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On the Structure of State-Space Models for Discrete-Time Stochastic Vector Processes

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***Abstract*—From a conceptual point of view, structural properties of linear stochastic systems are best understood in a geometric formulation which factors out the effects of the choice of coordinates. In this paper we study the structure of discrete-time linear systems with stationary inputs in the geometric framework of splitting subspaces set up in the work by Lindquist and Picci. In addition to modifying some of the realization results of this work to the discrete-time setting, we consider some problems which are unique to the discrete-time setting. These include the relations between models with and without noise in the observation channel, and certain degeneracies which do not occur in the continuous-time case. One type of degeneracy is related to the singularity of the state transition matrix, another to the rank of the observation noise and invariant directions of the matrix Riccati equation of Kalman filtering. We determine to what extent these degeneracies are properties of the output process. The geometric framework also accommodates infinite-dimensional state spaces, and therefore the analysis is not limited to finite-dimensional systems.**

I. Introduction

THIS paper is concerned with stochastic realization of discrete-time stationary vector processes and the structural properties of the resulting stochastic systems. Although our results provide new insight into the finite-dimensional case, the analysis is not restricted to finite-dimensional systems. The significance of a state-space theory for infinite-dimensional systems has been stressed by many authors in the deterministic context [1]–[4].

The stochastic realization problem is the centerpiece of any theory of stochastic systems. The early results in this field of study were developed in the context of spectral factorization and the positive-real lemma [5]–[8]. In more recent years, however, there has been a trend toward a more geometric approach [14]–[39]. This has several advantages from a conceptual point of view. First, there is no need to restrict the analysis to finite-dimensional systems: the geometric properties are in general (but not always) independent of dimension. Second, it allows us to factor out, in the first analysis, the properties of realizations which depend only on the choice of coordinates. In fact, the geometric approach is coordinate-free. Structural properties which look very complicated in their coordinate-dependent form are given geometric descriptions. Third, systems-theoretical concepts such as minimality, observability, constructibility, etc., can be defined and analyzed in geometric terms. We hasten to stress, however, that such theory does not *replace* the classical results. Indeed, we shall still need to do spectral factorization. The emphasis in the geometric approach is on the structural aspects of the problem rather than on the algorithmic ones, although the insights gained by this analysis may be helpful in providing better algorithms.

In this paper we use the geometric format laid out by Lindquist and Picci [19]–[24] to develop a theory of stochastic realization for discrete-time processes. Since much of the basic geometry is the same in continuous and discrete time, and hence is covered in [19]–[24], our emphasis here is on structural properties which are unique to the discrete-time setting, and which have not been covered elsewhere (such as in the work by Ruckebusch [28]–[32], which deals mainly with the discrete-time case). In addition to working out the details on difference-equation representations, we consider questions concerning the manner in which noise enters into the observation channel and the relations between models with and without observation noise. We study the types of degeneracy which manifest themselves either by the transition function being singular or the observation noise being deficient in rank. The first type of degeneracy occurs in the important class of moving-average processes, whereas the second one is related to

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the concept of *invariant directions*, a topic which has generated a rather extensive literature [40]–[43], [11].

The outline of the paper is as follows. The purpose of Section II is to define basic concepts and to motivate the reader for what is to follow. In Section III we review some basic geometric theory from [19]–[24], and in Section IV we introduce the noise processes by means of Wold decomposition. Section V is devoted to the construction of realizations. We follow the same pattern as in the continuous-time work [23], [24], but here the differences between continuous time and discrete time are nontrivial. The discrete-time case has also been studied by Ruckebusch [28]–[32], but his work contains no explicit construction of realizations. In Section VI we consider singularity in the observation noise, and Section VII contains a discussion of models without observation noise. In Section VIII we introduce Hardy space theory and, among other things, reformulate some of the results of [20]–[24] in the unit circle form required for the discrete-time case. We do this at a later point in the theory than in [20]–[24], and there is a reason for this. The purpose of the spectral theory is to provide the state-space description with more structure and to obtain a tool for deriving additional results, but the basic structural properties do not depend on this particular representation. Section X is devoted to a general discussion of the degeneracies mentioned above. We sort out to what extent these are properties of the given output process or merely of the individual realization (modulo coordinate transformations). Finally, in Sections IX and XI we illustrate the various results by examples.

This paper is a revised version of [44].

II. Preliminaries and Motivation

Let us, for the moment, consider a finite-dimensional stochastic system

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) & \text{(2.1a)} \\ y(t) = Cx(t) + Du(t) & \text{(2.1b)} \end{cases}$$

where x is the n-dimensional *state process*, y the m-dimensional *output process*, and u is p-dimensional white noise, i.e.,

$$E\{u(s)u(t)'\} = I\delta_{st} \tag{2.2}$$

(δ_{st} is the Kronecker symbol which is one when $s=t$ and zero otherwise, and prime (′) denotes transpose), and where A, B, C, and D are constant matrices of appropriate dimensions, with A having all its eigenvalues inside the unit circle. All processes are zero mean. The recursions (2.1) evolve forward in time t ranging over all integers ($\mathbb{Z}$), so that, for each t, $y(t)$ and $x(t+1)$ are linear functions of the past noises $\{u(t), u(t-1), u(t-2), \cdots\}$ and consequently uncorrelated to the future noises $\{u(t+1), u(t+2), \cdots\}$. Therefore, the $(n+m)$-dimensional vector process $\binom{x}{y}$ is (wide sense) stationary and (wide sense) Markov. For each $t \in \mathbb{Z}$, define the finite-dimensional space

$$X_t = \{a'x(t) \mid a \in \mathbb{R}^n\} \tag{2.3}$$

of all linear functionals (linear combinations of the components) of the state $x(t)$ at time t. The dimension of X is at most n.

The *stochastic realization problem* is the inverse problem in which the process y is given and we want to construct systems of type (2.1), called *stochastic realizations*, having y as its output.[1]

[1]There are two versions of this problem which are distinguished by whether y is a (strict sense) stochastic process defined on a probability space or merely a weak stationary process characterized by its spectral density. Although we shall take the first point of view here, the theory presented in this paper provides solutions of both problems.

Let $\hat{H}$ be the space of all finite linear combinations

$$\sum_{i=0}^{N} \sum_{j=1}^{m} \alpha_{ij} y_j(t_i) \tag{2.4}$$

of the stochastic variables $\{y_i(t);\ t \in \mathbb{Z}, i = 1,2,\cdots,m\}$. This is a pre-Hilbert space if we endow it with the inner product $\langle \xi, \eta \rangle = E\{\xi\eta\}$, where $E\{\cdot\}$ denotes mathematical expectation. Now closing $\hat{H}$ by amending to it all limit points of convergent sequences in $\hat{H}$ we obtain a Hilbert space H (whose dimension is countably infinite). Since y is stationary there is a unitary operator U: $H \to H$, called the *shift*, with the property

$$y_i(t+1) = Uy_i(t) \tag{2.5}$$

for all $t \in \mathbb{Z}$ and $i = 1,2,\cdots,m$ [47]. (A unitary operator is such that $U^{-1} = U^*$, the adjoint of U.)

Since y is the only thing given, it is reasonable to require that $X_t \subset H$ for all $t \in \mathbb{Z}$. A realization with this property is called *internal* or *output-induced*. Since $a'x(0) \in H$, $a'x(0) = \lim_{k\to\infty}\eta_k$, where $\{\eta_1, \eta_2, \eta_3, \cdots\}$ are random variables of the form (2.4). Then, by stationarity, $a'x(t) = \lim_{k\to\infty}\hat{\eta}_k$ where $\hat{\eta}_k$ is obtained from η_k by exchanging each $y_j(t_i)$ in the corresponding expression (2.4) by $y_j(t_i + t)$. Hence, $\hat{\eta}_k = U^t\eta_k$, and consequently, by continuity, $a'x(t) = U^t[a'x(0)]$. Therefore,

$$X_t = U^t X \tag{2.6}$$

for each $t \in \mathbb{Z}$, where, to simplify notations, we write X instead of X_0.

Next define two subspaces of H, the *past space* H^- and the *future space* H^+. Let H^- be the subspace generated by $\{y(-1), y(-2), y(-3), \cdots\}$, i.e., the closed linear hull of all expressions (2.4) with each t_i negative, and similarly, let H^+ be the subspace generated by $\{y(0), y(1), y(2), \cdots\}$. We shall investigate the relations between the spaces H^-, H^+, and X. To this end we shall need some notations. For each $\lambda \in H$ and subspace Z, let $E^Z\lambda$ be the orthogonal projection of λ onto Z. (We use this notation since, if Z is generated by a random variable or process z, $E^Z\lambda$ is the wide sense conditional mean of λ given z [47].) Moreover, we define the vector sum $Y \vee Z$ of two subspaces Y and Z as the closed linear hull of all sums $\eta + \zeta$ with $\eta \in Y$ and $\zeta \in Z$. Finally, let $Y \perp Z$ denote the fact that Y and Z are orthogonal. It is clear from (2.1) that each $\lambda \in H^+$ has the form

$$\lambda = b'x(0) + \sum_{i=0}^{\infty} f_i'u(i) \tag{2.7}$$

where $b, f_1, f_2, \cdots$ are constant vectors. Since λ and $b'x(0)$ both belong to H, so does the last term of (2.7). Now, due to the forward property of (2.1), this last term is orthogonal to both H^- and X, and is therefore canceled by the projections E^X and $E^{H^- \vee X}$. Hence, $E^{H^- \vee X}\lambda = b'x(0)$; consequently

$$\forall \lambda \in H^+, \qquad E^{H^- \vee X}\lambda = E^X\lambda. \tag{2.8}$$

This is equivalent to $(\lambda - E^X\lambda) \perp H^- \vee X$, or, since $\lambda - E^X\lambda$ is orthogonal to X by definition, $(\lambda - E^X\lambda) \perp H^-$. Therefore, $\langle \lambda - E^X\lambda, \mu \rangle = 0$ for all $\lambda \in H^+$ and $\mu \in H^-$, or, equivalently,

$$\langle E^X\mu, E^X\lambda \rangle = \langle \mu, \lambda \rangle \qquad \text{for all } \mu \in H^-,\ \lambda \in H^+, \tag{2.9}$$

i.e., H^- and H^+ are *conditionally orthogonal given* X; we write this as $H^- \perp H^+ \mid X$. Tracing the previous argument backwards it is easy to see that (2.8) and (2.9) are in fact equivalent and, by symmetry, also equivalent to

$$\forall \lambda \in H^-, \qquad E^{H^+ \vee X}\lambda = E^X\lambda. \tag{2.10}$$

A subspace X satisfying one of the three equivalent conditions (2.8)–(2.10) is called a *splitting subspace*. Clearly X serves as a "sufficient statistic" in the sense that it contains all the "information" about the past of y needed in prediction of the future of y and vice-versa.

It is by now well known [10], [11] that to each realization (2.1) of y there corresponds a *backward* realization

$$\begin{cases} \bar{x}(t)=\bar{A}\bar{x}(t+1)+\bar{B}\bar{u}(t) \\ y(t)=\bar{C}\bar{x}(t+1)+\bar{D}\bar{u}(t) \end{cases} \tag{2.11a}$$

with the same X-space as (2.1), i.e.,

$$\{a'\bar{x}(t)|a\in\mathbb{R}^n\}=X_t. \tag{2.12}$$

This system evolves backwards in time in the sense that $\bar{x}(t)$ and $y(t)$ are linear functions of $\{\bar{u}(t),\bar{u}(t+1),\cdots\}$ and are uncorrelated to $\{\bar{u}(t-1),\bar{u}(t-2),\cdots\}$; $\bar{A}$ has all its eigenvalues inside the unit circle. This realization is related to (2.10) as (2.1) is related to (2.8), and we can use (2.11) to derive (2.10) directly.

We define a shift $U(X)$ on the subspace X by restricting the domain of U to X and projecting the image back to X, i.e.,

$$U(X):=E^XU|_X \tag{2.13}$$

is a linear operator $X\to X$ defined as $U(X)\xi=E^XU\xi$. We call $U(X)$ the *Markov operator* of X. It is an abstract counterpart to A'. In fact, if $\xi:=a'x(0)$ is an arbitrary element in X, $U(X)\xi=E^Xa'x(1)=a'Ax(0)$, since $a'Bu(0)\perp X$. Likewise, $U(X)^2\xi=E^Xa'Ax(1)=a'A^2x(0)$, etc., in general, for $t\geq 0$,

$$U(X)^t\xi=a'A^tx(0). \tag{2.14}$$

On the other hand, if we project $U^t\xi=a'x(t)$ orthogonally onto $H^-\vee X$ we also obtain (2.14), since the components of $u(0),u(1),\cdots$, and $u(t)$ are orthogonal to $H^-\vee X$. Hence,

$$\forall\xi\in X,\qquad E^{H^-\vee X}U^t\xi=U(X)^t\xi\qquad \text{for } t=0,1,2,\cdots. \tag{2.15a}$$

By applying the same procedure to the backward system (2.11) we obtain the symmetric condition

$$\forall\xi\in X,\qquad E^{H^+\vee X}U^{-t}\xi=\left[U(X)^*\right]^t\xi\qquad \text{for } t=0,1,2,\cdots \tag{2.15b}$$

where

$$U(X)^*:=E^XU^{-1}|_X \tag{2.16}$$

is an abstract representation of $\bar{A}'$. It is easy to see that $\langle U(X)\xi,\eta\rangle=\langle\xi,U(X)^*\eta\rangle$ for all ξ and η in X, and therefore $U(X)^*$: $X\to X$ is the adjoint of $U(X)$. The two conditions (2.15) are *not* equivalent.

As we shall see below, (2.9) and (2.15) together completely characterize the Markovian property of the system (2.1), or that of (2.11), and therefore we shall call a splitting subspace X *Markovian* if it satisfies conditions (2.15). Just as we derived (2.8) and (2.9), we can show that

$$\left[H^-\vee(\vee_{t=-\infty}^{0}X_t)\right]\perp\left[H^+\vee(\vee_{t=0}^{\infty}X_t)\right]|X \tag{2.17}$$

where $\vee_{t\in I}Z_t$ denotes the vector sum of the spaces $\{Z_t;\ t\in I\}$. In fact, it can be shown that (2.9)+(2.15) is equivalent to (2.17) (also for noninternal realizations). Here we have chosen to express the Markov property in terms of the Markov operator $U(X)$.

To solve the stochastic realization problem we shall first determine the Markovian splitting subspaces of y; this is a geometric problem in the Hilbert space H generated by the given process. There are several reasons for adopting this procedure. First the system

$$\begin{cases} \tilde{x}(t+1)=TAT^{-1}\tilde{x}(t)+TBu(t) & (2.18a)\\ y(t)=CT^{-1}\tilde{x}(t)+Du(t) & (2.18b)\end{cases}$$

where T is an arbitrary nonsingular $n\times n$-matrix, has the same splitting subspace X as (2.1). The two systems can be obtained from each other by a trivial change of coordinates in X. In the first analysis we want to take a *coordinate-free* approach to the problem, and factor out properties of realizations connected with choice of coordinates in X.

Second, and more importantly, we want to consider also infinite-dimensional realizations in the same framework. In order for y to have a finite-dimensional representation (2.1) it is necessary that y has a rational spectral density. However, we wish to consider realizations of arbitrary stationary processes (subject to some technical conditions to be introduced below). This leads to infinite-dimensional X in general, and although we can construct systems of type (2.1) in this situation also (see Section V), it does require some care, and we need to decide how to define the state process x. In the basic geometric theory these difficulties do not occur.

Finally, we have the concept of *minimality*. In the classical (finite-dimensional) theory we say that a realization (2.1) of y is minimal if the dimension n of the state process is as small as possible. It is well-known that this requires that i) (A,B) is reachable, ii) (C,A) is observable, and iii) the spectral factor $W(z)=C(zI-A)^{-1}B+D$ is minimal in the sense that its McMillan degree is as small as possible [6]–[13]. Note that it is not sufficient that (A,B,C,D) be a minimal realization of W in the deterministic sense (i.e., that i) and ii) hold), but we must also have condition iii) fulfilled. Condition i) is equivalent to $x(0)$ being a basis in X. This can be seen by noting that the components of $x(0)=\sum_{k=-\infty}^{-1}A^kBu(k)$ are linearly independent if and only if $(B,AB,A^2B,\cdots\}$ has full rank, i.e., rank n. If (A,B) is not reachable, $n>\dim X$. Consequently reachability does not enter into the basic splitting subspace geometry. Once X has been determined, one is expected to choose $x(0)$ as a basis, and this will automatically take care of condition i). Now, having thus removed condition i) from consideration, the most natural way of defining minimality is simply to require that X is minimal in the sense that it contains no other Markovian splitting subspace as a *proper* subset. This is simple and has the advantage of also working in the infinite-dimensional case. Conditions ii) and iii) will then occur as geometric conditions which confine the size of X. In Section V we shall see that ii)+iii) is equivalent to requiring that *both* (C,A) and $(\bar{C},\bar{A})$ are observable.

III. A Review of Basic Geometric Theory

The problem of determining all Markovian splitting subspaces X leads to a geometric theory on Hilbert space [18]–[33]. We shall review some of the basic results from [21]–[24], mostly without proofs.

It should be noted that the theory reviewed in this section does not depend on the process y other than through H, H^-, H^+, and U. Hence, we could start out by merely assuming a Hilbert space H endowed with a unitary operator U: $H\to H$ and two subspaces H^- and H^- with the properties $U^*H^-\subset H^-$, $UH^-\subset H^-$, and $H^-\vee H^+=H$. This remark may seem a bit displaced in this paper since we have a particular application in mind, and indeed, H, H^-, H^+, and U will have the meanings assigned to them in Section II for the rest of the paper. However, in Section VII we shall redefine H^- *temporarily* to be UH^-, i.e., the space defined

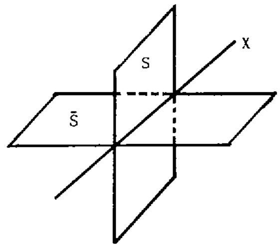

Fig. 1.

symmetrically to H^+ over the past, and we shall need to know that the results of this section hold also with this choice, as indeed the remark above implies. Such a symmetric choice of H^- and H^+ will lead to models (2.1) in which $D=0$, cf. [26].

Perpendicular intersection is a fundamental concept in the geometry of splitting subspaces. Two subspaces A and B are said to *intersect perpendicularly* if $E^A B = A \cap B$ or, equivalently, $E^B A = A \cap B$. It is shown in [22], [24], that A and B intersect perpendicularly if and, provided $A \vee B = H$, only if $B^\perp \subset A$, or, equivalently $A^\perp \subset B$. (Here $A^\perp$ is the orthogonal complement of A in H and $E^A B = \{E^A \beta | \beta \in B\}$.)

Theorem 3.1 [21], [22]: A subspace X is a splitting subspace if and only if $X = S \cap \bar{S}$ for some pair $(S, \bar{S})$ of perpendicularly intersecting subspaces such that $S \supset H^-$ and $\bar{S} \supset H^-$. The correspondence $X \leftrightarrow (S, \bar{S})$ is one to one, $(S, \bar{S})$ being uniquely determined by the relations $S = H^- \vee X$ and $\bar{S} = H^+ \vee X$.

We shall use the notation $X \sim (S, \bar{S})$ to recall this correspondence. The theorem is illustrated in Fig. 1 with obvious restrictions on dimensions.

From this geometry we immediately have the following corollary, in which $\oplus$ denotes orthogonal direct sum and $A \ominus B$ is the space C such that $A = B \oplus C$.

Corollary 3.1: Let S and $\bar{S}$ be subspaces such that $S^\perp \subset \bar{S}$ and set $X = S \cap \bar{S}$. Then

$$H = \bar{S}^\perp \oplus X \oplus S^\perp. \tag{3.1}$$

Equation (3.1) should be compared to the orthogonal decomposition in terms of incoming and outgoing subspaces in Lax–Phillips scattering theory [46]. We say that the splitting subspace $X \sim (S, \bar{S})$ is *proper* if both $S^\perp$ and $\bar{S}^\perp$ are full rank. (A subspace $A \subset H$ is full rank if $\bigvee_{t=-\infty}^{\infty} U^t A = H$.)

The Markovian property can now be characterized by simple invariance conditions on S and $\bar{S}$. In this paper we have a different definition of Markovian than in [21], and therefore we shall provide a proof of the next theorem, although it can be found in [21]. A proof can also be based on Lemma 0 in [48].

Theorem 3.2: Let $X \sim (S, \bar{S})$ be a splitting subspace. Then X is Markovian if and only if S and $\bar{S}$ satisfy the invariance conditions

$$U^{-1} S \subset S \tag{3.2a}$$

$$U \bar{S} \subset \bar{S}. \tag{3.2b}$$

Proof:

If: Let $t = 0,1,2,\cdots$ and $\xi \in X$ be arbitrary. Since $S = X \oplus \bar{S}^\perp$ (Corollary 3.1), $E^S U^t \xi = E^X U^t \xi + E^{\bar{S}^\perp} U^t \xi$. However, the last term in this expression is zero, for since $\xi \in X \subset \bar{S}$, (3.2b) implies that $U^t \xi \in \bar{S}$. Moreover, in view of (3.1), $E^X U^{t-1} \xi = U(X) E^X U^t \xi + E^X U E^{S^\perp} U^t \xi + E^X U E^{\bar{S}^\perp} U^t \xi$. Here the last term is zero for the same reason as above, and the second term is zero because, by (3.2a), $US^\perp \subset S^\perp \perp X$. (Here we have used the easily proven fact that if a subspace is invariant under some transformation, then its orthogonal complement is invariant under the adjoint transformation.) Then, by induction, we see that $E^S U^t \xi = U(X)^t \xi$ for all $\xi \in X$ and $t = 0,1,2,\cdots$. This is precisely (2.15a). In the same way we see that (2.15b) holds.

Only If: Since $S = X \oplus \bar{S}^\perp$ (Corollary 3.1), $E^S U\xi = E^X U\xi + E^{\bar{S}^\perp} U\xi$. But, if $\xi \in X$, (2.15a) implies that $E^S U \xi \in X$, and hence $E^{\bar{S}^\perp} U\xi = 0$. Consequently, $UX \subset \bar{S}$, from which we see that (3.2b) must hold, for $\bar{S} = H^+ \vee X$ and $UH^+ \subset H^+$. In the same way we see that (3.2a) must hold. □

The class of Markovian splitting subspaces is very wide. In fact, from Theorems 3.1 and 3.2 we see that H, H^-, and H^+ are all Markovian splitting subspaces (although they are not proper). However, we want X to be as small as possible in some sense. A (Markovian) splitting subspace is said to be *minimal* if it contains no proper subspace which is also a (Markovian) splitting subspace. It can be shown that a minimal Markovian splitting subspace is also a minimal splitting subspace [22], [24], [27], and therefore the two properties "Markovian" and "minimal" can be studied separately.

In analogy with deterministic realization theory [45, p. 52] we say that $\xi \in X$ is *unobservable* if $\xi \perp H^+$ and *unconstructible* if $\xi \perp H^-$. The subspaces $X \cap (H^+)^\perp$ and $X \cap (H^-)^\perp$ are the unobservable and unconstructible subspaces of X, respectively. The splitting subspace is said to be *observable* if $X \cap (H^+)^\perp = 0$ and *constructible* if $X \cap (H^-)^\perp = 0$. The simple proof of the following theorem is included for the benefit of the reader.

Theorem 3.3 [21]: A splitting subspace $X \sim (S, \bar{S})$ is observable if and only if

$$\bar{S} = H^+ \vee S^\perp \tag{3.3}$$

and constructible if and only if

$$S = H^- \vee \bar{S}^\perp. \tag{3.4}$$

Proof: The observability condition $X \cap (H^+)^\perp = 0$ is equivalent to $X^\perp \vee H^+ = H$. But, by Corollary 3.1, $X^\perp = S^\perp \oplus \bar{S}^\perp$. Therefore, since $\bar{S}^\perp \perp H^+$, observability is equivalent to $\bar{S}^\perp \oplus (S^\perp \vee H^+) = H$, which is the same as (3.3). The constructibility condition is derived in the same way. □

Conditions (3.3) and (3.4) are minimality conditions for $\bar{S}$ and S, respectively. In fact, $\bar{S} := H^+ \vee S^\perp$ is the smallest subspace containing H^+ which intersects S perpendicularly. Likewise, $S := H^- \vee \bar{S}^\perp$ is the smallest subspace containing H^- which intersects $\bar{S}$ perpendicularly. Therefore, as $X = S \cap \bar{S}$, the following theorem should come as no surprise. Ruckebusch, who was the first to use the terms observable and constructible in the sense described above, proved a version of this theorem in [30].

Theorem 3.4: A splitting subspace is minimal if and only if it is both observable and constructible.

Now, let us consider a splitting subspace $X \sim (S, \bar{S})$ with $S = H^-$. In view of (3.4), such an X is always constructible because, by perpendicular intersection, $\bar{S}^\perp \subset S = H^-$. Then, for X to be minimal, the observability condition (3.3) needs to be satisfied, i.e., we must have $\bar{S} = H^+ \vee (H^-)^\perp = (N^-)^\perp$, where

$$N^- := H^- \cap (H^+)^\perp. \tag{3.5}$$

(It is easy to see that, for any two subspaces A and B, $(A \vee B)^\perp = A^\perp \cap B^\perp$.) Hence, we have defined a minimal splitting subspace $X_- \sim (H^-, (N^-)^\perp)$ which satisfies (3.2); therefore X_- is Markovian. By the definition of perpendicular intersection, $X_- = E^S \bar{S} = E^{H^-}[H^+ \vee (H^-)^\perp]$, i.e., by orthogonality, X_- is the closure of $E^{H^-} H^+$ (which in the infinite-dimensional case may not be closed); we write this

$$X_- = \bar{E}^{H^-} H^+. \tag{3.6}$$

Therefore we call X_- the *predictor space*. Similarly, $X_+ \sim ((N^+)^\perp, H^+)$, where

$$N^+ = H^+ \cap (H^-)^\perp \tag{3.7}$$

is a minimal Markovian splitting subspace, called the *backward* *predictor space* with the property that

$$X_+ = \bar{E}^{H^+}H^-. \tag{3.8}$$

Since $N^+ \subset (N^-)^\perp$, $(N^+)^\perp$, and $(N^-)^\perp$ intersect perpendicularly, and therefore $H^\square := (N^+)^\perp \cap (N^-)^\perp$ is a splitting subspace which is Markovian but not (in general) minimal. Its decomposition (3.1) is

$$H = N^- \oplus H^\square \oplus N^+. \tag{3.9}$$

Let $X \sim (S, \bar{S})$ be any splitting subspace. Then, in view of (3.3) and the fact that $S^\perp \subset (H^-)^\perp$, we must have $\bar{S} \subset (N^-)^\perp$ for X to be observable. Similarly, we see that X constructible implies that $S \subset (N^+)^\perp$. Hence, any minimal splitting subspace must be contained in $H^\square$. Moreover, from (3.1) and (3.9) it is easy to see that

$$H^\square = X_- \vee X_- \tag{3.10}$$

and therefore $H^\square$ is the closed linear hull of all minimal splitting subspaces; it is called the *frame space*. Consequently, N^- and N^+ contain no useful information about the process y and could be discarded.

To have a nontrivial realization problem, N^- and N^+ need to be nontrivial so that there is some data reduction. In this case we say that y is *noncyclic*. If N^- and N^+ have full rank, we say that y is *strictly noncyclic*. This is the same as $H^\square$ being proper. Under this assumption any minimal splitting subspace $X \sim (S, \bar{S})$ is also proper, since $S^\perp \supset N^+$ and $\bar{S}^\perp \supset N^-$.

In the sequel quantities corresponding to the two splitting subspaces X_- and X_- will be marked by a plus or minus *subscript*.

IV. Generating Processes

From now on we shall assume that the given process y is strictly noncyclic and full rank, i.e., it has a spectral density $\Phi(e^{i\omega})$ which is full rank for almost all ω [47]. These assumptions are actually more than we need but they are convenient.

Let $X \sim (S, \bar{S})$ be a proper Markovian splitting subspace. In view of (3.2), X being proper is equivalent to

$$\begin{cases} \cap_{t=-\infty}^{0}(U^tS) = 0 & (4.1a) \\ \cap_{t=0}^{\infty}(U^t\bar{S}) = 0. & (4.1b) \end{cases}$$

Since X is Markovian, $S \subset US$, and it can be shown that $V := (US) \ominus S$ is a finite-dimensional subspace [49]. In fact, since y is full rank, $\dim V = m$, the dimension of the process y. Define $V_t := U^tV$ for all $t \in \mathbb{Z}$. Then we immediately have

$$S = V_{-1} \oplus V_{-2} \oplus \cdots \oplus V_{-t} \oplus (U^{-t}S)$$

and, in view of (4.1a), it can be shown that

$$S = V_{-1} \oplus V_{-2} \oplus V_{-3} \oplus \cdots. \tag{4.2}$$

Also, because of the full rank property of S,

$$H = \cdots \oplus V_{-2} \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus V_2 \oplus \cdots. \tag{4.3}$$

This is the so-called *Wold decomposition* [49].

Next choose an orthogonal basis $\{v_1, v_2, \cdots, v_m\}$ in V and, for each $t \in \mathbb{Z}$, define the m-dimensional random vector $u(t)$ with components $u_i(t) := U^tv_i$, $i = 1, 2, \cdots, m$. Then, since $U^sV \perp U^tV$ for all $s \neq t$, $E\{u_i(s)u_j(t)\} = \langle U^sv_i, U^tv_j\rangle = \langle v_i, U^{t-s}v_j\rangle = \delta_{st}\delta_{ij}$. Consequently,

$$E\{u(s)u(t)'\} = I\delta_{st}, \tag{4.4}$$

i.e., $\{u(t);\ t \in \mathbb{Z}\}$ is a normalized white noise process.

Now, for an arbitrary m-dimensional white noise process $\{w(t);\ t \in \mathbb{Z}\}$, define $H_t(w)$ to be the (finite-dimensional) space consisting of all linear combinations of the random variables $\{w_1(t), w_2(t), \cdots, w_m(t)\}$, and set $H(w) := \oplus_{t=-\infty}^{\infty} H_t(w)$, $H^-(w) := \oplus_{t=-\infty}^{-1} H_t(w)$, and $H^+(w) := \oplus_{t=0}^{\infty} H_t(w)$, much in analogy with the definitions of H, H^-, and H^+. Let $\mathscr{U}$ denote the class of all m-dimensional, normalized white noise processes w such that $H(w) = H$. Then, since $\{u_1(t), u_2(t), \cdots, u_m(t)\}$ is a basis in V_t, $H_t(u) = V_t$, and (4.2) can be written as

$$S = H^-(u) \tag{4.5}$$

and (4.4) as $H(u) = H$, i.e., $u \in \mathscr{U}$.

Similarly, (3.2b) gives us $U\bar{S} \subset \bar{S}$. Set $\bar{V} := \bar{S} \ominus U\bar{S}$ and $\bar{V}_t := U^t\bar{V}$. Then, as above, we see that $\dim \bar{V} = m$, that

$$\bar{S} = \bar{V}_0 \oplus \bar{V}_1 \oplus \bar{V}_2 \oplus \cdots \tag{4.6}$$

and that $H = \oplus_{t=-\infty}^{\infty} \bar{V}_t$. Hence, we can proceed as above to construct a normalized white noise process $\{\bar{u}(t);\ t \in \mathbb{Z}\}$ of class $\mathscr{U}$, such that

$$\bar{S} = H^+(\bar{u}). \tag{4.7}$$

Consequently, to any proper Markovian splitting subspace $X \sim (S, \bar{S})$ there corresponds a pair $(u, \bar{u})$ of normalized white noise processes of class $\mathscr{U}$ such that $S = H^-(u)$ and $\bar{S} = H^-(\bar{u})$. These processes, which are called the *generating processes* of X, are unique modulo trivial coordinate transformations in V and $\bar{V}$. In particular, let $(u_-, \bar{u}_-)$ and $(u_+, \bar{u}_+)$ be the generating processes of X_- and X_+, respectively. Then $H^-(u_-) = H^-$, i.e., u_- is the (steady-state) *innovation process* of y, and $H^+(\bar{u}_+) = H^+$, i.e., $\bar{u}_+$ is the (steady-state) *backward innovation process.*

V. Realizations

Let $X \sim (S, \bar{S})$ be a proper Markovian splitting subspace with generating processes $(u, \bar{u})$. We want to represent the given process y in terms of X.

Lemma 5.1: Set $\tilde{X} := (US) \cap \bar{S}$. *Then* $\tilde{X} \sim (US, \bar{S})$ *is a (nonminimal) proper Markovian splitting subspace such that* $y_i(0) \in \tilde{X}$ *for* $i = 1, 2, \cdots, m$. *Moreover,*

$$\tilde{X} = X \oplus H_0(u) = (UX) \oplus H_0(\bar{u}). \tag{5.1}$$

Proof: Since $\bar{S}^\perp \subset S \subset US$, US and $\bar{S}$ intersect perpendicularly. Also $US \supset S \supset H^-$ and $\bar{S} \supset H^+$. Therefore, by Theorem 3.1, $\tilde{X} \sim (US, \bar{S})$ is a splitting subspace. Clearly, it is also Markovian (Theorem 3.2) and proper, since $X \sim (S, \bar{S})$ is. Since $(UH^-) \cap H^+ \subset (US) \cap \bar{S}$, $y_i(0) \in \tilde{X}$ for $i = 1, 2, \cdots, m$. Now, by Corollary 3.1, $X := S \ominus \bar{S}^\perp$ and $\tilde{X} = (US) \ominus \bar{S}^-$. Inserting $US = H_0(u) \oplus S$ into the second relation and using the first we obtain $\tilde{X} = H_0(u) \oplus X$. Likewise, $X = \bar{S} \ominus S^\perp$, $\tilde{X} = \bar{S} \ominus (US)^\perp$, and $\bar{S} = H_0(\bar{u}) \oplus (U\bar{S})$ yield $\tilde{X} = H_0(\bar{u}) \oplus (UX)$. □

Therefore, using the first representation (5.1),

$$y_i(0) = E^X y_i(0) + E^{H_0(u)} y_i(0). \tag{5.2}$$

The last term can be written $\Sigma_{j=1}^m d_{ij}u_j(0)$ for some real numbers $d_{i1}, d_{i2}, \cdots, d_{im}$. Since $y_i(0) - \Sigma_{j=1}^m d_{ij}u_j(0)$ is orthogonal to $u_k(0)$ for $k = 1, 2, \cdots, m$, we have $\langle y_i(0), u_k(0)\rangle = \Sigma_{j=1}^m d_{ij}\langle u_j(0), u_k(0)\rangle$ for $i, k = 1, 2, \cdots, m$, and therefore, in view of (4.4),

$$d_{ij} = \langle y_i(0), u_j(0)\rangle. \tag{5.3}$$

Then, applying the operator U^t to (5.2), we obtain

$$y(t) = \eta(t) + Du(t) \tag{5.4}$$

where $\eta_i(t) := U^tE^Xy_i(0)$, $i = 1, 2, \cdots, m$ and D is the $m \times m$-matrix with components (5.3).

We need a representation for the vector process $\{\eta(t);\ t \in \mathbb{Z}\}$. The components $\eta_1(0), \eta_2(0), \cdots, \eta_m(0) \in X$. Therefore, in analogy with the deterministic case [1]–[4], [50], we construct a shift realization with respect to X. In doing so we follow the pattern of [23], [24].

Lemma 5.2: For any $\xi \in X$ and $t \in \mathbb{Z}$, we have the representation

$$U^t\xi = \sum_{k=-\infty}^{t-1} \sum_{i=1}^{m} \langle \xi, [U(X)^*]^{t-k-1} E^X u_i(-1) \rangle u_i(k). \tag{5.5}$$

Proof: Since $\xi \in X \subset S = H^-(u)$, $\xi = \sum_{k=-\infty}^{-1} \sum_{i=0}^{m} f_{ki} u_i(k)$ for some real numbers $\{f_{ki};\ i = 1, \cdots, m,\ k = -1, -2, \cdots,\}$, and therefore

$$U^t\xi = \sum_{k=-\infty}^{-1} \sum_{i=1}^{m} f_{ki} u_i(k+t) = \sum_{k=-\infty}^{t-1} \sum_{i=1}^{m} f_{k-t,i} u_i(k). \tag{5.6}$$

Now, (2.15a) can be written $E^S U^t \xi = U(X)^t \xi$ for $t = 0, 1, 2, \cdots$, and therefore, since $S = H^-(u)$,

$$U(X)^t \xi = \sum_{k=-\infty}^{-1} \sum_{i=1}^{m} f_{k-t,i} u_i(k).$$

Consequently, $\langle U(X)^{t-1}\xi, u_j(-1)\rangle = f_{-t,j}$. Since $[u_k(-1) - E^X u_k(-1)] \perp X$, we may write this

$$f_{ki} = \langle U(X)^{-k-1}\xi, E^X u_i(-1) \rangle. \tag{5.7}$$

(We can interpret $\langle \cdot, \cdot \rangle$ in (5.7) as the inner product $\langle \cdot, \cdot \rangle_X$ on X.) Then (5.7) inserted into (5.6) yields (5.5). □

If we set $A := U(X)^*$ and define B: $\mathbb{R}^m \to X$ to be the operator $Ba := E^X[a'u(-1)]$, (5.5) can be written

$$U^t\xi = \sum_{k=-\infty}^{t-1} \sum_{i=1}^{m} \langle \xi, A^{t-k-1} B e_i \rangle_X u_i(k) \tag{5.8}$$

where e_i is the i:th unit axis vector in $\mathbb{R}^m$. Since both arguments of the inner product belong to X, we have chosen to write $\langle \cdot, \cdot \rangle_X$ instead of $\langle \cdot, \cdot \rangle$. From a computational point of view this makes no difference, $\langle \cdot, \cdot \rangle_X$ is merely the restriction of $\langle \cdot, \cdot \rangle$ to X. However, X is a Hilbert space in its own right with inner product $\langle \cdot, \cdot \rangle_X$, and we want to emphasize that (5.8) is a factorization over X and that the space H plays no role whatsoever in (5.8) once the operators have been defined. Since $\eta_i(0) := E^X y_i(0) \in X$ and $\eta_i(t) = U^t \eta_i(0)$ for $i = 1, 2, \cdots, m$, (5.4) and (5.8) yield

$$y(t) = \sum_{k=-\infty}^{t-1} CA^{t-k-1} Bu(k) + Du(k) \tag{5.9}$$

where C: $X \to \mathbb{R}^m$ is defined by $(C\xi)_i = \langle E^X y_i(0), \xi \rangle_X$.

Now, (5.9) looks much like what we want to have. However, there is a possible source of confusion in this expression since X is used both as a *state space*, i.e., the space on which A is defined and takes its values, and as a *splitting subspace*, i.e., the space of all linear functionals of the state at $t = 0$. With regard to the finite-dimensional example in Section II, this means that X serves both as the state space $\mathbb{R}^n$ and as the space $X = \{a'x(0) | a \in \mathbb{R}^n\}$. This is of course all right, since $\mathbb{R}^n$ and X are isomorphic, and therefore, in an abstract sense, identical. However, in interpreting (5.9), we must remember that B: $\mathbb{R}^m \to X$ operates on the vector structure of $u(k)$, i.e., the projection E^X in the definition of B should *not* operate on the *random variables* $u_1(k), u_2(k), \cdots, u_k(k)$. There is no ambiguity in the definitions; we only need to work with two copies of the space X.

To avoid this confusion, we may choose any Hilbert space $\mathscr{X}$ of the same dimension as X as the state space. It is well known that $\mathscr{X}$ and X are isomorphic, i.e., there is a linear operator T which maps X onto $\mathscr{X}$ such that $\langle T\xi, T\eta \rangle_{\mathscr{X}} = \langle \xi, \eta \rangle_X$; see, e.g., [51, p. 213]. Then define the operators A: $\mathscr{X} \to \mathscr{X}$, B: $\mathbb{R}^m \to \mathscr{X}$ and C: $\mathscr{X} \to \mathbb{R}^m$ as

$$\begin{cases} A := TU(X)^* T^{-1} & (5.10\text{a}) \\ Ba = \sum_{i=1}^{m} a_i T E^X u_i(-1) & (5.10\text{b}) \\ (Cx)_i = \langle TE^X y_i(0), x \rangle_{\mathscr{X}}. & (5.10\text{c}) \end{cases}$$

The $m \times m$ matrix D is defined as before, i.e., by (5.3).

It is now a simple matter to check that (5.9) remains valid with this new choice of (A, B, C, D). If $\dim X = n < \infty$, we may choose the state space $\mathscr{X}$ to be Euclidean n-space $\mathbb{R}^n$ and interpret A, B, and C as matrices (choosing the usual axis vectors as a basis). In the same way we can choose $\mathscr{X}$ to be l_2 if X is infinite-dimensional. (Since H, and hence X, is separable, the dimension is always *countably* infinite.) In Section VIII we shall make another choice of $\mathscr{X}$ which is more suitable for analytic work.

It follows from (2.15b) and the fact that U and T are isometric that $\|A^t\xi\| = \|E^S U^{-t}\xi\| = \|E^{U^tS}\xi\|$, and therefore, in view of (4.1b), A^t tends strongly to zero as $t \to \infty$.

For the moment, let us assume that $\dim \mathscr{X} < \infty$. Then, it follows from the previous paragraph that all the eigenvalues of A are strictly inside the unit circle, and therefore we may define the *state process* $\{x(t);\ t \in \mathbb{Z}\}$ given by

$$x(t) = \sum_{k=-\infty}^{t-k} A^{t-k-1} Bu(k) \tag{5.11}$$

which is an $\mathscr{X}$-valued random process. It is then easy to see that

$$\begin{cases} x(t+1) = Ax(t) + Bu(t) & (5.12\text{a}) \\ y(t) = Cx(t) + Du(t). & (5.12\text{b}) \end{cases}$$

Since $X \perp H^+(u)$ (Corollary 3.1), this is a *forward* system and we shall call it the *standard forward realization with respect to X*. Moreover, in view of (5.8), each $\xi \in X$ can be written $\xi = \langle T\xi, x(0) \rangle_{\mathscr{X}}$. Therefore, since $f := T\xi$ varies over all of $\mathscr{X}$ as ξ varies over all of X,

$$\{\langle f, x(0) \rangle_{\mathscr{X}} | f \in \mathscr{X}\} = X, \tag{5.13}$$

i.e., the splitting subspace X consists of the linear functionals of the state vector at $t = 0$, precisely as required. (If $\mathscr{X} = \mathbb{R}^n$, (5.13) is the same as (2.3) for $t = 0$.)

When $\dim X = \infty$ we must be more careful when defining the state process. Then (5.12) and (5.13) must be interpreted in a weak sense; we refer the reader to [24] for details. Here it suffices to point out that (5.9) and (5.8) hold without restriction on dimension and that (5.13) and (5.12) may be interpreted via these.

To construct a backward realization with respect to X we use the second representation of $\bar{X}$ in Lemma 5.1 to obtain

$$y(t) = \bar{\eta}(t) + \bar{D}\bar{u}(t) \tag{5.14}$$

where $\bar{D}$ is the $m \times m$ matrix with components

$$\bar{d}_{ij} = \langle y_i(0), \bar{u}_j(0) \rangle \tag{5.15}$$

and $\bar{\eta}_i(t) := U^t E^{U X} y_i(0) = U^{t+1} E^X y_i(-1)$. The backward counterpart of Lemma 5.2 is as follows.

Lemma 5.3: *For any $\xi \in X$ and $t \in \mathbb{Z}$ we have the representation*

$$U^t\xi = \sum_{k=t}^{\infty}\sum_{i=1}^{m} \langle \xi, U(X)^{k-t}E^X\bar{u}_i(0)\rangle_X \bar{u}_i(k). \quad (5.16)$$

Proof: Since $\xi \in X \subset \bar{S} = H^-(\bar{u})$, $U^t\xi$ has a representation

$$U^t\xi = \sum_{k=t}^{\infty}\sum_{i=1}^{m} f_{k-t,i}\bar{u}_i(k) \quad (5.17)$$

and therefore (2.15b), i.e., $E^{H^-(\bar{u})}U^{-t}\xi = [U(X)^*]^t\xi$, for $t = 0,1,2,\cdots$, yields

$$\left[U(X)^*\right]^t\xi = \sum_{k=0}^{\infty}\sum_{i=1}^{m} f_{k+t,i}\bar{u}_i(k), \quad (5.18)$$

i.e., proceeding as in the proof of Lemma 5.2,

$$f_{ki} = \langle \xi, U(X)^k E^X \bar{u}_i(0)\rangle_X \quad (5.19)$$

which together with (5.17) yields the desired result. □

Now, taking $E^X y_i(-1)$ to be ξ in (5.16), (5.14) yields

$$y(t) = \sum_{k=t+1}^{\infty} \bar{C}\bar{A}^{k-t-1}\bar{B}\bar{u}(k) + \bar{D}\bar{u}(t) \quad (5.20)$$

where the operators $\bar{A}$: $\mathscr{X} \to \mathscr{X}$, $\bar{B}$: $\mathbb{R}^m \to \mathscr{X}$, and $\bar{C}$: $\mathscr{X} \to \mathbb{R}^m$ are defined as

$$\begin{cases} \bar{A} = TU(X)T^{-1} & (5.21a)\\ \bar{B}a = \sum_{i=1}^{m} a_i TE^X\bar{u}_i(0) & (5.21b)\\ (\bar{C}x)_i = \left\langle TE^X y_i(-1), x\right\rangle_{\mathscr{X}}. & (5.21c)\end{cases}$$

Note that $\bar{A} = A^*$ and that $\bar{A}^t$ tends strongly to zero as $t \to \infty$. Then, keeping in mind the comments we made above about the infinite-dimensional case, we may write (5.20) as

$$\begin{cases} \bar{x}(t) = \bar{A}\bar{x}(t+1) + \bar{B}\bar{u}(t) & (5.22a)\\ y(t) = \bar{C}\bar{x}(t+1) + D\bar{D}\bar{u}(t). & (5.12b)\end{cases}$$

Since $X \perp H^-(\bar{u})$ (Corollary 3.1), this is a *backward* system; we shall call (5.22) the *standard backward realization with respect to* X. From (5.16) we see that

$$\{\langle f, \bar{x}(0)\rangle_{\mathscr{X}} | f \in \mathscr{X}\} = X \quad (5.23)$$

as required.

We have seen that to each proper Markovian splitting subspace there corresponds two realizations of y, one evolving forwards and one backwards in time. We have made no assumptions about the minimality, observability, and constructibility of X, and we need to determine how these properties manifest themselves in the two systems (5.12) and (5.22). To this end, first recall that the forward system (5.12) is said to be *observable* if $\cap_{t=0}^{\infty}\ker CA^t = 0$ (where ker denotes null space), and *reachable* if the operator R given by $Rf = \sum_{t=0}^{\infty}A^tBf_t$ has a domain which is dense in the l_2 space of m-vector sequences $f := \{f_0, f_1, f_2, \cdots\}$ and a range which is dense in $\mathscr{X}$. If R is defined for all l^2-sequences and its range is all of $\mathscr{X}$, we say that (5.12) is *exactly reachable* [50, p. 243]. In the finite-dimensional case there is no difference between these two reachability concepts. Since direction of time is reversed in the backward system (5.22), it is consistent with standard notation [45] to say that (5.22) is *constructible* if $\cap_{t=0}^{\infty}\ker\bar{C}\bar{A}^t = 0$, *controllable* if the operator $\bar{R}$ given by $\bar{R}f = \sum_{t=0}^{\infty}\bar{A}^t\bar{B}f_t$ is densely defined and has dense range, and *exactly controllable* if $\bar{R}$ is defined everywhere and maps onto $\mathscr{X}$. Modulo details which depend on the particular problem formulation of this paper, the following theorem can be found in [23], [26].

Theorem 5.1 [23], [26]: The standard forward realization with respect to a proper Markovian splitting subspace X is always exactly reachable and it is observable if and only if X is observable. The standard backward realization w.r.t. X is always exactly controllable and it is constructible if and only if X is constructible.

This theorem tells us that we need never worry about reachability and controllability in the standard realizations. (In the finite-dimensional case this corresponds to the fact that $x(0)$ and $\bar{x}(0)$ are bases in X.) This is consistent with the fact that these properties do not occur in the geometric theory. Moreover, to test whether (say) the forward realization is minimal, it is not enough to ensure that it is observable, but we need also check that the backward realization is constructible. If this is not so, the state space $\mathscr{X}$ is too large and can be reduced.

The reachability/controllability part of the theorem can be found in [26]. The observability/constructibility part was first proven in the continuous-time case in [23] and modified for the discrete-time setting in [26]. Another observability/constructibility theorem was given by Ruckebusch [32] somewhat earlier, but his definitions of the C and $\bar{C}$ operators are not the ones used here. The setting and the problem formulation are here a bit different from that in [23], [26] and therefore, for the benefit of the reader, we shall provide a proof of the theorem. Although it does not contain any new ideas, there are a few details which need to be worked out.

Proof of Theorem 5.1: Applying E^X to both members of (2.15b) we obtain $[U(X)^*]^t\xi = E^XU^{-t}\xi$ for all $\xi \in X$ and $t = 0,1,2,\cdots$. Therefore, $A^tBe_i = TE^XU^{-t}E^Xu_i(-1) = TE^Xu_i(-t-1)$. The last step follows from the fact that $u_i(-1) \in S = X \oplus \bar{S}^\perp$ (Corollary 3.1) and $U^{-t}\bar{S}^\perp \subset \bar{S}^\perp \perp X$. In fact, $E^XU^{-t}E^Xu_i(-1) = E^XU^{-t}E^Su_i(-1) = E^XU^{-t}u_i(-1)$. Hence, since E^X is continuous, $Rf = TE^X\sum_{t=0}^{\infty}f_t'u(-t-1)$. Since the elements of S are precisely the sums of type $\sum_{t=0}^{\infty}f_t'u(-t-1)$, and since $X \subset S$, the operator R is defined for all l_2-sequences f and is surjective. This proves that (5.12) is exactly reachable. To prove the observability result, first note that, for any $\xi \in X$, $(CA^tT\xi)_i = \left\langle TE^Xy_i(0), T[U(X)^*]^t\xi\right\rangle_{\mathscr{X}} = \left\langle U(X)^tE^Xy_i(0), \xi\right\rangle_X$. But, by a similar argument as applied above [now instead using (2.15a) and the fact that $y_i(0) \in \bar{S} = X \oplus S^\perp$], we have $U(X)^tE^Xy_i(0) = E^Xy_i(t)$. Therefore, $(CA^tT\xi)_i = \langle y_i(t), \xi\rangle$, and consequently $\cap_{t=0}^{\infty}\ker CA^t = T\{X \cap (H^-)^\perp\}$. This proves the observability result. The proofs concerning the backward system are analogous. □

VI. Singularity and the Error Spaces

Given a finite-dimensional stochastic system (2.1), the *Kalman filtering problem* consists in determining recursively the linear least-squares estimate of $a'x(t)$ given $\{y(t_0), y(t_0+1), \cdots, y(t-1)\}$ for all $a \in \mathbb{R}^n$ and all $t > t_0$. Now, let $t_0 \to -\infty$. Then, setting $\xi = a'x(0)$, we need to find $E^{U^tH^-}U^t\xi$. Since $E^{U^tH^-}U^t\xi = U^tE^{H^-}\xi$, the problem is reduced to finding $E^{H^-}\xi$ for all $\xi \in X$. It can be shown [24] that, if $X \perp N^-$, all such estimates span the predictor space X_-, i.e., $E^{H^-}X = X_-$. (This holds also in the infinite-dimensional case if we only add a bar over the E to denote closure, and hence we shall consider the general case from now on.) Hence, the standard forward realization with respect to X_- is, modulo trivial coordinate transformations in X, the *steady-state Kalman filter* of all systems (2.1) such that $X \perp N^-$.

Now consider the space Z generated by all estimation errors $\{\xi - E^{H^-}\xi | \xi \in X\}$, i.e., $Z := \bar{E}^{(H^-)^\perp}X$. (We no longer need to assume that $X \perp N^-$.) Then, in view of the relation $S = H^- \vee X$, we have $Z = \bar{E}^{(H^-)^\perp}S$. But $H^- \subset S$, and therefore $(H^-)^\perp$ and S intersect perpendicularly. Consequently,

$$Z = (H^-)^\perp \cap S = S \ominus H^- \quad (6.1)$$

(Corollary 3.1). Hence, in analogy with X, Z has the forward generating process u (same as X) and the backward generating process u_-, since $H^-(u_-)=(H^-)^{\perp}$. Then, as in Section V, we can construct a forward representation

$$z(t+1)=Fz(t)+Gu(t) \tag{6.2}$$

where $z(t)$ takes values in some copy $\mathscr{Z}$ of Z and the operators F: $\mathscr{Z}\to\mathscr{Z}$ and G: $\mathbb{R}^m\to\mathscr{Z}$ are unitarily equivalent to $U(Z)^*$ and $a\mapsto E^Z[a'u(-1)]$, respectively. The system (6.2) has the property that $\{\langle\zeta,z(0)\rangle_{\mathscr{Z}}|\zeta\in\mathscr{Z}\}=Z$. We can also construct a backward representation in terms of the innovation process u_-, but here we shall have no use of that. We shall call Z the *forward error space* of X.

Similarly we can show that, if $X\perp N^+$, the estimates $\{E^{H^-}\xi|\xi\in X\}$ generate the backward predictor space X_+. Even without the condition $X\perp N^-$, we can define the *backward error space* of X, namely $\bar{Z}:=E^{(H^+)^{\perp}}X$, and see that it satisfies

$$\bar{Z}=(H^+)^{\perp}\cap\bar{S}=\bar{S}\ominus H^+. \tag{6.3}$$

As above, there is a backward representation

$$\bar{z}(t)=\bar{F}\bar{z}(t+1)+\bar{G}\bar{u}(t) \tag{6.4}$$

with analogous properties as (6.2); we could also construct a forward system generated by the backward innovation $\bar{u}_+$. Note that (6.2) and (6.4) are *not* a forward/backward pair in the sense of (5.12) and (5.22) but represent different spaces Z and $\bar{Z}$.

We can now express the conditions of observability and constructibility in terms of Z and $\bar{Z}$.

Proposition 6.1: Let X be a proper Markovian splitting subspace with error spaces Z and $\bar{Z}$. Then X is observable if and only if $X\cap\bar{Z}=0$ and constructible if and only if $X\cap Z=0$.

Proof: By definition, X is constructible if and only if $X\cap(H^-)^{\perp}=0$. But, since $X\subset S$, this condition is equivalent to $X\cap(H^-)^{\perp}\cap S=0$, which, in view of (6.1), is the same as $X\cap Z=0$. The proof of the observability part is analogous. □

In some applications it is required that one of the $m\times m$ matrices D and $\bar{D}$ in the standard realizations with respect to X is nonsingular. Such is the case in the usual implementation of the Kalman filter. We shall say that the standard forward [backward] realization with respect to X is *singular* if D [$\bar{D}$] is singular. If either the forward or the backward standard realization is singular, we say that X is *singular*. The next theorem shows that D has the same nullity as F^*, and $\bar{D}$ the same as $\bar{F}^*$. For any operator P, $\ker P$ denotes the null space of P, i.e., the space of all λ with $P\lambda=0$.

Theorem 6.1: Let X be a proper Markovian splitting subspace with error spaces Z and $\bar{Z}$ and standard realizations (5.12) and (5.22). Then

$$\operatorname{rank} D=m-\dim\ker U(Z) \tag{6.5a}$$

$$\operatorname{rank}\bar{D}=m-\dim\ker U(\bar{Z})^*. \tag{6.5b}$$

For the proof we need the following lemma.

Lemma 6.1: The null spaces of $U(Z)$ and $U(\bar{Z})^*$ are

$$\ker U(Z)=Z\cap H_{-1}(u) \tag{6.6a}$$

$$\ker U(\bar{Z})=\bar{Z}\cap H_0(\bar{u}). \tag{6.6b}$$

Proof: Let $\zeta\in Z$. Then $E^ZU\zeta=0$ if and only if $U\zeta\perp Z$, i.e., $\zeta\perp U^*Z$. Hence, $\ker U(Z)=Z\cap(U^*Z)^{\perp}$. However, by (6.1), $(U^*Z)^{\perp}=U^*H^-\oplus U^*S^{\perp}$. Here $U^*H^-\subset H^-\perp Z$, and $U^*S^{\perp}=S^{\perp}\oplus H_{-1}(u)$, where $S^{\perp}\perp Z$. Hence, (6.6a) holds. The proof of (6.6b) is analogous. □

Proof of Theorem 6.1: Since $Z=(H^-)^{\perp}\cap S$ and $H_{-1}(u)\subset S$, (6.6a) implies that $\ker U(Z)=(H^-)^{\perp}\cap H_{-1}(u)$, the dimension of which equals the number p of linearly independent $a_1,a_2,\cdots,a_p\in\mathbb{R}^m$ such that

$$a_k'u(-1)\perp H^- \quad \text{for } k=1,2,\cdots,p. \tag{6.7}$$

But, since $H_{-1}(u)\perp U^*S\supset U^*H^-$, (6.7) is equivalent to

$$a_k'u(-1)\perp H_{-1}(y) \quad \text{for } k=1,2,\cdots,p \tag{6.8}$$

which in turn is equivalent to $\langle y_i(-1),u(-1)'a_k\rangle=0$ for all $i=1,2,\cdots,m$ and $k=1,2,\cdots,p$, i.e., $Da_k=0$ for $k=1,2,\cdots,p$; cf. definition (5.3). This proves (6.5a). The proof of (6.5b) is analogous. □

Note that although in general Z is infinite-dimensional the null space of $U(Z)$ is finite-dimensional. Also note that, since the forward error space of X_- and the backward error space of X_+ are trivial, i.e., $Z_-=0$ and $\bar{Z}_+=0$, D_- and $\bar{D}_+$ are always nonsingular. In other words, the (forward and backward) steady-state Kalman filters are always nonsingular.

Singularity plays a key role in the theory of *invariant directions* [40]–[43], [11]. In Kalman filtering some linear combinations of the columns of the solution matrix of the Riccati equation may become constant after a finite number of steps. This happens if and only if X is singular. This follows from the fact, proved in Section X, that X is singular if and only if F_+ is invertible and Theorem 3.2 in [11]. (Note that Γ_* in [11] is a matrix representation of F_+.)

VII. Models Without Observation Noise

There is a certain lack of symmetry in our definition of H^- and H^+. If we redefine H^- so that it is the subspace generated by the components of $\{y(t);\ t=0,-1,-2,\cdots\}$, we obtain complete symmetry between the past space and future space. Theorem 3.1 still holds with this choice (as we pointed out in Section III), and therefore $H^-\cap H^+\subset S\cap\bar{S}=X$ for any splitting subspace. However, now $H^-\cap H^-$ contains the components of $y(0)$, so that $y_i(0)\in X$ for $i=1,2,\cdots,m$. Consequently, the construction in Section V will lead to a forward model of type

$$\begin{cases} x(t+1)=Ax(t)+Bu(t+1) & (7.1a)\\ y(t)=Cx(t) & (7.1b)\end{cases}$$

and a backward model of type

$$\begin{cases} \bar{x}(t-1)=\bar{A}\bar{x}(t)+\bar{B}\bar{u}(t-1) & (7.2a)\\ y(t)=\bar{C}\bar{x}(t) & (7.2b)\end{cases}$$

in lieu of (5.12) and (5.22), i.e., realizations without observation noises (cf. [26]).

Clearly any splitting subspace in the new setting is also a splitting subspace in the old, but the converse is not true. To fix notations, let us retain the old definition of H^- given in Section II, and refer to the two types of splitting subspaces as (H^-,H^+)-splitting and (UH^-,H^+)-splitting, respectively. We can construct a proper Markovian (UH^-,H^+)-splitting subspace $\tilde{X}\sim(\tilde{S},\bar{S})$ from any proper Markovian (H^-,H^-)-splitting subspace by merely shifting S to $\tilde{S}:=US$. According to Lemma 5.1 this amounts to amending the state process by including the observation noise in it. This is a standard construction in the finite-dimensional theory. The question is whether minimality is preserved under this transformation. (Note that, although $\tilde{X}$ is a nonminimal (H^-,H^-)-splitting subspace, it may very well be a minimal (UH^-,H^-)-splitting subspace.)

Theorem 7.1: Let $X\sim(S,\bar{S})$ be constructible (observable) in the (H^-,H^+) framework. Then $\tilde{X}\sim(US,\bar{S})$, given by Lemma 5.1, is constructible (observable) in the (UH^-,H^+) framework if and only if the standard forward (backward) realization w.r.t. X is *nonsingular. Hence, if X is minimal, $\tilde{X}$ is minimal if and only if X is nonsingular.*

Proof: Let us first prove the statement about constructibility. In view of Theorem 6.1, Lemma 6.1, and Proposition 6.1, we need to prove that $Z \cap H_{-1}(u) = 0$ if and only if $\tilde{X} \cap \tilde{Z} = 0$, where $\tilde{Z} := US \ominus UH^- = UZ$. Recall that $\tilde{X} := X \oplus H_0(u)$ (Lemma 5.1). If $Z \cap H_{-1}(u) \neq 0$, then the shifted space $\tilde{Z} \cap H_0(u) \neq 0$. But then, since $H_0(u) \subset \tilde{X}$, $\tilde{Z} \cap \tilde{X} \neq 0$. To prove the other direction, assume that there is a $\lambda \neq 0$ in $\tilde{Z} \cap \tilde{X}$. Then, since $\tilde{X} = X \oplus H_0(u)$, $\lambda = \xi + \eta$ with $\xi \in X$ and $\eta \in H_0(u)$. But $\lambda \in \tilde{Z} \perp UH^- \supset H^-$ and $\eta \in H_0(u) \perp S \supset H^-$. Hence, $\xi = \lambda - \eta \perp H^-$, i.e., $\xi \in X \cap (H^-)^{\perp}$. Therefore, since X is constructible, $\xi = 0$, i.e., $\lambda \in H_0(u)$. Then, $\lambda \in \tilde{Z} \cap H_0(u)$, and consequently $Z \cap H_{-1}(u) \neq 0$, for it contains $U^*\lambda \neq 0$. This proves the first part of the theorem. In the observability part we need to prove that $\bar{Z} \cap H_0(\bar{u}) = 0$ if and only if $\tilde{X} \cap \bar{Z} = 0$. Since $H_0(\bar{u}) \subset \tilde{X}$ (Lemma 5.1), the if-part is immediate. To prove the only-if-part assume that $\tilde{X} \cap \bar{Z}$ contains $\lambda \neq 0$. Then, since $\tilde{X} = UX \oplus H_0(\bar{u})$ (Lemma 5.1), $\lambda = U\xi + \eta$ where $\xi \in X$ and $\eta \in H_0(\bar{u})$. Then, since $\lambda \in \bar{Z} \perp H^+ \supset UH^-$ and $\eta \in H_0(\bar{u}) \perp U\bar{S} \supset UH^+$, we have $U\xi \perp UH^-$, i.e., $\xi \perp H^-$. But X is observable, and therefore $X \cap (H^+)^{\perp} = 0$. Hence, $\xi = 0$, and consequently $\lambda \in H_0(\bar{u})$. Since, in addition $\lambda \in \bar{Z}$, it follows that $\bar{Z} \cap H_0(\bar{u}) \neq 0$. This concludes the observability part of the proof. Then the minimality part follows from Theorem 3.4. ◻

In Section IX we shall give an example in which $\tilde{X}$ is nonminimal.

VIII. State-Space Representation in Hardy Space

In Section V we let the state-space $\mathscr{X}$ be an arbitrary Hilbert space of the same dimension as X; in fact we could have chosen (another copy of) the space X itself. In order to get more structure, in this section we shall choose $\mathscr{X}$ to be a space of functions.

First note that any normalized white noise process $\{u(t); t \in \mathbb{Z}\}$ has a spectral representation

$$u(t) = \int_{-\pi}^{\pi} e^{i\omega t}\, d\hat{u}(\omega) \tag{8.1}$$

where $\hat{u}$ is a Wiener process on $[-\pi, \pi]$ with incremental covariance

$$E\{d\hat{u}\, d\hat{u}^*\} = \frac{1}{2\pi} I\, d\omega \tag{8.2}$$

[47]. (Here * denotes transposition plus conjugation.) Now, all white noise processes in which we are interested have the property $H(u) = H$—we have denoted the class of such processes $\mathscr{U}$—and therefore any $\eta \in H$ can be written as

$$\eta = \sum_{k=-\infty}^{\infty} f'_{-k} u(k) \tag{8.3}$$

for some l_2-sequence $\{f_k\}_{-\infty}^{\infty}$ of m-dimensional vectors. This, together with (8.1), yields the representation

$$\eta = \int_{-\pi}^{\pi} f(e^{i\omega})'\, d\hat{u}(\omega) \tag{8.4}$$

where f, the sum of the Fourier series

$$f(z) = \sum_{k=-\infty}^{\infty} f_k z^{-k}, \tag{8.5}$$

belongs to the space $L_2(\mathbb{T})$ of all m-dimensional vector functions square-integrable on the unit circle $\mathbb{T}$ with respect to the measure $(1/2\pi)\, d\omega$ [47]. It is well known and easy to show that (8.4) defines a unitary operator T_u: $H \to L_2(\mathbb{T})$ such that $T_u \eta = f$. Next define the *Hardy spaces* $\mathscr{H}_2^-$ and $\mathscr{H}_2^+$ in the following way. Let $\mathscr{H}_2^-$ be the space of all functions (8.5) in $L_2(\mathbb{T})$ such that $f_k = 0$ for $k < 0$ and $\mathscr{H}_2^+$ the space of functions in $L_2(\mathbb{T})$ for which $f_k = 0$ for $k > 0$. These Hardy functions can be extended to the complex plane so that the functions in $\mathscr{H}_2^-$ are analytic outside the unit circle and those in $\mathscr{H}_2^+$ are analytic inside $\mathbb{T}$ [49], [50]. Then from (8.3) it is easy to see that

$$\begin{cases} T_u H^-(u) = z^{-1}\mathscr{H}_2^- & (8.6)\\ T_u H^+(u) = \mathscr{H}_2^+ & (8.7)\end{cases}$$

and that $T_u U T_u^{-1} = z$, where here z denotes multiplication by $e^{i\omega}$. The fact that the shift becomes a simple multiplication operator is one of the advantages with the Hardy space setting.

Let W be the $m \times m$ matrix function whose i:th column is $T_u y_i(0)$. Then

$$y(t) = \int_{-\pi}^{\pi} e^{i\omega t} W(e^{i\omega})'\, d\hat{u} \tag{8.8}$$

and therefore

$$W(z)'W(1/z) = \Phi(z) \tag{8.9}$$

is the spectral density of y, and W is a *spectral factor*. Conversely, under the given assumptions, y has a spectral representation

$$y(t) = \int_{-\pi}^{\pi} e^{i\omega t}\, d\hat{y}(\omega) \tag{8.10}$$

where $E\{d\hat{y}\, d\hat{y}^*\} = (2\pi)^{-1}\Phi(e^{i\omega})\, d\omega$. Then to any $m \times m$ spectral factor W, there corresponds a normalized white noise (8.1) with $d\hat{u} = (W')^{-1} d\hat{y}$. (Since Φ is full rank, then so is W.) Hence, there is a one-to-one correspondence between $m \times m$ spectral factors and processes in $\mathscr{U}$.

Now consider a proper Markovian splitting subspace X with generating processes $(u, \bar{u})$. Since these processes belong to $\mathscr{U}$, there is a pair $(W, \bar{W})$ of spectral factors, W corresponding to u and $\bar{W}$ corresponding to $\bar{u}$. However, by Theorem 3.1, it is required that $H^- \subset H^-(u)$ and $H^+ \subset H^+(\bar{u})$ and that $H^-(u)$ and $H^+(\bar{u})$ intersect perpendicularly. Using (8.6) it is easy to see that the first condition is equivalent to W having its columns in $\mathscr{H}_2^-$, i.e.,

$$W = \sum_{k=0}^{\infty} W_k z^{-k}. \tag{8.11}$$

Such a spectral factor is called *stable* because it is analytic outside the unit circle. The second condition is equivalent to $\bar{W}$ having all its columns in $\mathscr{H}_2^-$, i.e.,

$$\bar{W}(z) = \sum_{k=-\infty}^{0} \bar{W}_k z^{-k}. \tag{8.12}$$

Such a spectral factor is called *strictly unstable*. Finally, it is shown in [21], [22] that perpendicular intersection is equivalent to

$$K := W\bar{W}^{-1} \tag{8.13}$$

having all its columns in $\mathscr{H}_2^-$. In addition, it is clear from (8.9) that $K(e^{i\omega})$ is a unitary matrix for all ω. A function K with this property is called *inner*, or in engineering language, *stable allpass*. The last name is motivated by the fact that K is a stable transfer function transforming white noise into white noise, more specifically u to $\bar{u}$.

The $m \times m$ matrix function K is called the *structural function* of X. It is a rational function if and only if $\dim X < \infty$. If the process y is scalar $(m = 1)$ all minimal Markovian splitting subspaces have the same K, but this is not true in the vector case [25].

Following the procedure in [21]–[23] we can now transfer the splitting geometry to the Hardy space setting. Set $\mathscr{X}:=T_u X$. Then, applying T_u to $X=H^-(u)\ominus H^-(\bar{u})$ (Corollary 3.1), it is not hard to see that

$$\mathscr{X}=z^{-1}[\mathscr{H}_2^- \ominus (K\mathscr{H}_2^-)]. \tag{8.14}$$

To see this merely use (8.6) and the fact that $T_u T_{\bar{u}}^{-1}$ is multiplication by K.

Hence, we have reconstructed (in our discrete-time setting) one of the results of [21], [22]. The proper Markovian splitting subspaces are in one-to-one correspondence to the pairs $(W,\bar{W})$ of spectral factors such that W is stable, $\bar{W}$ is strictly unstable, and $K:=W\bar{W}^{-1}$ is inner. The splitting subspace X satisfies

$$X=\int_{-\pi}^{\pi}\mathscr{X}\,d\hat{u} \tag{8.15}$$

where $\mathscr{X}$ is given by (8.14) and $d\hat{u}=(W')^{-1}d\hat{y}$. Similarly, we can show that

$$X=\int_{-\pi}^{\pi}\bar{\mathscr{X}}\,d\hat{\bar{u}} \tag{8.16}$$

where $\bar{\mathscr{X}}:=T_{\bar{u}}X=\mathscr{H}_2^+\ominus(K^*\mathscr{H}_2^+)$ and $d\hat{\bar{u}}=(\bar{W}')^{-1}d\hat{y}$.

We can now use either $\mathscr{X}$ or $\bar{\mathscr{X}}$ as the state space. In fact, choosing T to be T_u in Section V we have $Af=P^{\mathscr{X}}z^{-1}f$, $Ba=P^{\mathscr{X}}z^{-1}a$, $Cf=(2\pi)^{-1}\int_{-\pi}^{\pi}W(e^{i\omega})f(e^{i\omega})\,d\omega$, and $D=W_0$, where $P^{\mathscr{X}}$ is the orthogonal projection onto $\mathscr{X}$. Moreover, choosing T to be $T_{\bar{u}}$, we obtain $\bar{A}f=P^{\bar{\mathscr{X}}}zf$, $\bar{B}a=P^{\bar{\mathscr{X}}}a$, $\bar{C}f=(2\pi)^{-1}\int_{-\pi}^{\pi}e^{i\omega}\bar{W}(e^{i\omega})f(e^{i\omega})\,d\omega$, and $\bar{D}=\bar{W}_0$.

We recall from Section VI that the forward error space Z of a proper Markovian splitting subspace is the intersection between the perpendicularly intersecting subspaces $H^-(u)$ and $H^-(u_-)$. Hence, as in the case of X, $Q:=WW_-^{-1}$ is inner. Therefore, we have the well-known decomposition

$$W=QW_- \tag{8.17}$$

of a spectral factor as a product of an inner (Q) and an *outer* (W_-) factor. The engineering name for outer is *minimum-phase*; in the finite-dimensional case, this is the spectral factor with all its poles *and* zeros inside the unit circle. Applying a symmetric argument to the backward error space $\bar{Z}$ we obtain the decomposition

$$\bar{W}=\bar{Q}\bar{W}_+ \tag{8.18}$$

where $\bar{Q}$ is conjugate inner, i.e., its inverse $\bar{Q}^*$ is inner, and $\bar{W}_+$ is the strictly unstable minimum phase spectral factor (having all its poles and zeros outside the unit circle in the finite-dimensional case). It was shown in [22] that X is observable if and only if K and $\bar{Q}^*$ are right coprime (i.e., they do not have any right inner factor in common) and constructible if and only if K and Q are left coprime. (In fact, this is easy to see by applying the map zT_u to either (3.3) or $X\cap\bar{Z}=0$ (Proposition 6.1) and $T_{\bar{u}}$ to either (3.4) or $X\cap Z=0$.) Henceforth, we shall refer to Q and $\bar{Q}^*$ as the forward and backward *spectral inner factors* of X.

As may be expected, singularity can also be characterized in terms of Q and $\bar{Q}$.

Theorem 8.1: Let X be a proper Markovian splitting subspace with spectral inner factors Q and $\bar{Q}^*$. Then the standard forward [backward] realization w.r.t. X is singular if and only if $Q(\infty)$ is singular [$\bar{Q}^*(\infty)$ is singular].

For the proof we need the following lemma, which is a variation of [53, Theorem 13]. It will be used in Section X also.

Lemma 8.1: Let u and v be two processes of class $\mathscr{U}$ such that $H^-(u)$ and $H^+(v)$ intersect perpendicularly, let W_u and W_v be the corresponding spectral factors, and let R be the inner function $R:=W_uW_v^{-1}$. Set $Y:=H^-(u)\cap H^+(v)$. Then the three conditions i) $\ker U(Y)=0$, ii) $\ker U(Y)^*=0$, and iii) $R(\infty)$ nonsingular are equivalent.

Proof: As we have pointed out above, the perpendicular intersection is equivalent to R being inner; see [21]–[24]. In exactly the same way as in Lemma 6.1 we see that $\ker U(Y)=Y\cap H_{-1}(u)$ and $\ker U(Y)^*=Y\cap H_0(v)$. Eliminating Y in these expressions, we obtain $\ker U(Y)=H^+(v)\cap H_{-1}(u)$ and $\ker U(Y)^*=H^-(u)\cap H_0(v)$. Condition ii) fails if and only if there is a vector $a\in\mathbb{R}^m$ such that $a'v(0)\in H^-(u)$. Since $T_uT_v^{-1}$ is multiplication by R and $T_v[a'v(0)]=a$, the isomorphic image of this under T_u is $Ra\in z^{-1}\mathscr{H}_2^-$. But, since $R(z)=\Sigma_{k=0}^{\infty}R_kz^{-k}$, this happens if and only if $R_0a=0$, i.e., if and only if $R(\infty)$ is singular. This establishes the equivalence between ii) and iii). In the same way we see that i) fails if and only if there is a vector $a\in\mathbb{R}^m$ such that $a'u(-1)\in H^+(v)$, the isomorphic image of which under T_v is $z^{-1}R^*a\in\mathscr{H}_2^+$ because $T_u[a'u(-1)]=z^{-1}a$ and $T_vT_u^{-1}$ is multiplication by $R^*=R^{-1}$. But this is equivalent to $R_0'a=0$, for $R^*(z)=R'(z^{-1})=\Sigma_{k=0}^{\infty}R_k'z^k$. Since $R(\infty)=R_0$, this establishes the equivalence between i) and iii). ☐

Proof of Theorem 8.1: Let $(u,\bar{u})$ be the generating processes of X. Then, taking the two processes in Lemma 8.1 to be u and the innovation process u_-, Y and R become Z and Q, respectively. By Theorem 6.1, the standard forward realization w.r.t. X is singular if and only if $\ker U(Z)\neq 0$, which, in view of Lemma 8.1, happens if and only if $Q(\infty)$ is singular. The "backward" part of the proof is analogous, taking $\bar{u}_-$ and $\bar{u}$ for the two processes. This yields $Y=\bar{Z}$ and $R=\bar{Q}^*$ so that the required result follows from (6.5b) and the equivalence between ii) and iii) in Lemma 8.1. ☐

IX. A Simple Example

Let us consider a scalar process y with spectral density

$$\Phi(z)=\frac{(z-\frac{2}{3})(z-\frac{1}{4})(z^{-1}-\frac{2}{3})(z^{-1}-\frac{1}{4})}{(z-\frac{1}{2})^2(z-\frac{1}{3})^2(z^{-1}-\frac{1}{2})^2(z^{-1}-\frac{1}{3})^2}. \tag{9.1}$$

The minimum phase spectral factor W_- must have all its poles and zeros inside the unit circle. Moreover, it must have the form (8.11) with $D=W_0\neq 0$ (Theorem 6.1), i.e., $W(\infty)\neq 0$. Consequently, W_- must be

$$W_-(z)=\frac{z^2(z-\frac{2}{3})(z-\frac{1}{4})}{(z-\frac{1}{2})^2(z-\frac{1}{3})^2}. \tag{9.2}$$

In the same way, the backward minimum phase spectral factor $\bar{W}_+$ has all its poles and zeros outside the unit circle and has the form (8.12) with $\bar{D}=\bar{W}_0=\bar{W}(0)\neq 0$, and therefore

$$\bar{W}_+(z)=\frac{(1-\frac{2}{3}z)(1-\frac{1}{4}z)}{(1-\frac{1}{2}z)^2(1-\frac{1}{3}z)^2}. \tag{9.3}$$

Note that $\bar{W}_+(z)=W_-(1/z)$. This always holds in the scalar case but, in general, not in the vector case. Now, the quotient $\bar{W}_+^{-1}W_-$ is not an inner function, and hence $(W_-,\bar{W}_+)$ does not define a splitting subspace. (In geometric terms this is equivalent to saying that H_-^- and H_+^+ do not intersect perpendicularly.) Instead choosing $\bar{W}_-$ to be

$$\bar{W}_-(z)=\frac{z^2(z-\frac{2}{3})(z-\frac{1}{4})}{(1-\frac{1}{2}z)^2(1-\frac{1}{3}z)^2} \tag{9.4}$$

we obtain the structural function

$$K_-(z)=\frac{(1-\frac{1}{2}z)^2(1-\frac{1}{3}z)^2}{(z-\frac{1}{2})^2(z-\frac{1}{3})^2} \tag{9.5}$$

which is inner. Hence, $(W_-, \bar{W}_-)$ defines a Markovian splitting subspace. Since K_- and $Q_- = 1$ are coprime, it is constructible. Moreover, K_- and

$$\bar{Q}_-^* = \frac{(1-\frac{2}{3}z)(1-\frac{1}{4}z)}{z^2(z-\frac{2}{3})(z-\frac{1}{4})} \tag{9.6}$$

are coprime, which establishes observability. Hence, the pair $(W_-, \bar{W}_-)$ corresponds to a minimal splitting subspace which must be X_-. Then, since y is scalar, all minimal Markovian splitting subspaces will have the structural function (9.5).

To pursue this point a bit further let us choose another stable spectral factor, say

$$W(z) = \frac{z(z-\frac{2}{3})(1-\frac{3}{4}z)}{(z-\frac{1}{2})^2(z-\frac{1}{3})^2}$$

and pair it with $\bar{W}_-$. Then

$$K(z) = \frac{(1-\frac{1}{2}z)^2(1-\frac{1}{3}z)^2(1-\frac{1}{4}z)}{z(z-\frac{1}{2})^2(z-\frac{1}{3})^2(z-\frac{1}{4})} \tag{9.7}$$

which is an inner function. Therefore, $(W, \bar{W}_-)$ defines a Markovian splitting subspace X. However, K and $\bar{Q}_-^*$ are not coprime, nor are K and

$$Q = \frac{1-\frac{1}{4}z}{z(z-\frac{1}{4})}. \tag{9.8}$$

Hence, X is neither observable nor constructible, and therefore not minimal ($\dim X = 6$).

To obtain the minimal splitting subspace corresponding to the conjugate outer factor $\bar{W}_-$, we merely multiply it by K_-. This yields

$$W_+(z) = \frac{(1-\frac{2}{3}z)(1-\frac{1}{4}z)}{(z-\frac{1}{2})^2(z-\frac{1}{3})^2} \tag{9.9}$$

which is the *maximum phase* stable spectral factor. (Note that W_+ has all its zeros outside the unit circle.) Of course, $(W_-, \bar{W}_+)$ corresponds to X_+.

Now, if X is a splitting subspace with a rational structural function $K = \bar{\psi}/\psi$, and where $\bar{\psi}$ and ψ are coprime polynomials, then it can be shown [50] that $\mathscr{X}$ consists of rational functions ρ/ψ, where ρ is an arbitrary polynomial with degree less than $n := \deg \psi$. We shall write this

$$X = \left\{ \frac{\rho}{\psi} \mid \deg \rho < n \right\}. \tag{9.10}$$

This is clearly on n-dimensional space. Now, by Lemma 5.1 (apply T_u to the relation $y_i(0) \in H_0(u) \oplus X$), the forward spectral factor W of X is the sum of a constant and an element in $\mathscr{X}$. Therefore,

$$W = \pi/\psi \tag{9.11}$$

for some polynomial π such that $\deg \pi \leq n$. Hence, in view of (8.15),

$$X = \int \left\{ \frac{\rho}{\pi} \mid \deg \rho < n \right\} d\hat{y}, \tag{9.12}$$

i.e., X is uniquely determined by the numerator polynominal π and the *degree* of the denominator polynomial ψ, as was pointed out in [18]. In particular, $\pi_-(z) = z^2(z-\frac{2}{3})(z-\frac{1}{4})$, and consequently, by partial fraction expansion, (9.12) yields $X_- = \text{span}\{\int z^{-1} d\hat{y}, \int z^{-2} d\hat{y}, \int (z-\frac{2}{3})^{-1} d\hat{y}, \int (z-\frac{1}{4})^{-1} d\hat{y}\}$, i.e., X_- is the linear span of $y(-1)$, $y(-2)$, $x_1^- := \sum_{k=-\infty}^{-1} (2/3)^{-k-1} y(k)$, and $x_2^- := \sum_{k=-\infty}^{-1} 4^{k-1} y(k)$. (To see this, use the geometric series expansion.) In the same way we see that X_- is the linear span of $y(1)$, $y(0)$, $x_1^+ := \sum_{k=0}^{\infty} (2/3)^k y(k)$, and $x_2^+ := \sum_{k=0}^{\infty} 4^{-k} y(k)$.

Consequently, the frame space is the eight-dimensional space

$$H^\square = \text{span}\{ y(1), y(0), y(-1), y(-2), x_1^-, x_2^-, x_1^+, x_2^+ \}. \tag{9.13}$$

We know that all minimal X are contained in $H^\square$, but what subspaces are they? To answer this, first note that all minimal X have the same K (since $m = 1$), and consequently W_- and W, the stable spectral factor of X, have the same ψ. Therefore, since they both satisfy (8.9),

$$\pi(z)\pi(z^{-1}) = \pi_-(z)\pi_-(z^{-1}). \tag{9.14}$$

Conversely, any π of degree $\leq \deg \psi = 4$ satisfying (9.14) provides a pair $(W, \bar{W}) = (\pi/\psi, \pi/\bar{\psi})$ defining an X with structural functions $K_- = \bar{\psi}/\psi$, i.e., a minimal X. There are exactly 12 minimal Markovian splitting subspaces, and we list them below together with the corresponding π.

$$\begin{aligned}
X_- &= \text{span}\{ y(-1), y(-2), x_1^-, x_2^- \} & \pi_-(z) &= z^2(z-\tfrac{2}{3})(z-\tfrac{1}{4}) \\
X_2 &= \text{span}\{ y(0), y(-1), x_1^-, x_2^- \} & \pi_2(z) &= z(z-\tfrac{2}{3})(z-\tfrac{1}{4}) \\
X_3 &= \text{span}\{ y(1), y(0), x_1^-, x_2^- \} & \pi_3(z) &= (z-\tfrac{2}{3})(z-\tfrac{1}{4}) \\
X_4 &= \text{span}\{ y(-1), y(-2), x_1^-, x_2^+ \} & \pi_4(z) &= z^2(z-\tfrac{2}{3})(1-\tfrac{1}{4}z) \\
X_5 &= \text{span}\{ y(0), y(-1), x_1^-, x_2^+ \} & \pi_5(z) &= z(z-\tfrac{2}{3})(1-\tfrac{1}{4}z) \\
X_6 &= \text{span}\{ y(1), y(0), x_1^-, x_2^+ \} & \pi_6(z) &= (z-\tfrac{2}{3})(1-\tfrac{1}{4}z) \\
X_7 &= \text{span}\{ y(-1), y(-2), x_1^+, x_2^- \} & \pi_7(z) &= z^2(1-\tfrac{2}{3}z)(z-\tfrac{1}{4}) \\
X_8 &= \text{span}\{ y(0), y(-1), x_1^+, x_2^- \} & \pi_8(z) &= z(1-\tfrac{2}{3}z)(z-\tfrac{1}{4}) \\
X_9 &= \text{span}\{ (y(1), y(0), x_1^+, x_2^- \} & \pi_9(z) &= (1-\tfrac{2}{3}z)(z-\tfrac{1}{4}) \\
X_{10} &= \text{span}\{ y(-1), y(-2), x_1^-, x_2^+ \} & \pi_{10}(z) &= z^2(1-\tfrac{2}{3}z)(1-\tfrac{1}{4}z) \\
X_{11} &= \text{span}\{ y(0), y(-1), x_1^-, x_2^+ \} & \pi_{11}(z) &= z(1-\tfrac{2}{3}z)(1-\tfrac{1}{4}z) \\
X_+ &= \text{span}\{ y(1), y(0), x_1^+, x_2^+ \} & \pi_+(z) &= (1-\tfrac{2}{3}z)(1-\tfrac{1}{4}z).
\end{aligned}$$

Now let us use this example to illustrate the results of Section VII. To each X_i, $i = 1, 2, \cdots, 12$, in the above list there corresponds a (UH^-, H^-)-splitting subspace $\tilde{X}_i := \text{span}\{X_i, u(0)\}$. However, none of these are *minimal* (UH^-, H^+)-splitting subspaces. In fact, $Q(\infty) = 0$ for X_2, X_3, X_5, X_6, X_8, X_9, X_{11}, and X_+ and $\bar{Q}^*(\infty) = 0$ for X_-, X_2, X_4, X_5, X_7, X_8, X_{10}, and X_{11}. Consequently, $\tilde{X}_3$, $\tilde{X}_6$, $\tilde{X}_9$, and $\tilde{X}_+$ fail to be constructible, $\tilde{X}_-$, $\tilde{X}_4$, $\tilde{X}_7$, and $\tilde{X}_{10}$ fail to be observable, and $\tilde{X}_2$, $\tilde{X}_5$, $\tilde{X}_8$, and $\tilde{X}_{11}$ are neither observable nor constructible (Theorems 7.1 and 8.1).

To obtain the minimal Markovian (UH^-, H^+)-splitting subspaces we clearly need to pair the spectral factors differently. It is not hard to see that X_2, X_3, X_5, X_6, X_8, X_9, X_{11}, and X_+ are all (UH^-, H^+)-splitting as well. This is manifested in the facts that their π-polynomials are of degree less than three and that they all contain $y(0)$.

It should be noted that, in general, the sets of minimal X for the two formulations will not overlap as here, since they may not even have the same dimension.

X. Degeneracy

We shall say that a proper Markovian splitting subspace X is *degenerate* if its structural function K is singular at infinity, i.e., $\det K(\infty) = 0$; *nondegenerate* otherwise. By Lemma 8.1, X is nondegenerate if and only if the two equivalent conditions $\ker U(X)=0$ and $\ker U(X)^*=0$ hold. If so, both $U(X)$ and the adjoint $U(X)^*$ are *quasi-invertible*, i.e., they map one to one and onto a dense subset of X [50]. This is the appropriate infinite-dimensional generalization of *invertible* and reduces to this when $\dim X<\infty$.

If X is nondegenerate, both A and $\bar{A}$ are quasi-invertible, for they are unitarily equivalent to $U(X)^*$ and $U(X)$, respectively. In the finite-dimensional case this implies that both the forward and the backward standard realization w.r.t. X can be reversed in the deterministic sense. The forward one, for example, can be written

$$\begin{cases} x(t)=A^{-1}x(t+1)-A^{-1}Bu(t) & (10.1a)\\ y(t)=CA^{-1}x(t+1)+(D-CA^{-1}B)u(t). & (10.1b)\end{cases}$$

However, the reader is warned that such a *reversed* system is not a backward system in the stochastic sense, since X is not orthogonal to $H^-(u)$. Nevertheless, in some analysis it is useful to perform this transformation. One case in point is the transformation from realizations w.r.t. X to realizations w.r.t. $\tilde{X}$ as discussed in Section VI; see [11] for details. Another is the theory of invariant directions of the matrix Riccati equation of Kalman filtering [40]–[43], [11].

The following lemma, the proof of which is analogous to that of Lemma 6.1, spreads some further light on the concept of degeneracy.

Lemma 10.1: The null spaces of $U(X)$ and its adjoint $U(X)^*$ are given by

$$\begin{cases} \ker U(X)=X\cap H_{-1}(u) & (10.2a)\\ \ker U(X)^*=X\cap H_0(\bar{u}). & (10.2b)\end{cases}$$

Consequently, a degenerate X contains some linear combinations of the components of $u(-1)$ and $\bar{u}(0)$, i.e., some linear functional of the state process $x(t)$ is white noise. (We can see this directly from the standard realizations by "premultiplying" by a vector in the null space of A^* or $\bar{A}^*$, respectively. For example, if $\xi\in\ker A^*$, $\langle\xi,x(t+1)\rangle_{\mathscr{X}}=\langle\xi,Bu(t)\rangle_{\mathscr{X}}$, which is white noise.)

One reason for considering degeneracy of splitting subspaces is that such a phenomenon occurs in an important subclass of finite-dimensional systems, namely those modeled by a *moving-average process*. As a simple example let us consider a process y with spectral density

$$\Phi(z)=5+2(z+z^{-1}). \qquad (10.3)$$

There are only two minimal Markovian splitting subspaces, namely X_-, corresponding to $W_-(z)=z^{-1}+2$ and $\bar{W}_-(z)=1+2z$, and X_+, corresponding to $W_-(z)=1+2z^{-1}$ and $\bar{W}_+(z)=z+2$. As must be, since y is scalar, X_- and X_+ have the same structural function $K(z)=z^{-1}$. Note that $K(\infty)=0$; hence both splitting subspaces are degenerate. The state space $\mathscr{X}$, which again is the same for both X_- and X_-, consists of all functions $f(z)=\alpha z^{-1}$, where α is a real number. Hence, $Af=P^{\mathscr{X}}zf=P^{\mathscr{X}}\alpha=0$, i.e., $\ker A^*\neq 0$.

We shall now assume that y is strictly noncyclic. Then the frame space $H^\square$ is proper, and *a fortiori* so are all minimal splitting subspaces. The frame space being degenerate is a property of the process y, and therefore we shall say that y is *degenerate* (*nondegenerate*) when $H^\square$ is. Since the generating processes of $H^\square$ are u_+ and $\bar{u}_-$, Lemma 10.1 and the facts that $H_0(u_+)\subset N^+$ and $H_{-1}(\bar{u}_-)\subset N^-$, consequences of (3.9), imply that y is degenerate if and only if the two equivalent conditions $(UH^\square)\cap N^+\neq 0$ and $(U^*H^\square)\cap N^-\neq 0$ hold. Now recall that $H^\square$ is the closed linear hull of all minimal splitting subspaces and that a state-space element in N^- is unobservable and one in N^+ unconstructible. Therefore, N^- and N^+ are the parts of H that we normally want to discard in state-space construction. As we can see from the two conditions just derived, degeneracy of y means that, if we shift one step forward or backward in time, some elements of the discarded spaces become part of the new frame space.

Degeneracy of minimal splitting subspaces is a property of the process y, as the following theorem shows.

Theorem 10.1: If one minimal Markovian splitting subspace is degenerate, then all are.

Proof: It was shown in [25] that minimal Markovian splitting subspaces have quasi-equivalent structural functions. In particular, this means that the structural functions have identical determinants, which consequently vanish simultaneously at infinity. □

We say that y is *state-space degenerate* if the minimal Markovian splitting subspaces are degenerate.

Next we shall tie up these concepts with the concept of *singularity* introduced in Section VI. Recall that X is singular if and only if D or $\bar{D}$ or both are singular and that this is connected to the observability and constructibility of $\tilde{X}$ in Section VII.

Theorem 10.2: If one minimal Markovian splitting subspace is singular, then all are.

Proof: By definition, $Q\bar{Q}^*K_+=WW_-^{-1}\bar{W}_+\bar{W}^{-1}W_+\bar{W}_-^{-1}$ and $Q_+K=W_+W_-^{-1}W\bar{W}^{-1}$, where quantities marked by + correspond to X_+ and those unmarked to an arbitrary X. Now, writing the determinants of these products as products of determinants, we obtain

$$\det Q\det\bar{Q}^*\det K_+=\det Q_+\det K.$$

But $\det K=\det K_+$ (see the proof of Theorem 10.1), and therefore

$$\det Q\det\bar{Q}^*=\det Q_+.$$

Then the theorem follows from Theorem 8.1 □

Consequently, singularity of minimal Markovian splitting subspaces is also a property of the process y. We say that y is *error-spaces degenerate* if the minimal X are singular. It is known [11] that conditions for invariant directions can be expressed in terms of F_+. (Note that Γ_* in [11] is a matrix representation of F_+.) Theorem 10.2 explains why this is so. It is easy to see that the process in the example of Section IX is error-space degenerate. In fact, this is the reason why the spaces $\tilde{X}$ are nonminimal.

To establish a connection between degeneracy, state-space degeneracy, and error-space degeneracy of a process y, first note that there are several ways in which $H^\square$ can be written as a sum of a minimal splitting subspace and an error space. Two are given by

$$H^\square=X_-\oplus Z_+=X_+\oplus\bar{Z}_- \qquad (10.4)$$

as the reader can easily check. Another is given by the following lemma.

Lemma 10.2: The frame space is given by the nonorthogonal decomposition

$$H^\square=X_+\vee Z_+. \qquad (10.5)$$

Proof: Since X_- is observable, $\bar{Z}_-\cap X_-=0$ (Proposition 6.1). Hence, taking orthogonal complements in $H^\square$ and using (10.4), we obtain the desired result. □

In the finite-dimensional case, Lemma 10.2. suggests that we construct a forward realization with respect to $H^\square$ by combining (5.12a) for X_+ and (6.2) for Z_+ to obtain

$$\begin{bmatrix} x_+(t+1)\\ z_+(t+1)\end{bmatrix}=\begin{bmatrix} A_+ & 0\\ 0 & F_+\end{bmatrix}\begin{bmatrix} x_+(t)\\ z_+(t)\end{bmatrix}+\begin{bmatrix} B_+\\ G_+\end{bmatrix}u_+(t).$$

In fact, since X_+ is constructible, $X_+\cap Z_+=0$ (Proposition 6.1).

Hence, in view of Lemma 10.1, $\{x_+(0), z_+(0)\}$ is a basis in $H^{\square}$. Moreover, both X_+ and Z_+ have u_+ as their forward generating process. Hence, y is degenerate if and only if either A_+ or F_+ is singular, i.e., if and only if y is either state-space or error-space degenerate. This argument works only in the finite-dimensional case, but the next theorem says that the conclusion holds in general.

Theorem 10.3: The process y is degenerate if and only if it is either state-space or error-space degenerate or both. If y is scalar $(m=1)$ both cannot happen at the same time.

Proof: The first statement follows from the fact that the structural function of $H^{\square}$ can be written $K_{\square} = K_- Q_+$. Hence, $K_{\square}(\infty)$ is singular if and only if either $K_-(\infty)$ or $Q_+(\infty)$ or both are singular. The first condition is equivalent to y being state-space degenerate, and, since $\overline{Q}_+ = I$, the second is equivalent to y being error-space degenerate. To prove the second statement, observe that, when $m=1$, $H_{-1}(u_+)$ is one-dimensional. Therefore, if $\ker U(X_+) \neq 0$, we must have $H_{-1}(u_+) \subset X_+$ (Lemma 10.1). In fact, if $X_+ \cap H_{-1}(u_+)$ contains an element $\xi \neq 0$, then it must contain $\alpha\xi$ for all $\alpha \in \mathbb{R}$ which is all of $H_{-1}(u_+)$. Likewise, if $\ker U(Z_+) \neq 0$, $H_{-1}(u_+) \subset Z_+$ (Lemma 6.1). However, this cannot happen at the same time, for, since X_+ is constructible, $X_- \cap Z_+ = 0$ (Proposition 6.1). □

At least in the rational case we can develop criteria for degeneracy in terms of the spectral density. If $m=1$, it is not hard to show that y is state-spaces degenerate if and only if $\Phi(\infty) = \infty$ and error-space degenerate if and only if $\Phi(\infty) = 0$ [44]. The corresponding vector results only hold in one direction [44], but if y is not state-space degenerate, it can be shown that $\Phi(\infty)$ is singular if and only if y is error degenerate [11].

XI. Another Example

Consider a vector process y with spectral density

$$\Phi(z) = \begin{bmatrix} 1 + \dfrac{(z-\frac{1}{2})(z-\frac{1}{3})(z^{-1}-\frac{1}{2})(z^{-1}-\frac{1}{3})}{(z-\frac{2}{3})(z^{-1}-\frac{2}{3})} & \dfrac{1}{1-\frac{1}{4}z} \\ \dfrac{1}{1-\frac{1}{4}z^{-1}} & \dfrac{1}{(z-\frac{1}{4})(z^{-1}-\frac{1}{4})} \end{bmatrix}. \tag{11.1}$$

The minimum phase spectral factor is

$$W_-(z) = \begin{bmatrix} \dfrac{(z-\frac{1}{2})(z-\frac{1}{3})}{z(z-\frac{2}{3})} & 0 \\ 1 & \dfrac{z}{z-\frac{1}{4}} \end{bmatrix} \tag{11.2}$$

and the backward minimum phase spectral factor is

$$\overline{W}_+(z) = \frac{1}{24\sqrt{17}} \begin{bmatrix} \dfrac{-4z^3+52z^2-132z+102}{(1-\frac{2}{3}z)(1-\frac{1}{4}z)} & \dfrac{-24z}{1-\frac{1}{4}z} \\ \dfrac{-z^2-55z+86}{(1-\frac{2}{3}z)(1-\frac{1}{4}z)} & \dfrac{96}{1-\frac{1}{4}z} \end{bmatrix}. \tag{11.3}$$

The predictor space X_- is defined by W_- and the strictly unstable spectral factor

$$\overline{W}_-(z) = \begin{bmatrix} \dfrac{(z-\frac{1}{2})(z-\frac{1}{3})}{1-\frac{2}{3}z} & \dfrac{z-\frac{1}{4}}{1-\frac{1}{4}z} \\ 0 & \dfrac{z}{1-\frac{1}{4}z} \end{bmatrix}. \tag{11.4}$$

To see this check that $K_- = W_- \overline{W}_-^{-1}$, given by

$$K_-(z) = \begin{bmatrix} \dfrac{1-\frac{2}{3}z}{z(z-\frac{2}{3})} & 0 \\ 0 & \dfrac{1-\frac{1}{4}z}{z-\frac{1}{4}} \end{bmatrix} \tag{11.5}$$

is inner and that K_- and $\overline{Q}_-^* := \overline{W}_+ \overline{W}_-^{-1}$ are right coprime. (K_- and $Q_- = I$ are of course always left coprime.)

Note that $K(\infty) = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}$ is singular, and consequently X_- is degenerate. Hence, in view of Theorem 10.1, we expect all other minimal X to be degenerate also. Let us check with X_+. The stable spectral factor of X_+ is

$$W_+(z) = \begin{bmatrix} \dfrac{(1-\frac{1}{2}z)(1-\frac{1}{3}z)}{z(z-\frac{2}{3})} & 0 \\ \dfrac{1}{z} & \dfrac{1}{z-\frac{1}{4}} \end{bmatrix}. \tag{11.6}$$

In fact, $W_+ \overline{W}_-^{-1}$ is given by

$$K_+(z) = \frac{1}{\sqrt{17}} \begin{bmatrix} \dfrac{4(1-\frac{2}{3}z)}{z(z-\frac{2}{3})} & \dfrac{1-\frac{2}{3}z}{z-\frac{2}{3}} \\ -\dfrac{1-\frac{1}{4}z}{z(z-\frac{1}{4})} & \dfrac{4(1-\frac{1}{4}z)}{z-\frac{1}{4}} \end{bmatrix} \tag{11.7}$$

and $W_- W_-^{-1}$ by

$$Q_+(z) = \begin{bmatrix} \dfrac{(1-\frac{1}{2}z)(1-\frac{1}{3}z)}{(z-\frac{1}{2})(z-\frac{1}{3})} & 0 \\ 0 & \dfrac{1}{z} \end{bmatrix} \tag{11.8}$$

and from this we can check that K_- is inner and the required coprimeness conditions are fulfilled; note that $\overline{Q}_-^* = I$. We now see that

$$K_+(\infty) = \frac{1}{\sqrt{17}} \begin{bmatrix} 0 & 2/3 \\ 0 & -1 \end{bmatrix}$$

which is singular as we expected.

Moreover, $Q_-(\infty) = \begin{bmatrix} 1/6 & 0 \\ 0 & 0 \end{bmatrix}$ is singular. Hence, X_+ is singular, and by Theorem 10.2, so are all other minimal X. This example shows that a *vector* process y can be *both* state-space and error-space degenerate.

Routine calculations show that K_- (or K_+) has McMillan degree 3, and therefore all minimal realizations are three-dimensional. The corresponding $\tilde{X}$-spaces have dimension 5; see Section VII. However, due to error space degeneracy, none of them is a minimal (UH^-, H^+)-splitting subspace. In fact, it can be seen that the minimal ones are four-dimensional, which is consistent with the fact that $\ker Q_+(\infty)$ is one-dimensional.

XII. Conclusions

In this paper we have investigated structural properties of discrete-time linear stochastic systems in geometric terms. This study has been undertaken in the framework of Markovian splitting subspaces laid out in the work by Lindquist and Picci. Some of the results of the latter work have been modified to the discrete-time setting, but to a large extent we have dealt with problems which are unique to the discrete-time setting, such as degeneracy, singularity, and their connections to models with and without observation noise. We have expressed properties of stochastic systems in geometric terms and determined to what extent they are properties of the individual splitting subspace or of the given process. Our results are conceptual rather than computational in nature, but we hope that the insights gained on the conceptual level will prove useful in better understanding algorithmic problems.

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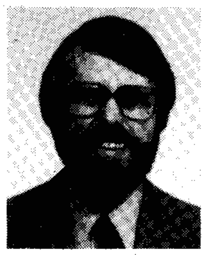

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On Controllability and Observability of Time Delay Systems

DIETMAR SALAMON

Abstract—**This paper deals with controllability and observability properties of time delay systems in the state space $\mathbb{R}^n \times L^p$. In particular, we prove the equivalence of spectral controllability and approximate null-controllability. Moreover, it is shown that the necessary condition for approximate F-controllability—obtained recently by Manitius—is also sufficient, and a verifiable and matrix type criterion for F-controllability is derived for systems with commensurate delays. Finally, we introduce the dual observability notion of approximate controllability and prove that the control system Σ is exactly null-controllable if and only if the transposed delay system Σ^T is continuously finally observable.**

Introduction

Controllability and observability of systems with delays in the state variables has become an area of active research in the last few years. On one hand, the algebraic systems theory, in particular that of linear systems over rings, has led to a clear connection between controllability over a ring and a spectrum assignability via feedback [12], [20], [28]. On the other hand, the functional analytic theory of infinite dimensional linear systems led to criteria for approximate controllability and observability in a function space [16]–[18], which are related to the ideas of state feedback and observers [4], [22], [24], [25]. In spite of this progress, there are still several gaps in the relations between the various concepts of controllability, stabilizability, and observability. In particular, some duality relations have not yet been clarified.

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In this paper we study linear control systems with delays in the state variables within the framework of the state space $R^n \times L^p$ $(1 < p < \infty)$. The aim is to establish relationships between the exact and approximate null-controllability and certain notions of observability and to generalize and extend recent results of Manitius [17] on approximate F-controllability. This latter effort is motivated in part by the fact that the F-controllability provides via duality a clear criterion for observability of retarded systems. For more motivation of the F-controllability concept the reader is referred to [14] and [17].

One of the key features of this paper is the use of the structural operators F and G [2], [15] associated with retarded systems. These operators give a clear characterization of the structure of the semigroup operator and eliminate the burden of cumbersome notation often encountered in some work on functional differential equations. As will be seen in this paper, the use of these operators allows us to obtain very concise proofs of all the results.

Function space controllability of retarded systems has been studied via several approaches. Banks *et al.* [1] considered the exact controllability in $W^{1,2}$ and showed that it led to a very restrictive condition on system matrices. Pandolfi [22] has proved a criterion for feedback stabilization in the state space $\mathscr{C}$. Analogous results on spectral observability have been derived by Bhat and Koivo [3]. The null-controllability has been investigated by several Soviet authors (see, e.g., [19]) and also in [11]. Manitius and Triggiani [18] and Manitius [14], [16], [17] have characterized the approximate controllability and F-controllability in the product space $\mathbb{R}^n \times L^2$, and a dual concept of observability. Dynamic observers for retarded systems have been investigated by Bhat and Koivo [4], Olbrot [21], and Salamon [24], [25].

The main results of this paper are as follows. In Section II it is shown that a general retarded functional differential system is spectrally controllable if and only if it is approximately null-con-

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