2C_3(1+r^m)}{C_1}$$

Since $(1/2)|DW|^2(\xi) = (1/2)|DW|^2(\xi)\varphi^2(\xi) \leq G(x_0)\varphi(x_0)$, we obtain the bound for $|DW|(\xi)$. □

We may normalize W_R as $W_R(0) = 0$ because (1.1) does not include zeroth term on W_R. Then, from Lemma 2.3, there exists $W \in C(\mathbb{R}^N)$ such that W_R converges to W on each compact sets as $R \to \infty$ by taking a subsequence if necessary. Also, since $\{W_R\}_{R>2r}$ is bounded in $H^1(B_r)$ by Lemma 2.3, W_R converges to W $L^2_{\rm loc}$-strongly and $H^1_{\rm loc}$-weakly. Furthermore, we can see that ∇W_R converges $L^2_{\rm loc}$-strongly in a similar way to [9], [13],

We rewrite (2.2), (2.3) in integral form :

$$-\frac{1}{2}\int a^{ij}D_iW_RD_j\varphi\ dx+\frac{1}{2}\int \hat{a}^{ij}D_iW_RD_jW_R\varphi\ dx \\ +\int b\cdot\nabla W_R\,\varphi\ dx+\int V\varphi\ dx=\int \Lambda\varphi\ dx,\quad \varphi\in C_0^\infty(B_R).$$

Fix $r>0$. Since W_R converges to W $H^1_{\rm loc}$-strongly, we obtain the following by sending R to ∞ :

$$-\frac{1}{2}\int a^{ij}D_iWD_j\varphi\ dx+\frac{1}{2}\int \hat{a}^{ij}D_iWD_jW\varphi\ dx \\ +\int b\cdot\nabla W\,\varphi\ dx+\int V\varphi\ dx=\int \Lambda\varphi\ dx,\quad \varphi\in C_0^\infty(B_r),\ r>0.$$

Owing to the regularity theorem of elliptic equations and imbedding theorem, we have W as a classical solution of (1.1). Therefore, we have proved that $\mathcal{A}\neq\emptyset$.

We shall state and prove the form of the set of Λ.

Theorem 2.4. *Under the assumptions* (A1)–(A4)*, there exists* $\Lambda^*\in(-\infty,\infty)$ *such that* $\mathcal{A}=[\Lambda^*,\infty)$.

Proof. In order to show $\inf\mathcal{A}>-\infty$, we suppose $\inf\mathcal{A}=-\infty$, *i.e.*, there exists $\{\Lambda_n\}\subset\mathcal{A}$ such that Λ_n tends to $-\infty$ as $n\to\infty$. Let W_n be a solution of (1.1) corresponding to Λ_n. Then, by the integral form of (1.1), we have

$$-\frac{1}{2}\int a^{ij}D_iW_nD_j\varphi\ dx+\frac{1}{2}\int \hat{a}^{ij}D_iW_nD_jW_n\,\varphi\ dx \\ +\int b\cdot\nabla W_n\,\varphi\ dx+\int V\varphi\ dx=\int \Lambda_n\varphi\ dx,\quad \varphi\in C_0^\infty(\mathbb{R}^N). \tag{2.14}$$

Take $\varphi\in C_0^\infty(\mathbb{R}^N)$ such that $\int\varphi dx\neq 0$. Since $\{\Lambda_n\}$ is bounded from above, it implies from Lemma 2.3 that

$$\sup_{B_r}|\nabla W_n|\le C_r, \tag{2.15}$$

where C_r is a constant independent of n and r is taken such that $\operatorname{supp}\varphi\subset B_r$. Therefore, the left hand side of (2.14) is bounded on n. On the other hand, the right hand side of (2.14) is unbounded because of the assumption which we made above. This leads to a contradiction.

We shall next prove if $\tilde{\Lambda}\in\mathcal{A}$, then $[\tilde{\Lambda},\infty)\subset\mathcal{A}$. Let $\tilde{W}$ be a solution corresponding to $\tilde{\Lambda}$. For arbitrary $\Lambda\ge\tilde{\Lambda}$, we have

$$\frac{1}{2}D_i(a^{ij}D_j\tilde{W})+\frac{1}{2}\hat{a}^{ij}D_i\tilde{W}D_j\tilde{W}+b\cdot\nabla\tilde{W}+V=\tilde{\Lambda}\le\Lambda\ \text{ in }\mathbb{R}^N. \tag{2.16}$$

By Remark 2.2, there exists $\tilde{W}_0$ such that

$$\frac{1}{2}D_i(a^{ij}D_j\tilde{W}_0) + \frac{1}{2}\hat{a}^{ij}D_i\tilde{W}_0 D_j\tilde{W}_0 + b\cdot\nabla\tilde{W}_0 + V \geq \tilde{\Lambda} \ \text{ in } \mathbb{R}^N. \tag{2.17}$$

Consider the Dirichlet problem (2.2) with boundary condition $W_R = \tilde{W}_0$ on ∂B_R. From (2.16), (2.17), the existence of a classical solution for this Dirichlet problem is guaranteed by [10]. In the same manner as that right after the proof of Lemma 2.3, we can see that there exists a smooth function W satisfying (1.1) for Λ.

We shall prove that $\Lambda^* \equiv \inf \mathcal{A}$ actually belongs to $\mathcal{A}$. $\{\Lambda_n\}$ is a sequence in $\mathcal{A}$ such that $\Lambda_n \to \Lambda^*$ and W_n is a solution of (1.1) corresponding to Λ_n normalized as $W_n(0) = 0$. Then, W_n satisfies (2.14). Since $\{\Lambda_n\}$ is bounded, it follows from Lemma 2.3 that (2.15) holds for some constant C_r independent of n. Following the same way as the discussion after Lemma 2.3, we can see that W_n converges to $W^* \in C(\mathbb{R}^N)$ uniformly on compact sets and H^1_{loc}-strongly. By taking a limit in (2.14) as $n \to \infty$, we have

$$\begin{aligned} -\frac{1}{2}\int a^{ij}D_iW^*D_j\varphi\,dx + \frac{1}{2}\int \hat{a}D_iW^*D_jW^*\,\varphi\,dx \\ + \int b\cdot\nabla W^*\varphi\,dx + \int V\varphi\,dx = \int \Lambda^*\varphi\,dx,\ \forall\varphi\in C_0^\infty(\mathbb{R}^N). \end{aligned}$$

Therefore, the existence of a classical solution W^* of (1.1) for Λ^* follows from the regularity theorems of elliptic equation and imbedding theorem. □

3 Classification of solutions

3.1 Transience and ergodicity of diffusion processes

In the last section, we proved that the set of Λ for which (1.1) has a smooth solution is $\mathcal{A} = [\Lambda^*, \infty)$ for some $\Lambda^* \in (-\infty, \infty)$. In the present section, we shall study the classification of Λ by global behavior of $\{X_t\}$ defined by (1.6) under stronger conditions. Instead of (A1), we assume the following:

(A1)$'$ Da^{ij} is bounded and there exists c_1, $c_2 > 0$ such that

$$|b(x)| \leq c_1(1+|x|),\ |Db(x)| \leq c_1,$$
$$|V(x)| \leq c_2(1+|x|^2),\ |DV(x)| \leq c_2(1+|x|).$$

(A1)$''$ There exist γ_1, $\gamma_2 > 0$ such that

$$x\cdot b(x) \leq -\gamma_1|x|^2 + \gamma_2,\ x\in\mathbb{R}^N.$$

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a filtered probability space on which N-dimensional Brownian motion $\{B_t\}$ is defined. For given $\Lambda \in [\Lambda^*, \infty)$, consider the SDE:

$$dX_t = (\tilde{b}(X_t) + \hat{a}\nabla W(X_t))dt + \sigma(X_t)dB_t, \quad X_0 = x, \tag{3.1}$$

where $W(x)$ is a solution of (1.1) corresponding to Λ. We shall classify Λ according to the global properties of $\{X_t\}$. More precisely, we shall prove that for $\Lambda > \Lambda^*$, $\{X_t\}$ is transient and for $\Lambda = \Lambda^*$, $\{X_t\}$ is ergodic.

First of all, we shall see that diffusion process $\{X_t\}$ defined in (3.1) does not explode in finite time. Let us define diffusion process $\{Y_t\}$ governed by the following SDE:

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t,\ Y_0 = x.$$

Under $(\mathrm{A1})'$, $\{Y_t\}$ is well-defined on $[0,\infty)$. Let us introduce a new measure as follows:

$$\left.\frac{d\bar{P}}{dP}\right|_{\mathcal{F}_T} \equiv \exp\left[\int_0^T \sigma\nabla W(Y_t)dB_t - \frac{1}{2}\int_0^T a\nabla W\cdot\nabla W(Y_t)dt\right]. \tag{3.2}$$

Indeed, as proved below, it implies from $(\mathrm{A1})'$, $(\mathrm{A1})''$, (A2), (A3) that $\bar{P}$ is a probability measure. Therefore, (3.1) has a solution on each closed interval $[0,T]$ by change of drift under $\bar{P}$.

Lemma 3.1. *Suppose that* (W,Λ) *is a solution of* (1.1) *and* $\{X_t\}$ *is a solution of* (3.1). *Under* $(\mathrm{A1})'$, $(\mathrm{A1})''$, (A2), (A3), $\bar{P}$ *defined in* (3.2) *is a probability measure on* $\mathcal{F}_T$.

Proof. It is sufficient to prove that there exists $\theta > 0$ such that

$$\sup_{0\le t\le T} E_x\left[e^{\theta|\nabla W(Y_t)|^2}\right] < \infty.$$

See [11], p.220. By Lemma 2.3, ∇W is at most linearly growing on x, *i.e.*, there exists $C_1 > 0$ such that

$$|\nabla W(x)| \le C_1(1+|x|),\ x\in\mathbb{R}^N. \tag{3.3}$$

Therefore, we need to see that for some $\theta > 0$,

$$\sup_{0\le t\le T} E_x\left[e^{\theta|Y_t|^2}\right] < \infty. \tag{3.4}$$

The way to prove (3.4) is relatively standard by using $(\mathrm{A1})'$, $(\mathrm{A1})''$ and its proof is given in Appendix. □

We shall next discuss transience of $\{X_t\}$ for $\Lambda\in(\Lambda^*,\infty)$. We introduce the operator associated to solution (W,Λ) of (1.1):

$$T_t^{W,\Lambda}f(x) \equiv E_x[f(X_t)],\ f\in C_0(\mathbb{R}^N),$$

where $\{X_t\}$ is a solution of (3.1) corresponding to (W,Λ).

Lemma 3.2. *Under* $(\mathrm{A1})'$, $(\mathrm{A1})''$, (A2)–(A4), *the following inequality holds for each solution* (W,Λ) *of* (1.1)

$$T_t^{W,\Lambda}f(x) \le ke^{-c(\Lambda-\Lambda^*)t},\quad f\in C_0(\mathbb{R}^N),\ f\ge 0,$$

where c *is in Remark* 2.1 *and* k *is a constant independent of* t.

Proof. Let W^* be a solution of (1.1) corresponding to Λ^*. We set $W_c \equiv cW$, $W_c^* \equiv cW^*$, where $c > 0$ is taken from Remark 2.1. Then, we have from (1.2)

$$\frac{1}{2}a^{ij}D_{ij}W_c + \frac{1}{2c}\hat{a}^{ij}D_iW_cD_jW_c + \tilde{b}\cdot\nabla W_c + cV = c\Lambda, \tag{3.5}$$

$$\frac{1}{2}a^{ij}D_{ij}W_c^* + \frac{1}{2c}\hat{a}^{ij}D_iW_c^*D_jW_c^* + \tilde{b}\cdot\nabla W_c^* + cV = c\Lambda^*. \tag{3.6}$$

Subtracting (3.6) from (3.5),

$$\frac{1}{2}a^{ij}D_{ij}(W_c - W_c^*) + (\tilde{b} + \hat{a}\nabla W^*)\cdot\nabla(W_c - W_c^*) + \frac{1}{2c}\hat{a}\nabla(W_c - W_c^*)\cdot\nabla(W_c - W_c^*) = c(\Lambda - \Lambda^*).$$

Setting $\bar{W} \equiv W_c - W_c^*$, we have

$$\frac{1}{2}a^{ij}D_{ij}\bar{W} + (b + \hat{a}\nabla W^*)\cdot\nabla\bar{W} + \frac{1}{2c}\hat{a}\nabla\bar{W}\cdot\nabla\bar{W} = c(\Lambda - \Lambda^*). \tag{3.7}$$

Let us define diffusion process $\{\tilde{X}_t\}$ satisfying the following SDE:

$$\begin{aligned} d\tilde{X}_t &= (\tilde{b}(\tilde{X}_t) + \hat{a}\nabla W(\tilde{X}_t))dt - a\nabla\bar{W}(\tilde{X}_t)dt + \sigma(\tilde{X}_t)dB_t \\ &= (\tilde{b}(\tilde{X}_t) + \hat{a}\nabla W^*(\tilde{X}_t))dt + \left(\frac{1}{c}\hat{a}\nabla\bar{W}(\tilde{X}_t) - a\nabla\bar{W}(\tilde{X}_t)\right)dt + \sigma(\tilde{X}_t)dB_t \\ \tilde{X}_0 &= x. \end{aligned}$$

By Girsanov theorem,

$$T_t^{W,\Lambda}f(x) = E_x[f(X_t)] = E_x\left[f(\tilde{X}_t)e^{\int_0^t \sigma\nabla\bar{W}(\tilde{X}_s)dB_s - \frac{1}{2}\int_0^t a\nabla\bar{W}\cdot\nabla\bar{W}(\tilde{X}_s)ds}\right] \tag{3.8}$$

Applying Ito formula to $\bar{W}(\tilde{X}_t)$,

$$\begin{aligned} &d\bar{W}(\tilde{X}_t) \\ &= \nabla\bar{W}\cdot\left(\tilde{b} + \hat{a}\nabla W^* + \frac{1}{c}\hat{a}\nabla\bar{W} - a\nabla\bar{W}\right)(\tilde{X}_t)dt + \frac{1}{2}a^{ij}D_{ij}\bar{W}(\tilde{X}_t)dt + \sigma\nabla\bar{W}(\tilde{X}_t)dB_t \\ &= \left(\frac{1}{2}a^{ij}D_{ij}\bar{W} + (\tilde{b} + \hat{a}\nabla W^*)\cdot\nabla\bar{W}\right)(\tilde{X}_t)dt + \left(\frac{1}{c}\hat{a}\nabla\bar{W}\cdot\nabla\bar{W} - a\nabla\bar{W}\cdot\nabla\bar{W}\right)(\tilde{X}_t)dt \\ &\qquad + \sigma\nabla\bar{W}(\tilde{X}_t)dB_t \\ &= \left(-\frac{1}{2c}\hat{a}\nabla\bar{W}\cdot\nabla\bar{W} + c(\Lambda - \Lambda^*)\right)(\tilde{X}_t)dt + \left(\frac{1}{c}\hat{a}\nabla\bar{W}\cdot\nabla\bar{W} - a\nabla\bar{W}\cdot\nabla\bar{W}\right)(\tilde{X}_t)dt \\ &\qquad + \sigma\nabla\bar{W}(\tilde{X}_t)dB_t \\ &= \sigma\nabla\bar{W}(\tilde{X}_t)dB_t - \frac{1}{2}a\nabla\bar{W}\cdot\nabla\bar{W}(\tilde{X}_t)dt + \frac{1}{2}\left(\frac{1}{c}\hat{a} - a\right)\nabla\bar{W}\cdot\nabla\bar{W}(\tilde{X}_t)dt + c(\Lambda - \Lambda^*)dt \end{aligned} \tag{3.9}$$

Here we used (3.7). Then, by (3.8) and (3.9), we have

$$\begin{aligned} T_t^{W,\Lambda}f(x) &= E_x\left[f(\tilde{X}_t)e^{-c(\Lambda-\Lambda^*)t + \bar{W}(\tilde{X}_t) - \bar{W}(x) + \frac{1}{2}\int_0^t\left(a - \frac{1}{c}\hat{a}\right)\nabla\bar{W}\cdot\nabla\bar{W}(\tilde{X}_s)ds}\right] \\ &\le \|f\|_\infty e^{\sup\{\bar{W}(y) - \bar{W}(x):\ y\in\operatorname{supp} f\}}e^{-c(\Lambda-\Lambda^*)t}E_x\left[e^{\frac{1}{2}\int_0^t\left(a - \frac{1}{c}\hat{a}\right)\nabla\bar{W}\cdot\nabla\bar{W}(\tilde{X}_s)ds}\right]. \end{aligned}$$

Since $ca(x) \leq \hat{a}(x)$, we have

$$T_t^{W,\Lambda} f(x) \leq k e^{-c(\Lambda-\Lambda^*)t}, \quad k = \|f\|_\infty \exp\big(\sup_{y \in \operatorname{supp} f} (\bar{W}(y) - \bar{W}(x))\big). \qquad \square$$

Now we have the result on transience.

Theorem 3.3. *Let* (W, Λ) *be a solution of* (1.1) *and* $\{X_t\}$ *be a solution of* (3.1) *corresponding to* (W, Λ)*. If* (A1)$'$, (A1)$''$, (A2)–(A4) *hold, then for* $\Lambda > \Lambda^*$, $\{X_t\}$ *is transient.*

Proof. Let $f \in C_0(\mathbb{R}^N)$ and $f \geq 0$. Since $\Lambda > \Lambda^*$, we can see that by Lemma 3.2,

$$\int_0^\infty T_t^{W,\Lambda} f(x) dt < \infty.$$

Therefore, $\{X_t\}$ is transient. $\square$

We proved that for $\Lambda > \Lambda^*$, $\{X_t\}$ defined by (3.1) is transient. We next show that if $\Lambda = \Lambda^*$, the corresponding diffusion process $\{X_t^*\}$ satisfying (3.1) is ergodic.

We have to show the following proposition

Proposition 3.4. *Let* (W, Λ) *be a solution of* (1.1) *and* $\{X_t\}$ *be the corresponding diffusion process defined by* (3.1). *Assume* (A1)$'$, (A1)$''$, (A2)–(A4). *If* $\{X_t\}$ *is transient, then there exists* $\alpha > 0$ *such that*

$$T_t^{W,\Lambda} f(x) \leq C e^{-\alpha t}, \quad f \in C_0(\mathbb{R}^N), \ f \geq 0, \ x \in \mathbb{R}^N,$$

where C *is a constant independent of* t.

We prepare two lemmas to prove the above proposition.

Let (W, Λ) be a solution of (1.1) and $\{X_t\}$ be a solution of (3.1). We define occupation measure for $\{X_t\}$ as follows:

$$\mu_t(B) \equiv \frac{1}{t} \int_0^t 1_B(X_s) ds, \ B \in \mathcal{B}(\mathbb{R}^N),$$

where $\mathcal{B}(\mathbb{R}^N)$ is Borel σ-field on $\mathbb{R}^N$. Let $\mathcal{M}_1(\mathbb{R}^N)$ be the set of probability measures on $\mathcal{B}(\mathbb{R}^N)$. We think of $\mathcal{M}_1(\mathbb{R}^N)$ as the topological vector space with topology compatible to weak convergence. Note that $\mu_t \in \mathcal{M}_1(\mathbb{R}^N)$.

The following lemma on large deviation type estimate is useful.

Lemma 3.5. *Let* $\{X_t\}$ *be a solution of* (3.1) *with no explosion in finite time. Then, the following estimate holds:*

$$\varlimsup_{t\to\infty} \frac{1}{t} \log P[\mu_t \in \mathcal{K}] \leq -\inf_{\mu \in \mathcal{K}} I^W(\mu), \quad \mathcal{K} \text{ is compact set in } \mathcal{M}_1(\mathbb{R}^N). \tag{3.10}$$

$I^W(\mu)$ *is defined as follows:*

$$I^W(\mu) \equiv -\inf_{u \in \mathcal{U}} \int \frac{Lu}{u}(x) \mu(dx), \ L \equiv \frac{1}{2} a^{ij} D_{ij} + (\tilde{b} + \hat{a} \nabla W) \cdot \nabla,$$

$$\mathcal{U} \equiv \{u \in C^2(\mathbb{R}^N) : \ Du, \ D^2 u \text{ are bounded and } \exists R > r > 0 \ s.t. \ r \leq u(x) \leq R\}.$$

Note that $I^W(\mu)$ takes values on $[0,\infty]$ and is convex, lower semi-continuous on $\mathcal{M}_1(\mathbb{R}^N)$. This type of estimate is well-known in large deviation theory. As noted in [18], even if the state space of X_t is not compact, (3.10) holds for compact set $\mathcal{K}$ (cf. [17], [18] §7).

We prove the following second lemma.

Lemma 3.6. *Let $\{X_t\}$ be a solution of* (3.1). *Suppose that $\{X_t\}$ does not explode in finite time. If $I^W(\mu^*)=0$, then μ^* is invariant measure for $\{X_t\}$.*

Proof. Since $I^W(\mu^*) = -\inf_{u\in\mathcal{U}}\int (Lu/u)(x)\mu^*(dx) = 0$,

$$\int \frac{Lu}{u}(x)\mu^*(dx) \geq 0, \ \forall u \in \mathcal{U}.$$

Setting $w = \log u$, we have

$$\int Lw(x) + \frac{1}{2}|\nabla w|^2(x)\mu^*(dx) \geq 0, \ u = e^w \in \mathcal{U}. \tag{3.11}$$

It is easy to see that if $u = e^w \in \mathcal{U}$, then $u_\lambda \equiv e^{\lambda w} \in \mathcal{U}$ for $\lambda \in \mathbb{R}$. Therefore, applying λw in (3.11) instead of w,

$$\int Lw(x) + \frac{\lambda}{2}|\nabla w|^2(x)\mu^*(dx) \geq 0, \ u = e^w \in \mathcal{U}, \ \lambda > 0.$$

Taking the limit as $\lambda \to 0$, we have

$$\int Lw(x)\mu^*(dx) \geq 0, \ u = e^w \in \mathcal{U}.$$

Since $u = e^w \in \mathcal{U}$ implies $u_{-1} \equiv e^{-w} \in \mathcal{U}$, we obtain the following equation:

$$\int Lw(x)\mu^*(dx) = 0, \ u = e^w \in \mathcal{U}.$$

Noting that $C_0^\infty(\mathbb{R}^N)$ is included in $\{w: \ u = e^w \in \mathcal{U}\}$, μ^* satisfies the following partial differential equation in distributional sense:

$$L^*\mu^* = 0 \text{ in } \mathbb{R}^N,$$

where L^* is formal adjoint of L. Since we assumed the coefficients of L are sufficiently smooth, μ^* has a density $p^*(x)$ and p^* satisfies

$$L^*p^* = 0 \text{ in } \mathbb{R}^N.$$

Then, by slight modifications of Theorem in p.243, [16] to the case that second order term of L is divergence form, $\mu^*(dx) = p^*(x)dx$ is actually invariant measure. $\square$

(*Proof of Proposition* 3.4) Let us define U_0 as follows:

$$U_0(x) = -\left(\frac{1}{2}a^{ij}D_{ij}W_0 + \frac{1}{2}\hat{a}\nabla W_0\cdot\nabla W_0 + \tilde{b}\cdot\nabla W_0 + V\right),$$

where we take W_0 from (A4). By setting $W_{0,c} \equiv cW_0$ and $W_c \equiv cW$, we have

$$\frac{1}{2}a^{ij}D_{ij}W_{0,c} + \frac{1}{2c}\hat{a}\nabla W_{0,c}\cdot\nabla W_{0,c} + \tilde{b}\cdot\nabla W_{0,c} + cV = -cU_0,$$
$$\frac{1}{2}a^{ij}D_{ij}W_c + \frac{1}{2c}\hat{a}\nabla W_c\cdot\nabla W_c + \tilde{b}\cdot\nabla W_c + cV = c\Lambda$$

where c is in Remark 2.1. In the above equations, subtracting each side of the equations,

$$\frac{1}{2}a^{ij}D_{ij}(W_{0,c}-W_c) + (\tilde{b}+\hat{a}\nabla W)\cdot(\nabla W_{0,c}-\nabla W_c) + \frac{1}{2c}\hat{a}(\nabla W_{0,c}-\nabla W_c)\cdot(\nabla W_{0,c}-\nabla W_c) = -c(U_0+\Lambda).$$

Define $\bar{\phi}$ as $\bar{\phi} = e^{W_{0,c}-W_c}$. Then, we have

$$\frac{1}{2}a^{ij}D_{ij}\bar{\phi} + (\tilde{b}+\hat{a}\nabla W)\cdot\nabla\bar{\phi} + \frac{1}{2c}\left((\hat{a}-ca)\nabla\bar{\phi}\cdot\nabla\bar{\phi}\right)\frac{1}{\bar{\phi}} = -c(U_0+\Lambda)\bar{\phi}. \tag{3.12}$$

Let $\{X_t\}$ be a solution of (3.1). By Ito formula and (3.12),

$$\begin{aligned} &d\left(\bar{\phi}(X_t)e^{\int_0^t c(U_0(X_s)+\Lambda)ds}\right)\\ &= \left[\frac{1}{2}a^{ij}D_{ij}\bar{\phi} + (\tilde{b}+\hat{a}\nabla W)\cdot\bar{\phi} + c(U_0+\Lambda)\right](X_t)e^{\int_0^t c(U_0(X_s)+\Lambda)ds}dt\\ &\quad + \sigma\nabla\bar{\phi}(X_t)e^{\int_0^t c(U_0(X_s)+\Lambda)ds}dB_t\\ &= -\frac{1}{2c}\left[\frac{1}{\bar{\phi}}(\hat{a}-ca)\nabla\bar{\phi}\cdot\nabla\bar{\phi}\right](X_t)e^{\int_0^t c(U_0(X_s)+\Lambda)ds}dt + \sigma\nabla\bar{\phi}(X_t)e^{\int_0^t c(U_0(X_s)+\Lambda)ds}dB_t. \end{aligned}$$

Since $ca(x) \le \hat{a}(x)$ and $\bar{\phi} > 0$, we obtain

$$\bar{\phi}(X_t)e^{\int_0^t c(U_0(X_s)+\Lambda)ds} \le \bar{\phi}(x) + \int_0^t \sigma\nabla\bar{\phi}(X_s)e^{\int_0^s c(U_0(X_r)+\Lambda)dr}dB_s. \tag{3.13}$$

Note that the stochastic integral in the right-hand side of (3.13) is super-martingale because the left-hand side of (3.13) is bounded from below. Then, we have

$$E_x\left[\bar{\phi}(X_t)e^{\int_0^t c(U_0(X_s)+\Lambda)ds}\right] \le \bar{\phi}(x). \tag{3.14}$$

Let $\mathcal{C}_m$ be subset in $\mathcal{M}_1(\mathbb{R}^N)$ defined as follows:

$$\mathcal{C}_m \equiv \{\mu\in\mathcal{M}_1(\mathbb{R}^N):\ \mu(B_n)\ge 1-\delta_n,\ \forall n\ge m\},\quad m\ge 1,$$

where $\{\delta_n\}$ is a sequence such that $\delta_n \to 0$ and determined later. Note that $\mathcal{C}_m$ is relative compact set in $\mathcal{M}_1(\mathbb{R}^N)$ because $\mathcal{C}_m$ is tight. From the definition of $T_t^{W,\Lambda}f$,

$$\begin{aligned} T_t^{W,\Lambda}f(x) &\le E_x[f(X_t);\ \mu_t\in\mathcal{C}_m] + E_x[f(X_t);\ \mu_t\notin\mathcal{C}_m]\\ &\le \|f\|_\infty P_x[\mu_t\in\mathcal{C}_m] + E_x[f(X_t);\ \mu_t\notin\mathcal{C}_m],\quad f\in C_0(\mathbb{R}^N),\ f\ge 0. \end{aligned} \tag{3.15}$$

By Lemma 3.5,

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log P_x[\mu_t\in\mathcal{C}_m]\le -\inf_{\mu\in\bar{\mathcal{C}}_m} I^W(\mu)$$

where $\bar{\mathcal{C}}_m$ is the closure of $\mathcal{C}_m$. Since $I^W(\mu)$ is lower semi-continuous and $\bar{\mathcal{C}}_m$ is compact in $\mathcal{M}_1(\mathbb{R}^N)$, $\inf_{\mu\in\bar{\mathcal{C}}_m} I^W(\mu)$ is attained at some $\mu^*\in\bar{\mathcal{C}}_m$. Since existence of invariant measure implies recurrence, it follows from Lemma 3.6 and transience of $\{X_t\}$

$$\inf_{\mu\in\bar{\mathcal{C}}_m} I^W(\mu)>0,\ \forall m.$$

Hence, we can find a positive constant $\alpha_m>0$ such that

$$P_x[\mu_t\in\mathcal{C}_m]\le Ce^{-\alpha_m t},\quad t>0. \tag{3.16}$$

Then, from (3.15) and (3.16), we obtain

$$\begin{aligned}T_t^{W,\Lambda}f(x)&\le\|f\|_\infty e^{-\alpha_m t}+E_x[f(X_t);\ \mu_t\notin\mathcal{C}_m]\\&\le\|f\|_\infty e^{-\alpha_m t}+\|f\bar{\phi}^{-1}\|_\infty E_x[\bar{\phi}(X_t);\ \mu_t\notin\mathcal{C}_m].\end{aligned}$$

We shall prove that $E_x[\bar{\phi}(X_t);\mu\notin\mathcal{C}_m]$ exponentially decays as $t\to\infty$. On the event $\{\mu_t\notin\ \mathcal{C}_m\}$, there exists $n\ge m$ such that

$$\mu_t(B_n)=\frac{1}{t}\int_0^t 1_{B_n}(X_s)ds\le 1-\delta_n \tag{3.17}$$

which is equivalent to

$$\mu_t(B_n^c)=\frac{1}{t}\int_0^t 1_{B_n^c}(X_s)ds>\delta_n. \tag{3.18}$$

Then, we have

$$\begin{aligned}\int_0^t c(U_0(X_s)+\Lambda)ds&=\int_0^t c(U_0(X_s)+\Lambda)1_{B_n}(X_s)ds+\int_0^t c(U_0(X_s)+\Lambda)1_{B_n^c}(X_s)ds\\&\ge\inf_x c(U_0(x)+\Lambda)\int_0^t 1_{B_n}(X_s)ds+\inf_{|x|\ge n}c(U_0(x)+\Lambda)\int_0^t 1_{B_n^c}(X_s)ds\\&=\beta_0\mu_t(B_n)t+\beta_n\mu_t(B_n^c)t,\end{aligned} \tag{3.19}$$

where we set $\beta_0=\inf_x c(U_0(x)+\Lambda)$, $\beta_n=\inf_{|x|\ge n}c(U_0(x)+\Lambda)$. By (A4), there exists $m\ge 1$ such that

$$\beta_n>0,\ \forall n\ge m.$$

So, we obtain from (3.17), (3.18), (3.19),

$$\int_0^t c(U_0(X_s)+\Lambda)ds\ge(-|\beta_0|(1-\delta_n)+\beta_n\delta_n)t.$$

Take a positive constant $M>0$. Then we choose δ_n such that $M=-|\beta_0|(1-\delta_n)+\beta_n\delta_n$. Indeed, δ_n is defined by

$$\delta_n\equiv\frac{M+|\beta_0|}{|\beta_0|+\beta_n}.$$

Then, we have

$$\int_0^t c(U_0(X_s)+\Lambda)ds \geq Mt \quad \text{on } \{\mu_t \notin \mathcal{C}_m\}. \tag{3.20}$$

By (3.14) and (3.20),

$$\bar{\phi}(x) \geq E_x\left[\bar{\phi}(X_t)e^{\,{}_0^t c(U_0(X_s)+\Lambda)ds};\ \mu_t \notin \mathcal{C}_m\right] \geq e^{Mt}E_x[\bar{\phi}(X_t);\ \mu_t \notin \mathcal{C}_m].$$

Therefore we obtain

$$E_x[\bar{\phi}(X_t);\ \mu_t \notin \mathcal{C}_m] \leq \bar{\phi}(x)e^{-Mt}, \quad t>0. \qquad \square$$

We are ready to prove that for $\Lambda = \Lambda^*$, the corresponding diffusion process $\{X_t^*\}$ is ergodic.

Theorem 3.7. *Let* (W^*,Λ^*) *be a solution of* (1.1) *corresponding to* $\Lambda^* = \inf \mathcal{A}$ *and* $\{X_t^*\}$ *be a solution of* (3.1) *for* (W^*,Λ^*)*. Under* (A1)$'$, (A1)$''$, (A2)–(A4), $\{X_t\}$ *is ergodic.*

Proof. Suppose that $\{X_t^*\}$ is transient. Then, by Proposition 3.4,

$$T_t^{W^*,\Lambda^*}f(x) \leq Ce^{-\alpha t}, \quad \forall f \in C_0(\mathbb{R}^N),\ f \geq 0.$$

Note that α is a positive constant independent of f and x. Taking $0<\epsilon<\alpha$, we see that

$$\int_0^\infty E_x[f(X_t^*)e^{\epsilon t}]dt = \int_0^\infty T_t^{W^*,\Lambda^*}f(x)e^{\epsilon t}dt = \int_0^\infty Ce^{-(\alpha-\epsilon)t}dt < \infty$$

Then, there exists Green function $G(x,y)$ for $(1/2)a^{ij}D_{ij} + (\tilde{b}+\hat{a}\nabla W^*)\cdot\nabla + \epsilon$ and $G(x,y)$ satisfies the following:

$$\frac{1}{2}a^{ij}D_{ij}G(\cdot,y) + (\tilde{b}+\hat{a}\nabla W^*)\cdot\nabla G(\cdot,y) + \epsilon G(\cdot,y) = 0 \text{ in } \mathbb{R}\backslash\{y\}. \tag{3.21}$$

We take a sequence $\{y_n\}$ in $\mathbb{R}^N$ such that $y_n \in B_{n+1}\backslash\bar{B}_n$. Define $\bar{\phi}_n(x)$ as follows:

$$\bar{\phi}_n(x) \equiv \frac{G(x,y_n)}{G(0,y_n)}, \ x \in \mathbb{R}^N\backslash\{y_n\}.$$

Then, we have from (3.21)

$$\frac{1}{2}a^{ij}D_{ij}\bar{\phi}_n + (\tilde{b}+\hat{a}\nabla W^*)\cdot\nabla\bar{\phi}_n + \epsilon\bar{\phi}_n = 0 \text{ in } \mathbb{R}^N\backslash\{y_n\}. \tag{3.22}$$

We note that by setting $\bar{W}_n \equiv (1/\bar{c})\log\bar{\phi}_n$, (3.22) is equivalent to the following:

$$\frac{1}{2}a^{ij}D_{ij}\bar{W}_n + \frac{\bar{c}}{2}a^{ij}D_i\bar{W}_nD_j\bar{W}_n + (\tilde{b}+\hat{a}\nabla W^*)\cdot\nabla\bar{W}_n + \frac{\epsilon}{\bar{c}} = 0 \text{ in } \mathbb{R}^N\backslash\{y_n\},$$

where $\bar{c}$ is taken from Remark 2.1. By Lemma 2.3, we have

$$\sup_{B_r}|\nabla\bar{W}_n| \leq C_r, \ r<n.$$

Thus, in the similar way to the proof of existence of solutions of (1.1), we can see that there exists smooth function $\bar{W}$ such that

$$\frac{1}{2}a^{ij}D_{ij}\bar{W} + (\tilde{b} + \hat{a}\nabla W^*) \cdot \nabla \bar{W} + \frac{\bar{c}}{2} a \nabla \bar{W} \cdot \nabla \bar{W} + \frac{\epsilon}{\bar{c}} = 0. \tag{3.23}$$

Since (W^*, Λ^*) is a solution of (1.2),

$$\frac{1}{2}a^{ij}D_{ij}W^* + \tilde{b} \cdot \nabla W^* + \frac{1}{2}\hat{a}\nabla W^* \cdot \nabla W^* + V - \Lambda^* = 0. \tag{3.24}$$

Adding (3.23) to (3.24), it follows from Remark (2.1) that

$$\begin{aligned}
0 &= \frac{1}{2}a^{ij}D_{ij}(W^* + \bar{W}) + \tilde{b} \cdot (\nabla W^* + \nabla \bar{W}) \\
&\quad + \frac{1}{2}\hat{a}\nabla W^* \cdot \nabla W^* + \hat{a}\nabla W^* \cdot \nabla \bar{W} + \frac{\bar{c}}{2} a \nabla \bar{W} \cdot \nabla \bar{W} + V - \left(\Lambda^* - \frac{\epsilon}{\bar{c}}\right) \\
&\geq \frac{1}{2}a^{ij}D_{ij}(W^* + \bar{W}) + \tilde{b} \cdot (\nabla W^* + \nabla \bar{W}) \\
&\quad + \frac{1}{2}\hat{a}\nabla W^* \cdot \nabla W^* + \hat{a}\nabla W^* \cdot \nabla \bar{W} + \frac{1}{2} \hat{a} \nabla \bar{W} \cdot \nabla \bar{W} + V - \left(\Lambda^* - \frac{\epsilon}{\bar{c}}\right) \\
&= \frac{1}{2}a^{ij}D_{ij}(W^* + \bar{W}) + \tilde{b} \cdot \nabla (W^* + \bar{W}) \\
&\quad + \frac{1}{2}\hat{a}\nabla (W^* + \bar{W}) \cdot \nabla (W^* + \bar{W}) + V - \left(\Lambda^* - \frac{\epsilon}{\bar{c}}\right)
\end{aligned}$$

Thus, $W^* + \bar{W}$ is a super-solution of (1.1) for $\Lambda = \Lambda^* - \epsilon/\bar{c}$. In the same way as the proof that $\mathcal{A} \neq \emptyset$ given in §2, we can see that there exists a smooth function $\tilde{W}$ such that

$$\frac{1}{2}a^{ij}D_{ij}\tilde{W} + \frac{1}{2}\hat{a}\nabla \tilde{W} \cdot \nabla \tilde{W} + \tilde{b} \cdot \nabla \tilde{W} + V = \Lambda^* - \frac{\epsilon}{\bar{c}}.$$

This contradicts to $\Lambda^* = \inf \mathcal{A}$. Therefore, $\{X_t^*\}$ is recurrent.

In order to see that $\{X_t^*\}$ is actually ergodic, we recall the proof of Proposition 3.4. If we suppose $\inf_{\mu \in \bar{\mathcal{C}}_m} I^{W^*}(\mu) > 0$ for each m, we can prove Proposition 3.4, which implies that $\{X_t^*\}$ is transient. Hence, there exists some m such that

$$\inf_{\mu \in \bar{\mathcal{C}}_m} I^{W^*}(\mu) = 0. \tag{3.25}$$

Since $\bar{\mathcal{C}}_m$ is compact, (3.25) is attained at $\mu^* \in \bar{\mathcal{C}}_m$. Then, it follows from Lemma 3.6 that μ^* is invariant measure for $\{X_t^*\}$. □

3.2 Uniqueness of solutions corresponding to the bottom

We proved that solution (W^*, Λ^*) of (1.1) for $\Lambda^* = \inf \mathcal{A}$ corresponds to ergodicity to $\{X_t^*\}$ of (3.1). Now we shall show that solution corresponding to Λ^* is unique up to additive constant. Note that the solution of (1.1) has ambiguity on additive constant.

Theorem 3.8. *Let W_i^*, $i = 1,2$ be solutions of* (1.1) *corresponding to* $\Lambda^* = \inf \mathcal{A}$*. Under* (A1)′, (A1)″, (A2)–(A4)*, there exists constant k such that $W_2^*(x) = W_1^*(x) + k$.*

Proof. Since W_i^*, $i=1,\,2$ are solutions of (1.2),

$$\frac{1}{2}a^{ij}D_{ij}W_1^* + \frac{1}{2}\hat{a}\nabla W_1^*\cdot\nabla W_1^* + \tilde{b}\cdot\nabla W_1^* + V = \Lambda^*,$$
$$\frac{1}{2}a^{ij}D_{ij}W_2^* + \frac{1}{2}\hat{a}\nabla W_2^*\cdot\nabla W_2^* + \tilde{b}\cdot\nabla W_2^* + V = \Lambda^*$$

Subtracting each side in the above equations, we have

$$\frac{1}{2}a^{ij}D_{ij}(W_1^*-W_2^*) + (\tilde{b}+\hat{a}\nabla W_2^*)\cdot(\nabla W_1^*-\nabla W_2^*) + \frac{1}{2}\hat{a}(\nabla W_1^*-\nabla W_2^*)\cdot(\nabla W_1^*-W_2^*) = 0 \tag{3.26}$$

Let us set $\phi(x)\equiv e^{c(W_1^*(x)-W_2^*(x))}$ where c is in Remark 2.1. Rewriting (3.26) in terms of ϕ, we have

$$\frac{1}{2}a^{ij}D_{ij}\phi + (\tilde{b}+\hat{a}\nabla W_2^*)\cdot\nabla\phi + \frac{1}{2c^2}(\hat{a}-ca)\frac{\nabla\phi}{\phi}\cdot\nabla\phi = 0$$

Hence it implies from Remark 2.1 that

$$L\phi \equiv \frac{1}{2}a^{ij}D_{ij}\phi + (\tilde{b}+\hat{a}\nabla W_2^*)\cdot\nabla\phi \le 0. \tag{3.27}$$

Let us take $x,\, y\in\mathbb{R}^N$ and consider the SDE of (3.1) for W_2^*:

$$dX_t^* = (\tilde{b}(X_t^*) + \hat{a}\nabla W_2^*(X_t^*))dt + \sigma(X_t^*)dB_t,\ X_0^* = x.$$

Define $\tau_{B_n} = \inf\{t:\ X_t^*\notin B_n\}$, $\sigma_{B_\epsilon(y)} = \inf\{t:\ X_t^*\in B_\epsilon(y)\}$. Note that $\{X_t^*\}$ is ergodic from Theorem 3.7, especially recurrent. It follows from Ito formula and (3.27) that

$$\begin{aligned}\phi(X^*_{t\wedge\tau_{B_n}\wedge\sigma_{B_\epsilon(y)}}) &= \phi(x) + \int_0^{t\wedge\tau_{B_n}\wedge\sigma_{B_\epsilon(y)}} L\phi(X_s^*)ds + \int_0^{t\wedge\tau_{B_n}\wedge\sigma_{B_\epsilon(y)}} \nabla\phi(X_s^*)\cdot\sigma(X_s^*)dB_s \\ &\le \phi(x) + \int_0^{t\wedge\tau_{B_n}\wedge\sigma_{B_\epsilon(y)}} \nabla\phi(X_s^*)\cdot\sigma(X_s^*)dB_s.\end{aligned}$$

Thus we have $E_x[\phi(X^*_{t\wedge\tau_{B_n}\wedge\sigma_{B_\epsilon(y)}})]\le\phi(x)$. By taking the limit as $n\to\infty$, it follows by Fatou's lemma that $E[\phi(X^*_{t\wedge\sigma_{B_\epsilon(y)}})]\le\phi(x)$. Noting that $P_x[\sigma_{B_\epsilon(y)}<\infty]=1$, we have by sending $t\to\infty$,

$$E_x[\phi(X^*_{\sigma_{B_\epsilon(y)}})]\le\phi(x).$$

We note again that $\{X_t^*\}$ hits the boundary of $B_\epsilon(y)$ in finite time with probability 1. Hence we can see that

$$\phi(x)\ge E_x[\phi(X^*_{\sigma_{B_\epsilon(y)}})]\ge\inf_{\partial B_\epsilon(y)}\phi.$$

Taking the limit as $\epsilon\to 0$, we obtain

$$\phi(y)\le\phi(x),\ x,y\in\mathbb{R}^N$$

which implies ϕ is constant. Therefore $W_1^*-W_2^*$ is also constant. $\square$

Appendix. Proof of (3.4)

Applying Ito formula to $e^{\theta|Y_t|^2}$

$$
\begin{aligned}
e^{\theta|Y_t|^2} &= e^{\theta|x|^2} + \int_0^t 2\theta Y_s e^{\theta|Y_s|^2} \cdot \tilde{b}(Y_s)ds + \int_0^t 2\theta Y_s e^{\theta|Y_s|^2} \cdot \sigma(Y_s)dB_s \\
&\quad + \frac{1}{2}\int_0^t a^{ij}(Y_s)(2\theta e^{\theta|Y_s|^2}\delta_{ij} + 4\theta^2 Y_s^i Y_s^j e^{\theta|Y_s|^2})ds \\
&= \int_0^t 2\theta e^{\theta|Y_s|^2} Y_s \cdot \tilde{b}(Y_s)ds + \int_0^t 2\theta^2 a(Y_s)Y_s \cdot Y_s e^{\theta|Y_s|^2}ds \\
&\quad + \int_0^t \theta \operatorname{tr} a(Y_s) e^{\theta|Y_s|^2}ds + \int_0^t 2\theta e^{\theta|Y_s|^2} Y_s \cdot \sigma(Y_s)dB_s \\
&\leq \int_0^t 2\theta e^{\theta|Y_s|^2} Y_s \cdot \tilde{b}(Y_s)ds + \int_0^t 2\theta^2 \nu_2 |Y_s|^2 e^{\theta|Y_s|^2}ds \\
&\quad + \int_0^t N\theta\nu_2 e^{\theta|Y_s|^2}ds + \int_0^t 2\theta e^{\theta|Y_s|^2} Y_s \cdot \sigma(Y_s)dB_s. \qquad \text{(A.1)}
\end{aligned}
$$

Here we used (A2). By (A1)$''$, it follows that

$$
\begin{aligned}
\text{RHS of (A.1)} &\leq \int_0^t 2(-\gamma_1\theta + \nu_2\theta^2)|Y_s|^2 e^{\theta|Y_s|^2}ds \\
&\quad + \int_0^t (2\gamma_2\theta + N\nu_2\theta)e^{\theta|Y_s|^2}ds + \int_0^t 2\theta e^{\theta|Y_s|^2} Y_s \cdot \sigma(Y_s)dB_s \\
&= \delta_1 \int_0^t |Y_s|^2 e^{\theta|Y_s|^2}ds + \delta_2 \int_0^t e^{\theta|Y_s|^2}ds + \int_0^t 2\theta e^{\theta|Y_s|^2} Y_s \cdot \sigma(Y_s)dB_s,
\end{aligned}
$$

where $\delta_1 \equiv 2(-\gamma_1\theta + \nu_2\theta^2)$, $\delta_2 \equiv 2\gamma_2\theta + N\nu_2\theta$. If we take θ as $0 < \theta < \gamma_1/\nu_2$, then $\delta_1 < 0$. Thus, we have

$$
e^{\theta|Y_t|^2} \leq e^{\theta|x|^2} + \delta_2 \int_0^t e^{\theta|Y_s|^2}ds + \int_0^t 2\theta e^{\theta|Y_s|^2} Y_s \cdot \sigma(Y_s)dB_s.
$$

Setting $\tau_{B_n} = \inf\{t : \ Y_s \notin B_n\}$, we obtain

$$
E_x\left[e^{\theta|Y_{t\wedge\tau_{B_n}}|^2}\right] \leq e^{\theta|x|^2} + \delta_2 \int_0^t E_x\left[e^{\theta|Y_{s\wedge\tau_{B_n}}|^2}\right]ds.
$$

By Gronwall's inequality, there exists some constant $C_{x,T} > 0$ such that

$$
E_x\left[e^{\theta|Y_{t\wedge\tau_{B_n}}|^2}\right] \leq C_{x,T}, \quad 0 \leq t \leq T.
$$

Taking the limit as $n \to \infty$, we obtain

$$
E_x\left[e^{\theta|Y_t|^2}\right] \leq C_{x,T}, \quad 0 \leq t \leq T. \qquad \square
$$

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