The Annals of Probability
1995, Vol. 23, No. 3, 1163–1187

ON THE STRUCTURE OF STATIONARY STABLE PROCESSES[1]

BY JAN ROSINSKI

University of Tennessee, Knoxville

A connection between structural studies of stationary non-Gaussian stable processes and the ergodic theory of nonsingular flows is established and exploited. Using this connection, a unique decomposition of a stationary stable process into three independent stationary parts is obtained. It is shown that the dissipative part of a flow generates a mixed moving average part of a stationary stable process, while the identity part of a flow essentially gives the harmonizable part. The third part of a stationary process is determined by a conservative flow without fixed points and by a related cocycle.

1. Introduction and preliminaries. The purpose of this work is to establish and exploit a connection between structural studies of stationary stable processes and the ergodic theory of nonsingular flows. Using this connection, we obtain a unique in distribution decomposition of a stationary symmetric non-Gaussian stable process $\{X_t\}_{t\in T}$ ($T = \mathbb{R}$ or $\mathbb{Z}$) into three independent parts,

$$X =_d X^{(1)} + X^{(2)} + X^{(3)},$$

where $\{X_t^{(1)}\}_{t\in T}$ is a superposition of moving averages [the so-called mixed moving average in the terminology of Surgailis, Rosinski, Mandrekar and Cambanis (1994)], $\{X_t^{(2)}\}_{t\in T}$ is a harmonizable process and $\{X_t^{(3)}\}_{t\in T}$ is a "third kind" of a stationary stable process described by a conservative nonsingular flow without fixed points and by a related cocycle. As we may see, this situation is quite different from the Gaussian case where all stationary processes are harmonizable. In this sense the class of stationary non-Gaussian stable processes is much richer than the corresponding Gaussian class.

In Section 2 we recall some results of Hardin (1981, 1982) on minimal representations of stable processes. These results combined with a result of Rosinski (1994), quoted below, are the basis for this work. In Section 3, we establish an explicit form of a spectral representation of a stationary stable process in terms of a nonsingular flow and a cocycle (Theorem 3.1). We show that such flow is determined uniquely by the process up to the usual equivalence relation of flows in ergodic theory (Theorem 3.6). In Section 4 we prove that a stationary stable process generated by a dissipative flow is a

Received January 1994; revised September 1994.
[1]Research supported in part by NSF Grant DMS-94-06294.
AMS 1991 *subject classifications*. Primary 60G10; secondary 60G07, 60E07, 60G57.
Key words and phrases. Stationary stable process, spectral representation, mixed moving average, harmonizable process, nonsingular flow, Hopf decomposition, cocycle.

mixed moving average (Theorem 4.4) and show how to extract the mixed moving average part from a stable process. Section 5 is devoted to stable processes generated by conservative flows. We prove that harmonizable processes are essentially these stationary processes which are generated by identity flows. We also show how to extract the harmonizable part from a stable process. In Section 6 we combine some results of the previous sections and discuss the general structure of stationary stable processes.

We will now provide some basic definitions and facts that will be used throughout this paper. Recall that a stochastic process $\{X_t\}_{t\in T}$ is said to be symmetric α-stable (SαS) if any linear combination $\Sigma a_i X_{t_i}$, $a_i \in \mathbb{R}$, $t_i \in T$ has a SαS distribution. A family of functions $\{f_t\}_{t\in T} \subset L^\alpha(S, \mathscr{B}, \mu)$, where $(S, \mathscr{B}, \mu)$ is a standard Lebesgue space [i.e., $(S, \mathscr{B})$ is a standard Borel space equipped with a σ-finite measure μ; a standard Borel space is a measurable space measurably isomorphic to a Borel subset of the real line], is said to be a *spectral representation* of a SαS process $\{X_t\}_{t\in T}$ if

$$\{X_t\}_{t\in T} =_d \left\{\int_S f_t(s) M(ds)\right\}_{t\in T}, \tag{1.1}$$

where M is an independently scattered random measure on $\mathscr{B}$ such that

$$E\exp\{iuM(A)\} = \exp\{-|u|^\alpha \mu(A)\}, \qquad u \in \mathbb{R},$$

for every $A \in \mathscr{B}$ with $\mu(A) < \infty$. We will also consider complex stable processes. However, in the complex case, we restrict our attention to those processes $\{X_t\}_{t\in T}$ for which all linear combinations $\Sigma a_i X_{t_i}$, $a_i \in \mathbb{C}$, $t_i \in T$, have *rotationally invariant* stable distributions. In such a case, a family of complex α-integrable functions $\{f_t\}_{t\in T}$ defined on a standard Lebesgue space $(S, \mathscr{B}, \mu)$ is called a spectral representation of the process $\{X_t\}_{t\in T}$ if (1.1) holds with a complex independently scattered random measure M such that

$$E\exp\{i\Re(u\overline{M(A)})\} = \exp\{-|u|^\alpha \mu(A)\}, \qquad u \in \mathbb{C}.$$

A stochastic process $\{X_t\}_{t\in T}$ is said to be separable in probability if there exists a countable set $T_0 \subset T$ such that the set of random variables $\{X_t\}_{t\in T_0}$ is a dense subset of $\{X_t\}_{t\in T}$ with respect to the topology of convergence in probability. It is well known that every separable in probability SαS process admits a spectral representation such that S is the unit interval and μ is the Lebesgue measure on S [see Kuelbs (1973) and Hardin (1982) for a discussion of the history of (1.1) and its extension to the complex case]. Conversely, if $\{X_t\}_{t\in T}$ has a spectral representation defined on a standard Lebesgue space, then it is separable in probability. Let T be a separable metric space. A spectral representation $\{f_t\}_{t\in T} \subset L^\alpha(S, \mathscr{B}, \mu)$ of a process $\{X_t\}_{t\in T}$ is said to be measurable if the map $(s,t) \to f_t(s)$ is measurable with respect to the product σ-algebra of $S \times T$. It is known that every measurable SαS process has a measurable spectral representation [see, e.g., Rosinski and Woyczynski (1986)]. If $\{X_t\}_{t\in T}$ is a measurable process, then the map $T \ni t \to X_t \in L^0(\Omega, P)$ is Borel, implying that $\{X_t\}_{t\in T}$ is separable in probability. In this paper, without further mention, we will only consider measurable stochastic processes.

Let $\{f_t\}_{t\in T} \subset L^\alpha(S, \mathscr{B}, \mu)$ be a collection of functions. By $\operatorname{supp}\{f_t\colon t \in T\}$ (support of $\{f_t\}_{t\in T}$) we denote a minimal (modulo μ) set $A \in \mathscr{B}$ such that $\mu\{s\colon f_t(s) \neq 0,\ s \in A^c\} = 0$ for every $t \in T$. The following result, which is a slight modification of Theorem 3.1 in Rosinski (1994), will be frequently used in this paper.

THEOREM 1.1. *Let $\{f_t^{(i)}\}_{t\in T} \subset L^\alpha(S_i, \mathscr{B}_i, \mu_i)$, $i = 1,2$, be two measurable spectral representations of a $S\alpha S$ process $\{X_t\}_{t\in T}$. Suppose that* $\operatorname{supp}\{f_t^{(2)}\colon t \in T\} = S_2$ *μ_2-a.e. Then for every σ-finite measure λ on T there exist Borel functions $\Phi\colon S_2 \to S_1$ and $h\colon S_2 \to \mathbb{R} - \{0\}$ ($\mathbb{C} - \{0\}$, in the complex case) such that*

$$f_t^{(2)}(s) = h(s) f_t^{(1)}(\Phi(s)), \qquad \lambda \otimes \mu_2\text{-}a.e. \tag{1.2}$$

PROOF. Let T_0 be a countable subset of T such that $\{f_t^{(2)}\}_{t\in T_0}$ is dense in the L^α-closure of $\{f_t^{(2)}\}_{t\in T}$. Let $\bar{\lambda} = \lambda + \sum_{t\in T_0}\delta_t$. Applying Theorem 3.1 in Rosinski (1994), we obtain the relation (1.2) with λ replaced by $\bar{\lambda}$, where h may vanish on some subset of S_2. This clearly implies (1.2) and also that

$$f_t^{(2)}(s) = h(s) f_t^{(1)}(\Phi(s)), \qquad \mu_2\text{-a.e. for each } t \in T_0.$$

Thus, for every $t \in T_0$, $f_t^{(2)} = 0$ μ_2-a.e. on $\{s \in S_2\colon h(s) = 0\}$. By the choice of T_0, the last condition must hold for every $t \in T$. Since $\operatorname{supp}\{f_t^{(2)}\colon t \in T\} = S_2$ μ_2-a.e., we have $\mu_2(\{s\colon h(s) = 0\}) = 0$. Replacing h by its modification h_1 given by $h_1(s) = 1$, if $h(s) = 0$ and $h_1(s) = h(s)$, otherwise, we end the proof. □

2. Minimal representations of SαS processes. Let $\{f_t\}_{t\in T} \subset L^\alpha(S, \mathscr{B}, \mu)$ be a collection of functions. Let $\rho(\{f_t\colon t \in T\})$ denote the smallest σ-field generated by extended-valued functions f_t/f_τ, $t, \tau \in T$. Following Hardin (1982) we give the following definition:

DEFINITION 2.1. A spectral representation $\{f_t\}_{t\in T} \subset L^\alpha(S, \mathscr{B}, \mu)$ of a SαS process is said to be minimal if $\operatorname{supp}\{f_t\colon t \in T\} = S$ μ-a.e. and for every $B \in \mathscr{B}$ there exists an $A \in \rho(\{f_t\colon t \in T\})$ such that $\mu(A \Delta B) = 0$.

It is rather difficult to verify whether a concrete representation is minimal (with the exception of some obvious cases). Nevertheless, the results on the existence and uniqueness of minimal representations due to Hardin (1982) are very useful in our study of structures of SαS processes.

THEOREM 2.2 [Hardin (1982), Theorems 5.1 and 5.2]. (a) *Every separable in probability $S\alpha S$ process has a minimal spectral representation. Moreover, one can always choose S as a unit interval or a countable discrete set or the union of the latter two and choose μ as the direct sum of Lebesgue measure acting on the unit interval and a counting measure acting on the discrete part of S.*

(b) *Let* $\{f_t^{(i)}\}_{t\in T} \subset L^\alpha(S_i, \mu_i)$ *be minimal spectral representations of a* $S\alpha S$ *process* $\{X_t\}_{t\in T}$, $i = 1,2$. *Then there exist unique modulo* μ_2 *functions* $\Phi: S_2 \to S_1$ *and* $h: S_2 \to \mathbb{R} - \{0\}$ ($\mathbb{C} - \{0\}$ *in the complex case*) *such that* Φ *is one-to-one and onto and, for each* $t \in T$,

$$f_t^{(2)}(s) = h(s) f_t^{(1)}(\Phi(s)) \quad \text{for } \mu_2\text{-a.a. } s \in S_2, \tag{2.1}$$

and

$$\frac{d(\mu_1 \circ \Phi)}{d\mu_2} = |h|^\alpha, \qquad \mu_2\text{-a.e.} \tag{2.2}$$

REMARK 2.3. Hardin (1982) states (2.1) and (2.2) in terms of a linear isometry between the $L^\alpha(S_i, \mu_i)$ spaces, $i = 1,2$. Every such isometry is generated by a regular set isomorphism [see Lamperti (1958)], which, under our assumptions, acts between Boolean σ-algebras of Borel σ-fields modulo null sets. By Theorem 32.5 in Sikorski (1964), every such set isomorphism is generated by a point isomorphism. This makes it possible to express (2.1) and (2.2) in the terms of a point mapping Φ instead of a set isomorphism.

REMARK 2.4. Relation (2.1) can be obtained under weaker conditions than minimality [see Rosinski (1994)]. However, further properties of Φ such as uniqueness, invertibility and (2.2) require the minimality of both representations.

REMARK 2.5. Suppose that in Theorem 2.2(b) the assumption of minimality of $\{f_t^{(2)}\}_{t\in T}$ is replaced by the condition $\operatorname{supp}\{f_t^{(2)}: t \in T\} = S_2$ μ_2-a.e. Then (2.1) holds for some nonvanishing function h and a measurable but not necessary invertible $\Phi: S_2 \to S_1$. Instead of (2.2) we now have

$$\mu_1 = \mu_{2,h} \circ \Phi^{-1} \quad \text{on } \mathscr{B}(S_1), \tag{2.3}$$

where $\mu_{2,h}(ds) = |h(s)|^\alpha \mu_2(ds)$. This result follows by Theorem 4.2 in Hardin (1981) combined with the above quoted Theorem 32.5 in Sikorski (1964).

3. Spectral representations of stationary processes. From now on T will denote either $\mathbb{R}$ or $\mathbb{Z}$. A stochastic process $\{X_t\}_{t\in T}$ is said to be stationary if for every $\tau \in T$, $\{X_{t+\tau}\}_{t\in T} =_d \{X_t\}_{t\in T}$. Our first goal is to establish a spectral representation of stationary $S\alpha S$ processes. To this end we will need several definitions.

Let $(S, \mathscr{B})$ be a standard Borel space. A family $\{\phi_t\}_{t\in T}$ of measurable maps from S onto S is said to be a flow on S if $\phi_{t_1+t_2}(s) = \phi_{t_1}(\phi_{t_2}(s))$ and $\phi_0(s) = s$ for all $s \in S$ and $t_1, t_2 \in T$. A flow $\{\phi_t\}_{t\in T}$ is said to be measurable if the map $T \times S \ni (t,s) \to \phi_t(s) \in S$ is measurable. Given a σ-finite measure μ on $(S, \mathscr{B})$, a flow $\{\phi_t\}_{t\in T}$ is said to be *nonsingular* if $\mu(\phi_t^{-1}(A)) = 0$ if and only if $\mu(A) = 0$ for every $t \in T$ and $A \in \mathscr{B}$.

Let A be a locally compact second countable group. A measurable map $T \times S \ni (t, s) \to a_t(s) \in A$ is said to be a *cocycle* for a measurable flow $\{\phi_t\}_{t \in T}$ if for every $t_1, t_2 \in T$,

$$a_{t_1+t_2}(s) = a_{t_2}(s) a_{t_1}(\phi_{t_2}(s)) \quad \text{for } \mu\text{-a.a. } s \in S. \tag{3.1}$$

A cocycle $\{a_t\}_{t \in T}$ is said to be a *coboundary* if there exists a measurable function b: $S \to A$ such that $a_t(s) = b(\phi_t(s))b(s)^{-1}$ μ-a.e. for each $t \in T$. The notions of cocycles and coboundaries come from cohomology theory; every $L^0(S, \mu; A)$-valued cocycle corresponding to an action $ts := \phi_t(s)$ of the group T on S has jointly measurable realization satisfying (3.1).

In the case $T = \mathbb{Z}$, every nonsingular flow is generated by a nonsingular Borel isomorphism V of S such that

$$\phi_n(s) = V^n(s), \qquad n \in \mathbb{Z}, s \in S.$$

Thus any cocycle $\{a_n\}_{n \in \mathbb{Z}}$ for $\{\phi_n\}_{n \in \mathbb{Z}}$ is determined by a measurable function u: $S \to A$ such that

$$a_n(s) = \begin{cases} u(s)u(Vs) \cdots u(V^{n-1}s), & \text{for } n \geq 1, \\ 1 \text{ (the identity of } A), & \text{for } n = 0, \\ u(V^{-1}s)^{-1} \cdots u(V^n s)^{-1}, & \text{for } n < 0. \end{cases}$$

Hardin (1982) expressed a minimal representation of a stationary stable process using a group of linear isometries on the $L^\alpha(S, \mu)$ space. We will write such representation in more explicit terms of a flow and an associated cocycle. The following result strengthens Theorem 5.1 in Rosinski (1994).

THEOREM 3.1. *Let $\{f_t\}_{t \in T} \subset L^\alpha(S, \mu)$ be a measurable minimal spectral representation of a measurable stationary $S\alpha S$ process $\{X_t\}_{t \in T}$. Then there exist a unique modulo μ nonsingular flow $\{\phi_t\}_{t \in T}$ on (S, μ) and a cocycle $\{a_t\}_{t \in T}$ for $\{\phi_t\}_{t \in T}$ taking values in $\{-1, 1\}$ ($\{|z| = 1\}$ in the complex case) such that, for each $t \in T$,*

$$f_t = a_t \left\{ \frac{d\mu \circ \phi_t}{d\mu} \right\}^{1/\alpha} (f_0 \circ \phi_t), \qquad \mu\text{-a.e.} \tag{3.2}$$

PROOF. Since $\{f_t\}_{t \in T}$ is minimal, then, for each $\tau \in T$, $\{f_{t+\tau}\}_{t \in T}$ is also a minimal representation of the same $S\alpha S$ process. Applying Theorem 2.2, there exist a one-to-one and onto function Φ_τ: $S \to S$ and a function h_τ: $S \to \mathbb{R} - \{0\}$ ($\mathbb{C} - \{0\}$, resp.) such that, for each $t \in T$,

$$f_{t+\tau} = (h_\tau)(f_t \circ \Phi_\tau), \qquad \mu\text{-a.e.}, \tag{3.3}$$

and

$$\frac{d(\mu \circ \Phi_\tau)}{d\mu} = |h_\tau|^\alpha, \qquad \mu\text{-a.e.} \tag{3.4}$$

Since, for every $t, \tau_1, \tau_2 \in T$, it is true that, μ-a.e.,

$$f_{t+\tau_1+\tau_2} = f_{(t+\tau_1)+\tau_2} = (h_{\tau_2})(f_{t+\tau_1} \circ \Phi_{\tau_2})$$
$$= (h_{\tau_2})(h_{\tau_1} \circ \Phi_{\tau_2})(f_t \circ \Phi_{\tau_1} \circ \Phi_{\tau_2})$$

and $f_{t+\tau_1+\tau_2} = (h_{\tau_1+\tau_2})(f_t \circ \Phi_{\tau_1+\tau_2})$, we infer from Theorem 2.2 that, for every $\tau_1, \tau_2 \in T$,

$$h_{\tau_1+\tau_2} = (h_{\tau_2})(h_{\tau_1} \circ \Phi_{\tau_2}), \qquad \mu\text{-a.e.,} \tag{3.5}$$

and

$$\Phi_{\tau_1+\tau_2} = \Phi_{\tau_1} \circ \Phi_{\tau_2}, \qquad \mu\text{-a.e.} \tag{3.6}$$

If $T = \mathbb{Z}$, then one can obviously modify Φ to have (3.6) everywhere for all τ_1 and τ_2. Therefore, $\phi_t = \Phi_t$ is a flow and putting $a_t = h_t/|h_t|$ ends the proof. To obtain a modification of $\{\Phi_t\}$ to a flow in the continuous case ($T = \mathbb{R}$), we will use a theorem of Mackey (1962). To this end we will verify that the σ-Boolean algebra $\mathscr{B}_\mu$ of the classes $[B] = \{A \in \mathscr{B}: \mu(A \Delta B) = 0\}$ together with the map

$$\mathbb{R} \times \mathscr{B}_\mu \ni (t, [B]) \to [\Phi_t^{-1}(B)] \in \mathscr{B}_\mu$$

is a Boolean G space [$G = \mathbb{R}$; see Mackey (1962)]. By (3.4), $[B] \to [\Phi_t^{-1}(B)]$ is a Boolean-algebra isomorphism and by (3.6), $\mathscr{B}_\mu$ is a G space. Therefore, we only need to check that the map

$$t \to \tilde{\nu}([\Phi_t^{-1}(B)])$$

is measurable for every finite measure $\tilde{\nu}$ on $\mathscr{B}_\mu$. Hence $\tilde{\nu}$ defines a finite measure ν on $\mathscr{B}$ such that $\nu(B) = \tilde{\nu}([B])$ and we have $\nu \ll \mu$. Put $k = d\nu/d\mu$. It is enough to show that

$$t \to \int_S 1_B(\Phi_t(s)) k(s) \mu(ds) \tag{3.7}$$

is measurable for each Borel set $B \in \mathscr{B}$. Choose a function $g \in \overline{\mathrm{sp}}\{f_t: t \in T\}_{L^\alpha(S,\mu)}$ whose support coincides with S. Choose $g_n \in \mathrm{sp}\{f_t: t \in T\}$, $g_n = \sum_i c_{ni} f_{t_{ni}}$, such that $g_n \to g$ μ-a.e. In view of (3.3), for each $\tau \in T$,

$$h_\tau(s) g_n(\Phi_\tau(s)) = \sum_i c_{ni} f_{t_{ni}+\tau}(s) \quad \text{for } \mu\text{-a.a. } s \in S.$$

Since the right-hand side is a measurable function of (τ, s) and the left-hand side converges μ-a.e. as $n \to \infty$, we infer that there exists a measurable function $(\tau, s) \to g_\tau(s)$ such that

$$h_\tau(s) g(\Phi_\tau(s)) = g_\tau(s), \qquad \mu\text{-a.e. for each } \tau \in T. \tag{3.8}$$

Since $\{f_t\}_{t \in T}$ is minimal, for every $B \in \mathscr{B}$ there exist $t_1, t_2 \ldots \in T$ and $A \subset \mathbb{R}^\infty$ such that $B = \{s: (f_{t_1}(s) g(s)^{-1}, f_{t_2}(s) g(s)^{-1}, \ldots) \in A\}$ modulo μ. In view of (3.3) and (3.8),

$$1_B(\Phi_\tau(s)) = 1_A\left(f_{t_1+\tau}(s)[g_\tau(s)]^{-1}, f_{\tau_2+t}(s)[g_\tau(s)]^{-1}, \ldots\right)$$

for μ-a.a. $s \in S$. Since the right-hand side is measurable in (τ, s), we infer that the map in (3.7) is measurable. Applying Theorem 1 in Mackey (1962), we get that there exist a Borel G space $((S', \mu'), \{\phi'_t\})$ (μ' is a finite Borel measure on S') and a Boolean-algebra isomorphism $T\colon \mathscr{B}_{\mu'} \to \mathscr{B}_{\mu}$ such that, for every $B' \in \mathscr{B}'$,

$$[\Phi_t^{-1}(T[B'])] = T([\phi_t'^{-1}(B')]).$$

By Sikorski (1964) there exists a Borel isomorphism $\Psi\colon S \to S'$ such that $T[B'] = [\Psi^{-1}(B')]$. Hence $\phi_t := \Psi^{-1} \circ \phi'_t \circ \Psi$ is a measurable flow on S and $\phi_t = \Phi_t$ μ-a.e. for each $t \in T$. One can now replace Φ_τ by ϕ_τ in (3.3)–(3.5) and (3.8).

It now follows from (3.8) that $(\tau, s) \to h_\tau(s)$ has a jointly measurable modification. Taking (3.3) with $t = 0$ gives

$$f_\tau = (h_\tau)(f_0 \circ \phi_\tau), \qquad \mu\text{-a.e. for each } \tau \in T.$$

Define $a_t = h_t/|h_t|$. Then (3.5) shows that $\{a_t\}_{t \in \mathbb{R}}$ is a cocycle for $\{\phi_t\}_{t \in \mathbb{R}}$ and (3.4) completes the proof. □

For any measurable nonsingular flow, a related cocycle and an arbitrary function $f \in L^\alpha(S, \mu)$, (3.2) defines a stationary SαS process. In this case, a representation need not be minimal. Given such a representation, one can change a flow and a cocycle to obtain other representations of the same process which may be more convenient to work with. To this end we will introduce an equivalence relation between pairs consisting of a flow and a cocycle.

DEFINITION 3.2. Pairs $(\phi^{(1)}, a^{(1)})$ and $(\phi^{(2)}, a^{(2)})$, where $\{a_t^{(i)}\}_{t \in T}$ is a cocycle for a measurable nonsingular flow $\{\phi_t^{(i)}\}_{t \in T}$ on (S_i, μ_i), $i = 1, 2$, are said to be equivalent $[(a^{(1)}, \phi^{(1)}) \sim (a^{(2)}, \phi^{(2)})]$ if there exists a measurable map $\Phi\colon S_2 \to S_1$ with the following properties:

(i) There exist $N_i \subset S_i$ with $\mu_i(N_i) = 0$ $(i = 1, 2)$ such that Φ is a Borel isomorphism between $S_2 - N_2$ and $S_1 - N_1$.

(ii) μ_1 and $\mu_2 \circ \Phi^{-1}$ are mutually absolutely continuous.

(iii) $\phi_t^{(1)} * \Phi = \Phi \circ \phi_t^{(2)}$ μ_2-a.e. for each $t \in T$.

(iv) The cocycle $\{a_t^{(1)} \circ \Phi\}_{t \in T}$ is cohomologous to $\{a_t^{(2)}\}_{t \in T}$. That is, there exists a measurable function $b\colon S_2 \to \{|z| = 1\}$ ($\{-1, 1\}$ in the real case) such that for each $t \in T$, $a_t^{(1)} \circ \Phi = a_t^{(2)}(b \circ \phi_t^{(2)})/b$ μ_2-a.a.

Conditions (i)–(iii) describe the usual equivalence of flows $\phi^{(1)}$ and $\phi^{(2)}$ in ergodic theory. Condition (iv) is trivially satisfied if $a^{(2)} = a^{(1)} \circ \Phi$.

Proposition 3.3. *Let $\{X_t\}_{t\in T}$ be a measurable stationary $S\alpha S$ process with a spectral representation $\{f_t^{(1)}\}_{t\in T}$ such that, for every $t\in T$,*

$$f_t^{(1)} = a_t^{(1)}\left\{\frac{d(\mu_1\circ\phi_t^{(1)})}{d\mu_1}\right\}^{1/\alpha}\left(f^{(1)}\circ\phi_t^{(1)}\right),\qquad \mu_1\text{-a.e.,} \tag{3.9}$$

where $f^{(1)}\in L^\alpha(S_1,\mu_1)$ and $a^{(1)}$ is a cocycle for a nonsingular flow $\phi^{(1)}$ on (S_1,μ_1). Suppose that we have another pair $(\phi^{(2)},a^{(2)})$ defined on (S_2,μ_2) such that $(\phi^{(2)},a^{(2)})\sim(\phi^{(1)},a^{(1)})$ and let Φ and b be the functions specified in Definition 3.2. Put

$$f^{(2)} = b\left\{\frac{d(\mu_1\circ\Phi)}{d\mu_2}\right\}^{1/\alpha}\left(f^{(1)}\circ\Phi\right) \tag{3.10}$$

and let

$$f_t^{(2)} = a_t^{(2)}\left\{\frac{d(\mu_2\circ\phi_t^{(2)})}{d\mu_2}\right\}^{1/\alpha}\left(f^{(2)}\circ\phi_t^{(2)}\right),\qquad \mu_2\text{-a.e., } t\in T. \tag{3.11}$$

Then $\{f_t^{(2)}\}_{t\in T}$ is another spectral representation of the process $\{X_t\}_{t\in T}$. Moreover, if $\{f_t^{(1)}\}_{t\in T}$ is minimal, then $\{f_t^{(2)}\}_{t\in T}$ is minimal as well.

Proof. First we notice that for each $t\in T$,

$$\frac{d(\mu_2\circ\phi_t^{(2)})}{d\mu_2}\,\frac{d(\mu_1\circ\Phi)}{d\mu_2}\circ\phi_t^{(2)} = \frac{d(\mu_1\circ\phi_t^{(1)})}{d\mu_1}\circ\Phi\,\frac{d\mu_1}{d(\mu_2\circ\Phi^{-1})}\circ\Phi \tag{3.12}$$

holds μ_2-a.e. Then a proof of the first part of the proposition follows by a simple verification of the equality

$$\left\|\sum c_j f_{t_j}^{(2)}\right\|_{L^\alpha(S_2,\mu_2)} = \left\|\sum c_j f_{t_j}^{(1)}\right\|_{L^\alpha(S_1,\mu_1)}$$

for all $c_1,c_2,\ldots\in\mathbb{C}$ ($\mathbb{R}$, resp.), $t_1,t_2\ldots\in T$ and $n\ge 1$. The second part follows by an observation that $f_t^{(2)}/f_\tau^{(2)} = (f_t^{(1)}/f_\tau^{(1)})\circ\Phi$ μ_2-a.e. for every $t,\tau\in T$. $\square$

One can view (3.10) and (3.11) as change of flow and cocycle formulas in the spectral representations. They can be used to simplify a representation of a stationary $S\alpha S$ process as is indicated in the following example.

Example 3.4. Let $f\in L^\alpha((0,\infty),\text{Leb})$ and $\beta\neq 0$. It is easy to see that

$$f_t(s) = e^{(\beta/\alpha)t}f(se^{\beta t}),\qquad s\in(0,\infty),\ t\in\mathbb{R},$$

is a spectral representation of a stationary $S\alpha S$ process, say, $\{X_t\}_{t\in\mathbb{R}}$, which is of the form (3.9) with $\phi_t^{(1)}(s) = se^{\beta t}$, $s\in(0,\infty)=S_1$, $t\in\mathbb{R}$, a cocycle $a_t^{(1)}=1$ and $\mu_1=\text{Leb}$.

Let Φ: $\mathbb{R} \to (0, \infty)$ be given by $\Phi(x) = e^{\beta x}$. Define $\phi_t^{(2)} = \Phi^{-1} \circ \phi_t^{(1)} \circ \Phi$ and $a_t^{(2)} = 1$. Then $\phi_t^{(2)}(x) = t + x$ is the usual translation flow on $(S_2, \mu_2) = (\mathbb{R}, \text{Leb})$. Applying Proposition 3.3, we get that

$$g_t(x) = g(t + x)$$

is another representation of the process $\{X_t\}$, where $g = |\beta|^{1/\alpha} e^{\beta x/\alpha} f(e^{\beta x})$. Thus $\{X_t\}_{t \in \mathbb{R}}$ is a moving average process.

REMARK 3.5. One can always select a minimal representation with $f_0 \geq 0$ (even when the representation is complex-valued). Indeed, let $\{f_t\}_{t \in T}$ be a minimal representation satisfying (3.2). Define

$$b(s) = \begin{cases} |f_0(s)|/f_0(s), & \text{if } f_0(s) \neq 0, \\ 1, & \text{otherwise} \end{cases}$$

and

$$u_t(s) = a_t(s) \frac{b(s)}{b(\phi_t(s))}.$$

Clearly $(\phi, u) \sim (\phi, a)$ (the equivalence holds with respect to the identity map), so that by Proposition 3.3, $\{g_t\}_{t \in T}$ is also a minimal representation and

$$g_0 = b f_0 = |f_0| \geq 0.$$

The following theorem shows that a stationary stable process determines a flow and a cocycle in its minimal representation up to the equivalence relation.

THEOREM 3.6. *Let $\{f_t^{(1)}\}_{t \in T}$ and $\{f_t^{(2)}\}_{t \in T}$ be minimal measurable representations of a stationary $S\alpha S$ process $\{X_t\}_{t \in T}$ satisfying* (3.9) *and* (3.11), *respectively. Then $(\phi^{(1)}, a^{(1)}) \sim (\phi^{(2)}, a^{(2)})$.*

PROOF. By Theorem 2.2,

$$f_t^{(2)} = (h)\left(f_t^{(1)} \circ \Phi\right), \qquad \mu_2\text{-a.e., } t \in T.$$

Put

$$g_\tau^{(i)} = a_\tau^{(i)} \left\{ \frac{d\left(\mu_i \circ \phi_\tau^{(i)}\right)}{d\mu_i} \right\}^{1/\alpha}, \qquad \tau \in T,$$

$i = 1, 2$. For every $t, \tau \in T$ we have μ_2-a.e,

$$\begin{aligned} f_{t+\tau}^{(2)} &= (h)\left(f_{t+\tau}^{(1)} \circ \Phi\right) = (h)\left(g_\tau^{(1)} \circ \Phi\right)\left(f_t^{(1)} \circ \phi_\tau^{(1)} \circ \Phi\right) \\ &= (h)\left(g_\tau^{(1)} \circ \Phi\right)\left[h \circ \Phi^{-1} \circ \phi_\tau^{(1)} \circ \Phi\right]^{-1}\left(f_t^{(2)} \circ \Phi^{-1} \circ \phi_\tau^{(1)} \circ \Phi\right). \end{aligned}$$

Since we also have $f_{t+\tau}^{(2)} = (g_\tau^{(2)})(f_t^{(2)} \circ \phi_\tau^{(2)})$ μ_2-a.e., for each τ and all $t \in T$, by the uniqueness result of Theorem 2.2 we obtain μ_2-a.e.,

$$\phi_\tau^{(2)} = \Phi^{-1} \circ \phi_\tau^{(1)} \circ \Phi$$

and

$$g_\tau^{(2)} = (h)\left(g_\tau^{(1)} \circ \Phi\right)\left[h \circ \Phi^{-1} \circ \phi_\tau^{(1)} \circ \Phi\right]^{-1}.$$

Since $a_\tau^{(i)} = g_\tau^{(i)}/|g_\tau^{(i)}|$, $a^{(1)} \circ \Phi$ is cohomologous to $a^{(2)}$. □

We conclude this section with an example of cocycles for a flow in continuous time.

EXAMPLE 3.7. Let k: $S \to \mathbb{R}$ be a measurable function such that $\int_{-N}^{N} |k(\phi_t(s))|\, dt < \infty$ for every $N > 0$ μ-a.e. Then

$$a_t(s) = \exp\left\{ i \int_0^t k(\phi_u(s))\, du \right\}$$

is a cocycle for a flow $\{\phi_t\}_{t \in \mathbb{R}}$. More generally, if F: $\mathbb{R} \times S \to \mathbb{R}$ is an additive process (in the sense of Kingman), that is, F is jointly measurable and for every $t_1, t_2 \in \mathbb{R}$, $F(t_1 + t_2, s) = F(t_2, s) + F(t_1, \phi_{t_2}(s))$ μ-a.e., then

$$a_t(s) = \exp\{iF(t, s)\}$$

is a cocycle for $\{\phi_t\}_{t \in \mathbb{R}}$.

4. Decomposition of stationary processes: dissipative and conservative parts. We will say that a stationary SαS measurable process $\{X_t\}_{t \in T}$ is *generated by a nonsingular measurable flow* $\{\phi_t\}_{t \in T}$ on (S, μ) if it has a spectral representation of the form

$$f_t = a_t \left\{ \frac{d(\mu \circ \phi_t)}{d\mu} \right\}^{1/\alpha} (f \circ \phi_t), \qquad \mu\text{-a.e.,} \tag{4.1}$$

where $f \in L^\alpha(S, \mu)$, $\{a_t\}_{t \in T}$ is a cocycle for $\{\phi_t\}_{t \in T}$ taking values in $\{|z| = 1\}$ (or $\{-1, 1\}$ in the real case) and

$$\operatorname{supp}\{f \circ \phi_t \colon t \in T\} = S, \qquad \mu\text{-a.e.} \tag{4.2}$$

In view of Theorems 3.1 and 2.2(a) every measurable stationary SαS process is generated by a nonsingular flow. Our main goal is to show that certain standard decompositions of flows in ergodic theory induce natural decompositions of stationary SαS processes. To this end we will recall basic definitions and facts concerning nonsingular maps and flows.

A nonsingular map V: $S \to S$ is said to be conservative if there is no wandering set of positive μ measure (a set B is called wandering if the sets $V^{-k}B$, $k \geq 0$, are disjoint). Given a nonsingular map V, there exists a decomposition of S into two disjoint measurable sets C and D—the conservative and the dissipative parts—such that: (i) C and D are V-invariant, (ii) the restriction of V to C is conservative and (iii) $D = \bigcup_{k=-\infty}^{\infty} V^k B$, for some wandering set B. The decomposition of S into C and D is unique (modulo μ) and is called the *Hopf decomposition*. Given a nonsingular flow $\{\phi_t\}_{t \in T}$ ($T = \mathbb{R}$ or $\mathbb{Z}$), for each $t \in T - \{0\}$ one has the Hopf decomposition of S, $S = C_t \cup D_t$, generated by the map ϕ_t. Since all C_t (D_t, resp.) are equal to each other modulo μ [see Krengel (1969, 1985)], one can choose a set C that is invariant under $\{\phi_t\}_{t\in T}$ and such that $C = C_t$ and $D := S - C = D_t$ modulo μ for every $t \in T - \{0\}$. This is the Hopf decomposition of S corresponding to the flow $\{\phi_t\}_{t\in T}$. A flow is called *dissipative* if $S = D$ and *conservative* if $S = C$ (modulo μ).

THEOREM 4.1. *If a stationary $S\alpha S$ process $\{X_t\}_{t\in T}$ is generated by a conservative (dissipative, resp.) flow, then in any other representation* (4.1) *and* (4.2) *of $\{X_t\}_{t\in T}$, the flow must be conservative (dissipative, resp.). Hence the classes of stationary $S\alpha S$ processes generated by conservative and dissipative flows are disjoint.*

PROOF. Suppose that $\{X_t\}_{t\in T}$ is generated by a flow $\{\phi_t\}_{t\in T}$. Let $S = C \cup D$ be the Hopf decomposition for $\{\phi_t\}_{t\in T}$. We will show that the following equalities hold μ-a.e.:

$$C = \left\{s \in S: \int_T |f(\phi_t(s))|^\alpha w_t(s)\,\lambda(dt) = \infty\right\} \tag{4.3}$$

and

$$D = \left\{s \in S: \int_T |f(\phi_t(s))|^\alpha w_t(s)\,\lambda(dt) < \infty\right\}, \tag{4.4}$$

where $w_t = d(\mu\circ\phi_t)/d\mu$ and λ is the Lebesgue measure if $T = \mathbb{R}$ and the counting measure if $T = \mathbb{Z}$. Let us denote the sets on the right-hand sides of (4.3) and (4.4) by C_0 and D_0, respectively.

We first consider the case $T = \mathbb{Z}$. Since ϕ_1 is dissipative on D and $|f_0|^\alpha \in L^1(S,\mu)$, we have $D = \bigcup_{k\in\mathbb{Z}}\phi_k B$ for some wandering set B and

$$\int_B \sum_{k\in\mathbb{Z}} |f\circ\phi_k|^\alpha w_k\, d\mu = \int_D |f|^\alpha\, d\mu < \infty.$$

Hence $\sum_{k\in\mathbb{Z}}|f\circ\phi_k|^\alpha w_k < \infty$ a.e. on B. The same argument holds when B is replaced by $\phi_j B$, $j\in\mathbb{Z}$. This proves that $\sum_{k\in\mathbb{Z}}|f\circ\phi_k|^\alpha w_k < \infty$ a.e. on D, implying $D \subset D_0$ μ-a.e. Let now $p \in L^1(S,\mu)$ be a strictly positive function. By a result of Halmos (1946), $\sum_{k\in\mathbb{Z}}(p\circ\phi_k)w_k = \infty$ μ-a.e. on C, and by the Chacon–Ornstein theorem [see Krengel (1985)],

$$\lim_{n\to\infty} \frac{\sum_{|k|\le n}(p\circ\phi_k)w_k}{\sum_{|k|\le n}|f\circ\phi_k|^\alpha w_k}$$

exists and is finite on $\{\sum_{k\in\mathbb{Z}}|f\circ\phi_k|^\alpha w_k > 0\} \cap C$. By (4.2) this intersection equals C μ-a.e.; hence, $C \subset C_0$ μ-a.e. This completes the proof of (4.3) and (4.4) in the case $T = \mathbb{Z}$.

We will now consider $T = \mathbb{R}$. First we will prove that

$$C \cap \operatorname{supp}\{f\} \subset C_0 \tag{4.5}$$

and

$$D \subset D_0 \tag{4.6}$$

μ-a.e. By a version of Wiener's local ergodic theorem for nonsingular flows [see Krengel (1985)],

$$\lim_{t\to 0^+} t^{-1}\int_0^t |f(\phi_s(x))|^\alpha w_s(x)\,ds = |f(x)|^\alpha$$

μ-a.e. Therefore, for μ-a.a. $x \in \operatorname{supp}\{f\}$,

$$g(x) := \int_0^1 |f(\phi_s(x))|^\alpha w_s(x)\,ds > 0.$$

Clearly $g \in L^1(S, \mu)$ and

$$\sum_{k\in\mathbb{Z}} g(\phi_k(x))w_k(x) = \int_{-\infty}^{\infty} |f(\phi_t(x))|^\alpha w_t(x)\,dt$$

μ-a.e. By the first part of this proof the series on the left-hand side is finite μ-a.e. on D (dissipative part of ϕ_1) and infinite μ-a.e. on $C \cap \operatorname{supp}\{f\circ\phi_k: k \in \mathbb{Z}\}$. This proves (4.5) and (4.6). Now one can replace f by $(f\circ\phi_u)w_u^{1/\alpha}$ in the above argument and (4.5) yields $C \cap \operatorname{supp}\{(f\circ\phi_u)w_u^{1/\alpha}\} \subset C_0$ μ-a.e., for any fixed $u \in T$. Using $w_u > 0$ μ-a.e. and (4.2), we get $C \subset C_0$ μ-a.e. This and (4.6) complete the proof of (4.3) and (4.4) in the case $T = \mathbb{R}$.

Suppose now that the process $\{X_t\}_{t\in T}$ is generated by a conservative flow $\{\phi_t\}_{t\in T}$ on (S, μ) and let $\{\psi_t\}_{t\in T}$ be another flow defined on (Y, ν) which also generates $\{X_t\}_{t\in T}$. Therefore,

$$g_t = u_t\left\{\frac{d(\nu\circ\psi_t)}{d\nu}\right\}^{1/\alpha}(g\circ\psi_t)$$

is another representation of $\{X_t\}_{t\in T}$ satisfying (4.1) and (4.2) and $\{u_t\}_{t\in T}$ is a cocycle for $\{\psi_t\}_{t\in T}$. We will show that $\{\psi_t\}_{t\in T}$ must be conservative as well.

Since $\{\phi_t\}_{t\in T}$ is conservative, we have $\mu(C - C_0) = 0$, where C_0 is defined by the right-hand side of (4.3). Let f_t^0 be the restriction of f_t to C_0, $t \in T$. By Theorem 1.1 there exist measurable functions $\Phi: Y \to C_0$ and $h: Y \to \mathbb{R} - \{0\}$ ($\mathbb{C} - \{0\}$, resp.) such that

$$g_t(y) = h(y)f_t^0(\Phi(y)), \qquad \lambda\otimes\nu\text{-a.e.}$$

Because $\Phi(y) \in C_0$, we obtain ν-a.e.,

$$\begin{aligned}\int_T |g(\psi_t(s))|^\alpha \frac{d(\nu\circ\psi_t)}{d\nu}(y)\,\lambda(dt) &= \int_T |g_t(y)|^\alpha \lambda(dt)\\ &= |h(y)|^\alpha \int_T |f(\phi_t(\Phi(y)))|^\alpha w_t(\Phi(y))\,\lambda(dt) = \infty.\end{aligned}$$

Hence $\{\psi_t\}_{t\in T}$ is a conservative flow by (4.3) and (4.4).

A proof in the case when $\{X_t\}_{t\in T}$ is generated by a dissipative flow is similar. □

The following corollary gives a simple criterion to identify SαS processes generated by conservative and dissipative flows.

COROLLARY 4.2. *The process* $\{X_t\}_{t \in T}$ *is generated by a conservative (dissipative, resp.) flow if and only if for some (any) measurable representation* $\{f_t\}_{t \in T} \subset L^\alpha(S, \mu)$ *such that* $\operatorname{supp}\{f_t\colon t \in T\} = S$ *the integral*

$$\int_T |f_t(s)|^\alpha \lambda(dt)$$

is infinite (finite, resp.) μ*-a.e. Here* λ *is the Lebesgue measure if* $T = \mathbb{R}$ *and the counting measure if* $T = \mathbb{Z}$.

PROOF. We apply Remark 2.5 by choosing an arbitrary minimal representation $\{f_t^{(1)}\}_{t \in T}$ of $\{X_t\}_{t \in T}$ (which is of the form (4.1) and (4.2) by Theorem 3.1) and $f_t^{(2)} = f_t$. By Remark 2.5 the a.e. finiteness of the integral in Corollary 4.2 transfers to the a.e. finiteness of a similar integral with f replaced by $f^{(1)}$. Therefore, we may assume that $\{f_t\}_{t \in T}$ is of the form (4.1) and (4.2). In this case, the corollary follows immediately from (4.3) and (4.4). □

Let $\{X_t\}_{t \in T}$ be a stationary SαS process. Every such process is generated by some nonsingular flow $\{\phi_t\}_{t \in T}$ [Theorems 3.1 and 2.2(a)] on a standard Lebesgue space (S, μ). The Hopf decomposition of S induces the following decomposition (in distribution) of the process X:

$$X =_d X^D + X^C, \tag{4.7}$$

where

$$X_t^D = \int_D f_t \, dM,$$

$$X_t^C = \int_C f_t \, dM$$

and M is a SαS random measure with the control measure μ. Clearly the processes X^D and X^C are mutually independent and, since D and C are invariant under the flow, X^D and X^C are both stationary. X^D is generated by a dissipative flow and X^C by a conservative one.

THEOREM 4.3. *Decomposition* (4.7) *is unique in distribution.*

PROOF. We need to show that decomposition (4.7) does not depend on $\{f_t\}_{t \in T} \subset L^\alpha(S, \mu)$. Let $\{g_t\}_{t \in T}$ be a minimal representation of the process $\{X_t\}_{t \in T}$. Assume that this representation is defined by a flow $\{\psi_t\}_{t \in T}$ on (Y, ν). Denote by C_ψ and D_ψ the conservative and dissipative parts of Y, respectively. By Remark 2.5 and (4.2) there exist measurable $\Phi\colon S \to Y$ and $h\colon S \to \mathbb{R} - \{0\}$ ($\mathbb{C} - \{0\}$, resp.) such that

$$f_t(s) = h(s) g_t(\Phi(s)), \qquad \mu\text{-a.e. for each } t \in T,$$

and $\nu = \mu_h \circ \Phi^{-1}$, where $\mu_h(ds) = |h(s)|^\alpha \mu(ds)$. Similarly as in the proof of Theorem 4.1, we obtain $\Phi^{-1}(C_\psi) \subset C$ a.e. and $\Phi^{-1}(D_\psi) \subset D$ a.e., which yield $\Phi^{-1}(C_\psi) = C$ and $\Phi^{-1}(D_\psi) = D$, respectively (μ-a.e.). For every $a_1, \dots, a_n \in \mathbb{R}$ (C, resp.) and $t_1, \dots, t_n \in T$, $n \geq 1$, we have

$$\begin{aligned}\int_C \Big|\sum_i a_i f_{t_i}\Big|^\alpha d\mu &= \int_C \Big|\sum_i a_i g_{t_i} \circ \Phi\Big|^\alpha |h|^\alpha \, d\mu \\ &= \int_{\Phi^{-1}(C_\psi)} \Big|\sum_i a_i g_{t_i} \circ \Phi\Big|^\alpha |h|^\alpha \, d\mu \\ &= \int_{C_\psi} \Big|\sum_i a_i g_{t_i}\Big|^\alpha d\nu.\end{aligned}$$

Hence the process X^C has the same distribution as X^{C_ψ} [defined analogously by (4.7) for $\{g_t\}_{t \in T}$]. Similarly, we show that $X^D =_d X^{D_\psi}$. This completes the proof. □

In the next result we will give a complete description of processes generated by dissipative flows. We will show that such processes are superpositions of the usual moving averages, or mixed moving averages in the terminology of Surgailis, Rosinski, Mandrekar and Cambanis (1994). The class of mixed moving averages was introduced in Surgailis, Rosinski, Mandrekar and Cambanis (1994) because it nontrivially extends the class of the usual moving averages while still retaining such important properties as ergodicity. Our next result and (4.7) show that a mixed moving average is a natural component of every stationary SαS process.

THEOREM 4.4. *Let $\{X_t\}_{t \in T}$ be a measurable stationary SαS process generated by a dissipative flow. Then there exist a Borel space W, a σ-finite measure ν on W and a function $g \in L^\alpha(W \times T, \nu \otimes \lambda)$ such that*

$$\{X_t\}_{t \in T} =_d \left\{ \int_W \int_T g(x, t+u) N(dx, du) \right\}_{t \in T}.$$

Here N is a SαS random measure on $W \times T$ with the control measure $\nu \otimes \lambda$ and λ is the Lebesgue measure if $T = \mathbb{R}$ and the counting measure if $T = \mathbb{Z}$. Moreover, one can always choose (W, ν) and g such that the representation $g_t(x, u) := g(x, t+u)$ is minimal.

PROOF. By a result of Krengel (1969), for every dissipative flow $\{\phi_t\}_{t \in T}$ on (S, μ), there exists a finite standard Lebesgue space (W, ν) such that the flow $\{\phi_t\}_{t \in T}$ is null isomorphic to a flow $\{\beta_t\}_{t \in T}$ defined on $(W \times T, \nu \otimes \lambda)$ by

$$\beta_t(x, u) = (x, t+u), \qquad (x, u) \in W \times T, \, t \in T.$$

That is, there exists a nonsingular invertible map Φ: $W \times T \to S$ such that $\Phi \circ \beta_t = \phi_t \circ \Phi$ for all $t \in T$. Since $d[(\nu \otimes \lambda) \circ \beta_t]/d(\nu \otimes \lambda) = 1$ for every $t \in T$, in view of Proposition 3.3 it is enough to show that $\{a_t \circ \Phi\}_{t \in T}$ is cohomologous to 1 [we take $f_t^{(1)} = f_t$ of the form (4.1) and $\phi_t^{(2)} = \beta_t$].

Put $c_t(x,u) = a_t(\Phi(x,u))$, $t \in T$, $(x,u) \in W \times T$. Since $\{c_t\}_{t \in T}$ is also a cocycle, we have, for every $t_1, t_2 \in T$,

$$c_{t_1+t_2}(x,u) = c_{t_2}(x,u)c_{t_1}(x, t_2+u), \qquad \nu \otimes \lambda\text{-a.e.} \tag{4.8}$$

By Fubini's theorem, the set

$$E = \left\{(x,u)\colon c_{t_1+t_2}(x,u) = c_{t_2}(x,u)c_{t_1}(x,t_2+u) \text{ for } \lambda^{\otimes 2}\text{-a.a. } (t_1,t_2)\right\}$$

is of full $\nu \otimes \lambda$ measure. Using Kuratowski's theorem on the existence of Borel cross sections, there exists a Borel function $k\colon W \to T$ such that the set $W_0 = \{x \in W\colon (x, k(x)) \in E\}$ is of full ν measure. Define

$$b(x,u) = c_{u-k(x)}(u, k(x)), \qquad (x,u) \in W \times T.$$

Then we have

$$\begin{aligned} b(\beta_t(x,u))b^{-1}(x,u) &= b(x, t+u)b^{-1}(x,u) \\ &= c_{t+u-k(x)}(x,k(x))c^{-1}_{u-k(x)}(x,k(x)), \end{aligned}$$

and since for each $x \in W_0$, $c_{t+u-k(x)}(x,k(x)) = c_{u-k(x)}(x,k(x))c_t(x,u)$ for $\lambda^{\otimes 2}$-a.a. $(t,u) \in T^2$, we obtain

$$b(\beta_t(x,u))b^{-1}(x,u) = c_t(x,u) \quad \text{for almost all } (t,x,u). \tag{4.9}$$

To finish the proof we need to show that the above equality holds for each $t \in T$, $\nu \otimes \lambda$-a.e. To this end, consider the following one-parameter groups of linear isometries on $L^1(W \times T, \nu \otimes \lambda)$:

$$(U_t z)(x,u) = b(x, t+u)b^{-1}(x,u)z(x,t+u)$$

and

$$(V_t z)(x,u) = c_t(x,u)z(x,t+u),$$

$z \in L^1(W \times T,\ \nu \otimes \lambda)$, $t \in T$ [the fact that V_t satisfies the group property follows from (4.8)]. Since these one-parameter groups are measurable, they must be continuous in the strong operator topology. By (4.9) and Fubini's theorem, $U_t = V_t$ for λ-a.a. $t \in T$, so that by continuity, $U_t = V_t$ for all $t \in T$. This proves (4.9) for each $t \in T$ and $\mu \otimes \lambda$-a.a. (x,u). □

EXAMPLE 4.5. Let $\phi_t(s) = se^{\beta t}$, $s,t \in \mathbb{R}$ and $\beta \neq 0$. Then $\{\phi_t\}_{t \in T}$ is a nonsingular dissipative flow on $(\mathbb{R}, \text{Leb})$. Therefore, a SαS process $\{X_t\}_{t \in T}$ with the representation

$$f_t(s) = e^{(\beta/\alpha)t} f(se^{\beta t}), \qquad s,t \in \mathbb{R},$$

is a mixed moving average process provided $f \in L^\alpha(\mathbb{R})$. It is easy to see that $\{\phi_t\}_{t \in T}$ is equivalent (or null isomorphic) to a flow $\beta_t(x,u) = (x, t+u)$ defined on $W \times T = \{-1, 1\} \times \mathbb{R}$ equipped with a measure $(\delta_{-1} + \delta_1) \otimes \text{Leb}$. Hence the considered process is the sum of two independent moving averages.

COROLLARY 4.6. *Suppose that a SαS process $\{X_t\}_{t \in T}$ is generated by a dissipative ergodic flow. Then $\{X_t\}_{t \in T}$ is the usual moving average process.*

PROOF. Since a null isomorphism preserves ergodicity and $\{\phi_t\}_{t\in T}$ is null isomorphic to $\{\beta_t\}_{t\in T}$, specified in the proof of Theorem 4.4, $\{\beta_t\}_{t\in T}$ is ergodic as well. However, this clearly implies that W is a one-point set. □

We note that the converse to Corollary 4.5 trivially holds.

5. Stable processes generated by conservative flows. Now we turn our attention to processes generated by conservative flows. The *identity flow*, defined by $\phi_t(s) = s$ for all $t \in T$ and $s \in S$, is the simplest conservative flow. A process $\{X_t\}_{t\in T}$ is said to be *harmonizable* if it admits the representation

$$\{X_t\}_{t\in T} =_d \left\{\int_S e^{itx}N(dx)\right\}_{t\in T}, \tag{5.1}$$

where $S = [0, 2\pi)$ if $T = \mathbb{Z}$ and $S = \mathbb{R}$ if $T = \mathbb{R}$, and N is a complex-valued rotationally invariant SαS random measure with finite control measure ν on S. Notice that the representation (5.1) of $\{X_t\}_{t\in T}$ is minimal. A harmonizable process is an example of a complex-valued process generated by an identity flow acting on S [and $a(t) = \exp\{itx\}$ is the corresponding cocycle]. We will prove the converse:

PROPOSITION 5.1. *Let $\{X_t\}_{t\in T}$ be a measurable complex-valued stationary $S\alpha S$ process generated by an identity flow. Then $\{X_t\}_{t\in T}$ is harmonizable.*

PROOF. We have $f_t = a_t f$ μ-a.e., $t \in T$, and $a_{t_1+t_2} = a_{t_1}a_{t_2}$, μ-a.e., for each $t_1, t_2 \in T$. Let

$$S_0 = \left\{s\colon a_{t_1+t_2}(s) = a_{t_1}(s)a_{t_2}(s) \text{ for } \lambda^{\otimes 2}\text{-a.a. } (t_1, t_2)\right\},$$

where, as before, λ is the Lebesgue measure if $T = \mathbb{R}$ and the counting measure if $T = \mathbb{Z}$. By Fubini's theorem, $\mu(S - S_0) = 0$. We will show that for each $s \in S_0$ there exists a unique $k(s) \in \mathbb{R}$ [$k(s) \in [0, 2\pi)$ when $T = \mathbb{Z}$] such that

$$a_t(s) = e^{itk(s)} \quad \text{for } \lambda\text{-a.a. } t \in T. \tag{5.2}$$

Fix $s \in S_0$ and define a continuous linear functional A_s on $L^1(T, \lambda)$ by

$$A_s(g) = \int_T g(t)a_t(s)\lambda(dt).$$

Since $s \in S_0$, we have $A_s(g_1 * g_2) = A_s(g_1)A_s(g_2)$ for every $g_1, g_2 \in L^1(T, \lambda)$, proving that A_s is a complex homomorphism of $L^1(T, \lambda)$ which yields (5.2) [see Rudin (1962), Theorem 1.2.2]. This proof of (5.2) was suggested to us by T. Byczkowski. The measurability of k follows from a measurability of the map $s \to A_s$. Define now a finite measure $\mu_0(ds) = |f(s)|^\alpha\mu(ds)$ on S. The continuity of the group of linear operators $U_t z = a_t z$, $z \in L^1(S, \mu)$ and Fubini's theorem imply that $a_t = \exp\{itk(\cdot)\}$ μ-a.e., for every $t \in T$ [$\mu \sim \mu_0$ since $f \neq 0$ by (4.2)]. Therefore, (5.1) holds with $\nu = \mu_0 \circ k^{-1}$. □

The following proposition shows that there are not many *real-valued* stationary SαS processes generated by identity flows.

PROPOSITION 5.2. *Let $\{X_t\}_{t\in T}$ be a measurable real-valued stationary SαS process generated by an identity flow. If $T = \mathbb{R}$, then $X_t = X_0$ a.s. for every $t \in \mathbb{R}$. If $T = \mathbb{Z}$, then there exist two independent SαS random variables Z_1 and Z_2 such that $X_n = Z_1 + (-1)^n Z_2$ a.s. for every $n \in \mathbb{Z}$.*

PROOF. We have $f_t = a_t f$ and $a_{t_1+t_2} = a_{t_1}a_{t_2}$ μ-a.e. Viewing a_t as a $\{|z| = 1\}$-valued function, we get from the previous proof that $a_t = \exp\{itk(\cdot)\}$ μ-a.e. However, since $a_t \in \{-1, 1\}$, we obtain $k = 0$ μ-a.e. when $T = \mathbb{R}$ and $k = 0$ or π when $T = \mathbb{Z}$. If $T = \mathbb{R}$, we have $f_t = f$ μ-a.e. and the proof is complete. Consider the case $T = \mathbb{Z}$.

Put

$$S_0 = \{s: k(s) = 0\} \quad \text{and} \quad S_1 = \{s: k(s) = \pi\}.$$

Since the process $Y_n = \int_S f_n\, dM = \int_{S_0} f_0\, dM + (-1)^n \int_{S_1} f_0\, dM$, $n \in \mathbb{Z}$, has the same distribution as $\{X_n\}_{n\in\mathbb{Z}}$, the conclusion of this proposition holds with $Z_1 = (X_0 + X_1)/2$ and $Z_2 = (X_0 - X_1)/2$. □

In order to simplify statements of further results, we will call all SαS processes that are generated by identity flows *harmonizable*. The next theorem characterizes all possible spectral representations of complex harmonizable SαS processes.

THEOREM 5.3. *Let $\{f_t\}_{t\in T}$ be a spectral representation of the form (4.1) and (4.2) for a complex-valued SαS process $\{X_t\}_{t\in T}$. If $\{X_t\}_{t\in T}$ is harmonizable, then there exists a $\{\phi_t\}_{t\in T}$-invariant function $k: S \to \mathbb{R}$ such that, for every $t \in T$,*

$$f_t(s) = e^{itk(s)} f(s), \qquad \mu\text{-a.e.} \tag{5.3}$$

Furthermore, $\{\phi_t\}_{t\in T}$ preserves a finite measure $\mu_0(ds) = |f|^\alpha \mu(ds)$. Conversely, for any measurable $k: S \to \mathbb{R}$ and $f \in L^\alpha(S, \mu)$, a family of functions $\{f_t\}_{t\in T}$ given by (5.3) is a spectral representation of a harmonizable process.

PROOF. Since $\{X_t\}_{t\in T}$ has representations (4.1), (4.2) and (5.1), by Remark 2.5 there exist measurable functions $\Phi: S \to \mathbb{R}$ and $h: S \to \mathbb{C} - \{0\}$ such that

$$a_t(s)\left\{\frac{d(\mu\circ\phi_t)}{d\mu}(s)\right\}^{1/\alpha} f(\phi_t(s)) = h(s)e^{it\Phi(s)} \tag{5.4}$$

μ-a.e., for each $t \in T$. Putting $t = 0$ we get $f = h \neq 0$ μ-a.e. Taking the modulus on both sides of (5.4) yields

$$\left\{\frac{d(\mu\circ\phi_t)}{d\mu}\right\}^{1/\alpha} |f\circ\phi_t| = |f|, \qquad \mu\text{-a.e.}, \tag{5.5}$$

for each $t \in T$. Hence

$$u_t(s) = \left\{ \frac{d(\mu \circ \phi_t)}{d\mu}(s) \right\}^{1/\alpha} \frac{f(\phi_t(s))}{f(s)}$$

is a cocycle for $\{\phi_t\}_{t \in T}$ taking values in $\{|z| = 1\}$. From (5.4) we have

$$a_t(s) u_t(s) = e^{it\Phi(s)}, \qquad \mu\text{-a.e.},$$

for each $t \in T$. Also $v_t = a_t u_t$ is a cocycle for $\{\phi_t\}_{t \in T}$ and

$$v_t(s) = e^{it\Phi(s)}, \qquad \mu\text{-a.e. for each } t \in T.$$

Therefore, for each $t_1, t_2 \in T$ we have, μ-a.e.,

$$\begin{aligned} \exp(i(t_1 + t_2)\Phi(s)) &= v_{t_1 + t_2}(s) = v_{t_2}(s) v_{t_1}(\phi_{t_2}(s)) \\ &= \exp(it_2\Phi(s)) \exp\big(it_1\Phi(\phi_{t_2}(s))\big), \end{aligned}$$

implying

$$\exp(it_1\Phi(s)) = \exp\big(it_1\Phi(\phi_{t_2}(s))\big), \qquad \mu\text{-a.e., for every } t_1, t_2 \in T.$$

Hence $\Phi(s) = \Phi(\phi_{t_2}(s))$ μ-a.e. for each $t_2 \in T$. By a standard argument, there exists a $\{\phi_t\}_{t \in T}$-invariant modification k of Φ. That is, $k(\phi_t(s)) = k(s)$ for all $t \in T$, $s \in S$ and $k = \Phi$ μ-a.e. Since $f_t = v_t f$ μ-a.e., (5.3) holds. Now we will show that $\{\phi_t\}_{t \in T}$ preserves μ_0. Indeed, this follows from (5.5) since for every measurable set A we have

$$\mu_0(\phi_t(A)) = \int_S (1_A \circ \phi_{-t}) |f|^\alpha \, d\mu = \int_S 1_A |f \circ \phi_t|^\alpha d(\mu \circ \phi_t) = \mu_0(A).$$

Conversely, (5.3) clearly implies that $\{X_t\}_{t \in T}$ is harmonizable. The proof is complete. □

The above theorem indicates that a nonidentity flow can also generate a harmonizable process. The next example shows that this indeed can be a case.

EXAMPLE 5.4. Let $\{\phi_t\}_{t \in T}$ be a measure-preserving conservative flow on a finite measure space (S, μ). Let $k\colon S \to \mathbb{R}$ be an arbitrary $\{\phi_t\}_{t \in T}$-invariant function. Define

$$a_t(s) = e^{itk(s)}.$$

Since $a_t(s) = a_t(\phi_\tau(s))$ for every $t, \tau \in T$ and $s \in S$, $\{a_t\}_{t \in T}$ is a cocycle for $\{\phi_t\}_{t \in T}$. Let $f = g \circ k$, where g is arbitrary measurable function such that $g \circ k \in L^1(S, \mu)$. Then

$$f_t(s) = a_t(s) f(\phi_t(s)) = e^{itk(s)} f(s)$$

is of the form (4.1) and (4.2) and of the form (5.3). Now, if in addition $\{\phi_t\}_{t \in T}$ is a nonergodic flow without fixed points, we may choose a nonconstant function k in the above construction obtaining a nontrivial harmonizable process that is not generated by an identity flow.

Our next goal is to extract the harmonizable part from a stationary SαS process. Fix a representation $\{f_t\}_{t\in T}$ satisfying (4.1) and (4.2) of $\{X_t\}_{t\in T}$ and set

$$S_H = \left\{s\colon f_{t_1+t_2}(s)f_0(s) = f_{t_1}(s)f_{t_2}(s) \text{ for } \lambda^{\otimes 2}\text{-a.a. } (t_1,t_2)\right\}. \tag{5.6}$$

LEMMA 5.5. $\mu(S_H \cap \{f_0 = 0\}) = 0$.

PROOF. Let $s \in S_H$ and $f_0(s) = 0$. We have

$$0 = \lambda^{\otimes 2}\left(\left\{(t_1,t_2)\colon f_{t_1}(s)f_{t_2}(s) \neq 0\right\}\right) = \lambda\left(\{t\colon f_t(s) \neq 0\}\right)^2.$$

Hence,

$$\int_T \mu\left(\{s \in S_H \cap \{f_0 = 0\}\colon f_t(s) \neq 0\}\right)\lambda(dt) = 0.$$

Consequently, there exists $T_0 \subset T$ with $\lambda(T - T_0) = 0$ such that, for every $t \in T_0$,

$$f_t = 0, \qquad \mu\text{-a.e. on } S_H \cap \{f_0 = 0\}. \tag{5.7}$$

By the L^α-continuity of the map $t \to f_t$, (5.7) holds for all $t \in T$. In view of (4.2), we get $\mu(S_H \cap \{f_0 = 0\}) = 0$. □

LEMMA 5.6. *For every* $t \in T$, $\mu(S_H \Delta \phi_t^{-1}(S_H)) = 0$.

PROOF. Put $S_H^0 = S_H \cap \{f_0 \neq 0\}$. By Lemma 5.5, we have $\mu(S_H - S_H^0) = 0$. Let

$$T_1 = \left\{t\colon \lambda \otimes \mu\left(\left\{(\tau,s) \in T \times S_H^0\colon f_{t+\tau}(s)f_0(s) \neq f_t(s)f_\tau(s)\right\}\right) = 0\right\}.$$

By Fubini's theorem, $\lambda(T - T_1) = 0$. Now we will show that, for each $t \in T_1$,

$$f \circ \phi_t \neq 0, \qquad \mu\text{-a.e. on } S_H. \tag{5.8}$$

Suppose that this is not the case, so that there exist $t \in T_1$ and a set $A \subset S_H^0$ of positive measure such that $f_t = 0$ on A. Then

$$\begin{aligned}
0 &= \int_A \lambda\left(\{\tau\colon f_{t+\tau}(s)f_0(s) \neq f_t(s)f_\tau(s)\}\right)\mu(ds)\\
&= \int_A \lambda\left(\{\tau\colon f_{t+\tau}(s) \neq 0\}\right)\mu(ds)\\
&= \int_T \mu\left(\{s \in A\colon f_{t+\tau}(s) \neq 0\}\right)\lambda(d\tau)\\
&= \int_T \mu\left(\{s \in A\colon f_u(s) \neq 0\}\right)\lambda(du).
\end{aligned}$$

Hence, there exists $T_2 \subset T$ such that $f_u = 0$ μ-a.e. on A, for each $\mu \in T_2$. Again by the L^α-continuity of the map $u \to f_u$, f_u vanishes on A μ-a.e., for every $u \in T$, and that contradicts (4.2). Thus (5.8) is proven.

Since $T_1 \cap (-T)$ is also of full λ measure, we may further assume that T_1 is symmetric about zero. We will now show that $\mu(S_H \Delta \phi_t^{-1}(S_H)) = 0$ for every $t \in T_1$. By the flow property and the symmetry of T_1, it is enough to prove that $\mu(S_H - \phi_t^{-1}(S_H)) = 0$.

Fix $t \in T_1$. By (5.8) there exists $S_H^t \subset S_H$ such that $\mu(S_H - S_H^t) = 0$ and $f_0 \circ \phi_t \neq 0$ on S_H^t. By the definition of T_1, there exists $B_t \subset S_H^0$ with $\mu(S_H^0 - B_t) = 0$ such that for every $s \in B_t$,

$$f_{t+\tau}(s) f_0(s) = f_t(s) f_\tau(s) \quad \text{for } \lambda\text{-a.a. } \tau \in T.$$

Denote by $T_{(t,s)}$ the set of τ's which satisfy the above equality. We have $\lambda(T - T_{(t,s)}) = 0$ for $s \in B_t$. Since λ is shift-invariant, the set

$$A_{(t,s)} := \{(t_1, t_2) \colon t_1, t_2, t_1 + t_2 \in T_{(t,s)}\}$$

is of full $\lambda^{\otimes 2}$ measure. Let

$$u_t(s) := a_t(s) \left\{ \frac{d(\mu \circ \phi_t)}{d\mu}(s) \right\}^{1/\alpha}.$$

Since $\{u_t\}_{t \in T}$ is a cocycle with values in $\mathbb{C} - \{0\}$, there exists a set $E_t \subset S$ with $\mu(S - E_t) = 0$ such that, for every $s \in E_t$,

$$u_{t+\tau}(s) = u_t(s) u_\tau(\phi_t(s)) \quad \text{for } \lambda\text{-a.a. } \tau \in T.$$

Denote by $F_{(t,s)}$ the set of τ's satisfying this equality. We have $\lambda(T - F_{(t,s)}) = 0$ for $s \in E_t$. The set

$$G_{(t,s)} := \{(t_1, t_2) \colon t_1, t_2, t_1 + t_2 \in F_{(t,s)}\}$$

is of full $\lambda^{\otimes 2}$ measure, $s \in E_t$. Also, by the definition of S_H, we have that

$$I_s := \{(t_1, t_2) \colon f_{t_1+t_2}(s) f_0(s) = f_{t_1}(s) f_{t_2}(s)\}$$

is of full $\lambda^{\otimes 2}$ measure for each $s \in S_H$.

Let $K_t = S_H^0 \cap S_H^t \cap B_t \cap E_t$, where $t \in T_1$. Clearly $K_t \subset S_H$ and $\mu(S_H - K_t) = 0$. We will prove that $K_t \subset \phi_t^{-1}(S_H)$, which implies that $\mu(S_H - \phi_t^{-1}(S_H)) = 0$. Let $s \in K_t$ and $(t_1, t_2) \in A_{(t,s)} \cap G_{(t,s)} \cap I_s$. We have

$$f_{t_1+t_2}(\phi_t(s)) f_0(\phi_t(s)) = u_{t_1+t_2}(\phi_t(s)) f_0(\phi_{t+t_1+t_2}(s)) f_0(\phi_t(s)),$$

and since $t_1 + t_2 \in F_{(t,s)}$, the right-hand side equals

$$\begin{aligned} &u_{t+t_1+t_2}(s) u_t(s)^{-1} f_0(\phi_{t+t_1+t_2}(s)) f_0(\phi_t(s)) \\ &\quad = f_{t+t_1+t_2}(s) u_t(s)^{-1} f_0(\phi_t(s)). \end{aligned}$$

Since $t_1 + t_2 \in T_{(t,s)}$, the right-hand side in the above equality equals

$$\begin{aligned} &f_t(s) f_{t_1+t_2}(s) f_0(s)^{-1} u_t(s)^{-1} f_0(\phi_t(s)) \\ &\quad = f_t(s) f_{t_1}(s) f_{t_2}(s) f_0(s)^{-2} u_t(s)^{-1} f_0(\phi_t(s)), \end{aligned}$$

because $(t_1, t_2) \in I_s$. Using the facts that $t_1, t_2 \in T_{(t,s)}$ and $f_t(s) \neq 0$ since $s \in S_H^t$, the last expression is equal to

$$f_t(s)^{-1} f_{t+t_1}(s) f_{t+t_2}(s) u_t(s)^{-1} f_0(\phi_t(s)),$$

which in view of $t_1, t_2 \in F_{(t,s)}$ equals

$$f_t(s)^{-1} u_{t_1}(\phi_t(s)) f_0(\phi_{t+t_1}(s)) u_{t_2}(\phi_t(s)) f_0(\phi_{t+t_2}(s)) u_t(s) f_0(\phi_t(s))$$
$$= f_{t_1}(\phi_t(s)) f_{t_2}(\phi_t(s)).$$

Since $A_{(t,s)} \cap G_{(t,s)} \cap I_s$ is of full $\lambda^{\otimes 2}$ measure, this proves that $\phi_t(s) \in S_H$. We have established $\mu(S_H \Delta \phi_t^{-1}(S_H)) = 0$ for every $t \in T_1$.

To conclude the proof of this lemma, consider a probability measure μ_1 on S, which is equivalent to μ, and a measurable (thus continuous) group of linear isometries $\{U_t\}_{t \in T}$ on $L^1(S, \mu_1)$ given by

$$U_t(z) = \frac{d(\mu_1 \circ \phi_t)}{d\mu_1} z \circ \phi_t.$$

It follows from the first part of the proof that $U_t(1_{S_H}) = [d(\mu_1 \circ \phi_t)/d\mu_1] 1_{S_H}$ for every $t \in T_1$. This equality extends to all $t \in T$ by a continuity argument since T_1 is dense in T. Therefore, $1_{S_H} \circ \phi_t = 1_{S_H}$ μ-a.e. for every $t \in T$, which completes the proof. □

In view of Lemma 5.6 we can choose a $\{\phi_t\}_{t \in T}$-invariant set $\tilde{S}_H$ such that $\mu(S_H \Delta \tilde{S}_H) = 0$. This leads to the following decomposition of the process $\{X_t\}_{t \in T}$ with the spectral representation $\{f_t\}_{t \in T}$:

$$X =_d X^H + X^3, \tag{5.9}$$

where

$$X_t^H = \int_{\tilde{S}_H} f_t \, dM$$

and

$$X_t^3 = \int_{S - \tilde{S}_H} f_t \, dM.$$

Since $\tilde{S}_H$ is $\{\phi_t\}_{t \in T}$-invariant, both X^H and X^3 are measurable stationary SαS processes and obviously they are independent of each other.

THEOREM 5.7. *The decomposition* (5.9) *is unique in distribution. Moreover,* $\{X_t^H\}_{t \in T}$ *is a harmonizable process and* $\{X_t^3\}_{t \in T}$ *does not have a harmonizable component.*

PROOF. A proof of uniqueness is similar to the proof of Theorem 4.3. Let $\{g_t\}_{t \in T}$ be a minimal representation of the process $\{X_t\}_{t \in T}$. Assume that this representation is defined by a flow $\{\psi_t\}_{t \in T}$ on (Y, ν). By Remark 2.5 there exist measurable functions Φ: $S \to Y$ and h: $S \to \mathbb{R} - \{0\}$ ($\mathbb{C} - \{0\}$, resp.) such that

$$f_t(s) = h(s) g_t(\Phi(s)), \qquad \mu\text{-a.e. for every } t \in T,$$

and $\nu = \mu_h \circ \Phi^{-1}$, where $\mu_h(ds) = |h(s)|^\alpha \mu(ds)$. Consider a set Y_H defined by (5.6) for $\{g_t\}_{t \in T}$. We have $\Phi^{-1}(Y_H) \subset S_H$ μ-a.e. and $\Phi^{-1}(Y - Y_H) \subset S - S_H$

μ-a.e. Hence $\Phi^{-1}(Y_H) = S_H$ μ-a.e. The rest of the proof of the uniqueness is identical with the last part of the proof of Theorem 4.3.

Now we will show that $\{X_t^H\}_{t \in T}$ is harmonizable. To this end, define for $s \in \tilde{S}_H \cap S_H \cap \{f_0 \neq 0\}$,

$$v_t(s) = f_t(s) f_0(s)^{-1}, \qquad t \in T.$$

Since $s \in S_H$, we have $v_{t_1+t_2}(s) = v_{t_1}(s) v_{t_2}(s)$ for $\lambda^{\otimes 2}$-a.a. (t_1, t_2). Hence, using the same argument as in Proposition 5.1, we get for every $t \in T$,

$$v_t(s) = \exp(t[j(s) + ik(s)]), \qquad \mu\text{-a.e. on } \overline{S}_H \cap S_H \cap \{f_0 \neq 0\},$$

where $j, k: \tilde{S}_H \cap S_H \cap \{f_0 \neq 0\} \to \mathbb{R}$ are measurable functions (k takes values in $[0, 2\pi)$ if $T = \mathbb{Z}$). Extending j and k to functions on the whole of $\tilde{S}_H$ by defining them arbitrarily on the subset of measure zero (see Lemma 5.5), we have for every $t \in T$,

$$f_t(s) = \exp(t[j(s) + ik(s)]) f_0(s), \qquad \mu\text{-a.e. on } \tilde{S}_H.$$

We will show that $j = 0$ μ-a.e. Indeed, since $\tilde{S}_H$ is invariant under $\{\phi_t\}_{t \in T}$,

$$\int_{\tilde{S}_H} |f_t|^\alpha \, d\mu = \int_{\tilde{S}_H} |f_0|^\alpha \, d\mu < \infty \quad \text{for every } t \in T.$$

Hence

$$\int_{\tilde{S}_H} e^{\alpha t j} |f_0|^\alpha \, d\mu = \int_{\tilde{S}_H} |f_0|^\alpha \, d\mu,$$

which implies

$$\int_{\tilde{S}_H} [\cosh(\alpha t j) - 1] |f_0|^\alpha \, d\mu = 0.$$

Since $\cosh(x) - 1 \geq x^2/2$, we get

$$\int_{\tilde{S}_H} j^2 |f_0|^\alpha \, d\mu = 0,$$

and since $f_0 \neq 0$ a.e. on S_H (Lemma 5.5), we obtain $j = 0$ μ-a.e. Thus $f_t = \exp[itk(s)] f_0(s)$, implying that $\{X_t^H\}_{t \in T}$ is a harmonizable process by Theorem 5.3.

Assume now that $\{X_t^3\}_{t \in T}$ admits a harmonizable component, that is, that there exist mutually independent stationary SαS processes $\{V_t\}_{t \in T}$ and $\{W_t\}_{t \in T}$ such that

$$X^3 =_d V + W$$

and $\{V_t\}_{t \in T}$ is harmonizable. Let f_t^3 be the restriction f_t to $S - S_H$, $t \in T$. Using Theorem 1.1, we obtain the equality

$$e^{itx} = h(x) f_t^3(\Phi(x)), \qquad \lambda \otimes \nu\text{-a.e.,} \tag{5.10}$$

where ν is the control measure for the harmonizable process $\{V_t\}_{t \in T}$ given by (5.1). Then ν must be a zero measure; otherwise, (5.10) and the fact that $\Phi(x) \in S - S_H$ give a contradiction. □

The next proposition shows that in the case of minimal representations the set $\tilde{S}_H$ coincides with the set of fixed points for the flow.

PROPOSITION 5.8. *If the representation* (4.1)–(4.2) *is minimal, then*

$$\tilde{S}_H = \{s \in S: \phi_t(s) = s \text{ for all } t \in T\}$$

μ-a.e. In the real case we have the following: If $T = \mathbb{R}$, then $\tilde{S}_H$ coincides modulo μ with a one-point subset of $\tilde{S}_H$ or with an empty set; if $T = \mathbb{Z}$, then $\tilde{S}_H$ coincides modulo μ either with a one-point subset or with a two-point subset or with an empty set.

PROOF. Notice that if the representation $\{f_t\}_{t \in T}$ is minimal, then its restriction to the set $\tilde{S}_H$ is a minimal representation for $\{X_t^H\}_{t \in T}$. Since $\{X_t^H\}_{t \in T}$ is harmonizable by Theorem 5.7, it has another minimal representation (5.1), which is given by an identity flow. In addition, in the real case, such a minimal representation will be defined either on a one-point or a two-point set (see Proposition 5.2). By Theorem 3.6, the flow $\{\phi_t\}_{t \in T}$ restricted to $\tilde{S}_H$ is equivalent to an identity flow [see (i)–(iii) of Definition 3.2], which gives, for every $t \in T$, $\phi_t(s) = s$ μ-a.e. on $\tilde{S}_H$. Therefore, the set

$$B = \{s \in \tilde{S}_H: \phi_t(s) = s \text{ for } \lambda\text{-a.a. } t \in T\}$$

differs from $\tilde{S}_H$ by a μ null set. The set $\{t \in T: \phi_t(s) = s\}$ is a measurable subgroup of T. Since it is of full λ measure if $s \in B$, it must coincide with T for $s \in B$. This proves $B = \tilde{S}_H$ μ-a.e. □

6. Concluding remarks: structure of stationary SαS processes. In this section we combine results of the previous sections to discuss the general structure of stationary SαS processes. Let $\{X_t\}_{t \in T}$ be a stationary SαS process. By Theorems 2.2 and 3.1, every such process has a spectral representation $\{f_t\}_{t \in T} \subset L^\alpha(S, \mu)$ of the form (4.1)–(4.2). If $\{f_t\}_{t \in T}$ is minimal, then a flow $\{\phi_t\}_{t \in T}$ and a cocycle are determined up to an equivalence relation (Theorem 3.6). Put

$$D = \left\{s \in S: \int_T |f_t(s)|^\alpha \lambda(dt) < \infty\right\}$$

and

$$C = S - D,$$

where λ is the Lebesgue measure if $T = \mathbb{R}$ and the counting measure if $T = \mathbb{Z}$. Next we define

$$C_H = \{s \in C: f_{t_1+t_2}(s) f_0(s) = f_{t_1}(s) f_{t_2}(s)\ \lambda^{\otimes 2}\text{-a.e.}\}.$$

If $\{f_t\}_{t \in T}$ is minimal, then C_H coincides a.e. with the set of fixed points of the flow $\{\phi_t\}_{t \in T}$ (Proposition 5.8). In addition, if $\{X_t\}_{t \in T}$ is a real process and the representation $\{f_t\}_{t \in T}$ is minimal, then C_H is at most a two-point set (modulo μ). Define

$$X_t^{(1)} = \int_D f_t \, dM,$$

$$X_t^{(2)} = \int_{C_H} f_t \, dM$$

and

$$X_t^{(3)} = \int_{S-D-C_H} f_t \, dM.$$

Since D and C_H are invariant under the flow (see Lemma 5.6), the processes $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ are stationary and obviously they are independent of each other. Combining the results of Theorems 4.1, 4.3, 4.4 and 5.7, we obtain the following theorem:

THEOREM 6.1. *Every stationary $S\alpha S$ process $\{X_t\}_{t \in T}$ admits a unique in distribution decomposition*

$$X =_d X^{(1)} + X^{(2)} + X^{(3)}$$

into three mutually independent stationary $S\alpha S$ processes such that $X^{(1)}$ is a mixed moving average process, $X^{(2)}$ is harmonizable and $X^{(3)}$ does not admit a mixed moving average or a harmonizable component.

If the flow $\{\phi_t\}_{t \in T}$ is ergodic, then in the decomposition

$$S = D \cup C_H \cup (S - D - C_H)$$

two terms on the right-hand side must have measure zero. Hence we obtain the next theorem.

THEOREM 6.2. *Suppose that $\{X_t\}_{t \in T}$ is generated by an ergodic nonsingular flow. Then only one of the following cases holds:*

(i) *$\{X_t\}_{t \in T}$ is the usual moving average process.*

(ii) *There exists $b \in \mathbb{R}$ ($b \in [0, 2\pi)$ if $T = \mathbb{Z}$) such that, for every $t \in T$, $X_t = e^{ibt} X_0$ a.s.*

(iii) *$\{X_t\}_{t \in T}$ does not admit a mixed moving average or a harmonizable component.*

PROOF. If $S = D$ a.e., then $\{X_t\}_{t \in T}$ is the usual moving average by Corollary 4.6. If $S = C_H$, then $\{X_t\}_{t \in T}$ is harmonizable and Theorem 5.3 implies that $f_t = \exp(itk) f$ for some $\{\phi_t\}_{t \in T}$-invariant function $k\colon S \to \mathbb{R}$ (Theorem 5.3 holds in the real case as well). Since the flow is ergodic, k is constant μ-a.e. This ends the proof of (ii). Case (iii) follows from Theorem 6.1. □

Theorem 6.1 indicates that many problems concerning stationary $S\alpha S$ processes can be reduced to separate studies of the parts $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$.

Moving average and harmonizable SαS processes have been extensively studied by many authors and many results are known [see Samorodnitsky and Taqqu (1994)]. An introduction and some basic facts on mixed stable moving averages can be found in Surgailis, Rosinski, Mandrekar and Cambanis (1994). Some processes of type $X^{(3)}$ have been investigated in the past, most noticeably sub-Gaussian SαS processes and certain doubly stationary stable processes [introduced in Cambanis, Hardin and Weron (1987)]. A systematic study of processes of type $X^{(3)}$ is needed in order to understand the class of all stationary SαS processes. In particular, the role and probabilistic meaning of a cocycle term in (4.1) ought to be clarified beforehand.

REFERENCES

CAMBANIS, S., HARDIN, C. D., JR. and WERON, A. (1987). Ergodic properties of stationary stable processes. *Stochastic Process. Appl.* **24** 1–18.

HALMOS, P. (1946). An ergodic theorem. *Proc. Nat. Acad. Sci. U.S.A.* **32** 156–161.

HARDIN, C. D., JR. (1981). Isometries on subspaces of L^p. *Indiana Univ. Math. J.* **30** 449–465.

HARDIN, C. D., JR. (1982). On the spectral representation of symmetric stable processes. *J. Multivariate Anal.* **12** 385–401.

KRENGEL, U. (1969). Darstellungssätze für Strömungen und Halbströmungen. II. *Math. Ann.* **182** 1–39.

KRENGEL, U. (1985). *Ergodic Theorems.* de Gruyter, Berlin.

KUELBS, J. (1973). A representation theorem for symmetric stable processes and stable measures. *Z. Wahrsch. Verw. Gebiete* **26** 259–271.

LAMPERTI, J. (1958). On the isometries of certain function spaces. *Pacific J. Math.* **8** 459–466.

MACKEY, G. W. (1962). Point realizations of transformation groups. *Illinois J. Math.* **6** 327–335.

ROSINSKI, J. (1994). On the uniqueness of the spectral representation of stable processes. *J. Theoret. Probab.* **7** 615–634.

ROSINSKI, J. and WOYCZYNSKI, W. A. (1986). On Itô stochastic integration with respect to p-stable motion: Inner clock, integrability of sample paths, double and multiple integrals. *Ann. Probab.* **14** 271–286.

RUDIN, W. (1962). *Fourier Analysis on Groups.* Wiley, New York.

SAMORODNITSKY, G. and TAQQU, M. S. (1994). *Non-Gaussian Stable Processes.* Chapman and Hall, London.

SIKORSKI, R. (1964). *Boolean Algebras.* Academic Press, New York.

SURGAILIS, D., ROSINSKI, J., MANDREKAR, V. and CAMBANIS, S. (1993). Stable mixed moving averages. *Probab. Theory Related Fields* **97** 543–558.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TENNESSEE
KNOXVILLE, TENNESSEE 37996