

ON THE STRUCTURE OF THE IDELE GROUPS OF ALGEBRAIC NUMBER FIELDS, II

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In this paper, we make further and more precise investigation into the idele groups of algebraic number fields than in our previous paper [4].

Here we state, as an example, a theorem obtained in §7 in a somewhat weakened and simplified form, which, even so, includes the main result [4, Theorem 2] as a special case:

THEOREM. *Let L be a finite Galois extension of an algebraic number field F , and V an open subgroup of the idele group L_A^\times of L which contains $L^\times \cdot L_{\infty+}^\times$ and satisfies (*) $V^\sigma = V$ for any $\sigma \in \text{Gal}(L/F)$ and (**) $L_A^\times = F_A^\times \cdot V \cdot N_{L/F}^{-1}(F^\times)$. Then $F_A^\times \cap V = F_A^\times \cap V \cdot N_{L/F}^{-1}(F^\times)$.*

Our basic tool is Terada's theorem on transfers of a finite group, which is generalized in §4.

In the final section, we point out a few results on capitulation of ideals easily derived from what we obtain the previous sections.

1. Preliminaries. For an algebraic number field F , we denote the ring of adèles of F by F_A , and the idele group by F_A^\times . Let $F_A^\times = F_f^\times \cdot F_\infty^\times$ be the decomposition of F_A^\times into the product of its non-Archimedean part F_f^\times and its Archimedean part F_∞^\times . The connected component of the unity of F_∞^\times is denoted by $F_{\infty+}^\times$, and the topological closure of $F^\times \cdot F_{\infty+}^\times$ in F_A^\times by $F^\#$. Let F_{ab} be the maximal abelian extension of F in the algebraic closure of F . The Artin map $[\cdot, F]: F_A^\times \rightarrow \text{Gal}(F_{ab}/F)$ of class field theory is an open, continuous and surjective homomorphism, whose kernel is $F^\#$.

Let K be a finite abelian extension of F , and put $\mathfrak{g} = \text{Gal}(K/F)$. Then \mathfrak{g} acts on K_A^\times naturally. Let $G_{K,F}$ be the Weil group of the extension K over F . This is the extension of the idele class group K_A^\times/K^\times by \mathfrak{g} , which corresponds to the canonical class $\xi_{K/F}$ in the cohomology group $H^2(\mathfrak{g}, K_A^\times/K^\times)$. (See Weil [7] and Hochschild and Nakayama [2], or Iyanaga [3, Ch. 5, §6].) There exists a surjective homomorphism $\phi_{K,F}: G_{K,F} \rightarrow \text{Gal}(K_{ab}/F)$ whose kernel is $K^\#/K^\times$ and whose restriction to the subgroup K_A^\times/K^\times coincides with the homomorphism induced by the

Artin map $[\cdot, K]: K_A^\times \rightarrow \text{Gal}(K_{\text{ab}}/K)$. Let $V_{K,F}$ be the transfer of $G_{K,F}$ to the abelian subgroup K_A^\times/K^\times . Then it may naturally be regarded as a homomorphism of $G_{K,F}/G'_{K,F}$ to K_A^\times/K^\times where $G'_{K,F}$ is the commutator subgroup of $G_{K,F}$. The homomorphism $\phi_{K,F}$ induces a homomorphism

$$\bar{\phi}_{K,F}: G_{K,F} \rightarrow \text{Gal}(F_{\text{ab}}/F) = \text{Gal}(K_{\text{ab}}/F)/\text{Gal}(K_{\text{ab}}/F)' .$$

By the properties (A) and (D) of Weil groups, we have a commutative diagram

$$\begin{array}{ccccc} K_A^\times & \longrightarrow & K_A^\times/K^\times & \xleftarrow{V_{K,F}} & G_{K,F}/G'_{K,F} \\ \uparrow & & \uparrow & \searrow & \downarrow \bar{\phi}_{K,F} \\ F_A^\times & \longrightarrow & F_A^\times/F^\times & \xrightarrow{[\cdot, F]} & \text{Gal}(F_{\text{ab}}/F) . \end{array}$$

2. Our problem. If an open subgroup U of K_A^\times contains $K^\times \cdot K_{\infty+}^\times$, then it contains the kernel K^* of the Artin map of K , and determines a finite abelian extension K_U of K . By [4, Theorem 1], we immediately see:

PROPOSITION 1. *The abelian extension K_U of K is a Galois extension of F if and only if*

$$(*) \quad U^\sigma = U \quad \text{for any } \sigma \in \mathfrak{g} = \text{Gal}(K/F) .$$

We only consider an open subgroup U of K_A^\times which contains $K^\times \cdot K_{\infty+}^\times$ and satisfies (*) in this paper.

Put $G_U = G_{K,F}/(U/K^\times)$ and $\mathfrak{G}_U = \text{Gal}(K_U/F)$. Then $\phi_{K,F}$ induces an isomorphism $\phi_U: G_U \rightarrow \mathfrak{G}_U$. The natural map of K_A^\times/K^\times onto K_A^\times/U induces a homomorphism of $H^2(\mathfrak{g}, K_A^\times/K^\times)$ to $H^2(\mathfrak{g}, K_A^\times/U)$. Let ξ_U be the image of the canonical class $\xi_{K/F}$ by the homomorphism. Then G_U is the extension of K_A^\times/U by \mathfrak{g} corresponding to ξ_U . Put $K'_U = K_U \cap F_{\text{ab}}$. Then by class field theory, we see that $F_A^\times/F^\times \cdot N(U)$ is isomorphic to $\text{Gal}(K'_U/F)$ by the Artin map of F . Here $N = N_{K/F}: K_A^\times \rightarrow F_A^\times$ is the norm map of K over F .

PROPOSITION 2. *If K is an abelian extension of F , then the commutator subgroup G'_U of G_U is equal to $U \cdot N^{-1}(F^\times)/U$.*

PROOF. Since K is abelian over F , the maximal abelian extension K'_U of F in K_U contains K . Therefore, the abelian extension $K'_U = K_U \cap F_{\text{ab}}$ of K corresponds to the open subgroup $U \cdot N^{-1}(F^\times)$ of K_A^\times by class field theory. Note that $U \cdot N^{-1}(F^\times) = U \cdot N^{-1}(F^*)$. The isomorphism ϕ_U now establishes the proposition.

The isomorphism $\phi_v: G_v \rightarrow \text{Gal}(K_v/F)$ induces an isomorphism $\bar{\phi}_v: G_v/G'_v \rightarrow \text{Gal}(K'_v/F)$. Put $\psi_v = \bar{\phi}_v^{-1} \circ [\cdot, K'_v/F]$. Let V_v be the transfer of G_v to K'_v/U . This may be considered a homomorphism of G_v/G'_v to K'_v/U . One can easily see the following by the diagram at the end of the previous section:

PROPOSITION 3. *The following diagram is commutative:*

$$\begin{array}{ccccc}
 K_A^\times & \longrightarrow & K'_A/U & \xleftarrow{V_v} & G_v/G'_v \\
 \uparrow \iota & & \uparrow \iota_v & & \uparrow \bar{\phi}_v^{-1} \\
 F_A^\times & \longrightarrow & F'_A/F^\times \cdot N(U) & \xrightarrow{[\cdot, K'_v/F]} & \text{Gal}(K'_v/F)
 \end{array}$$

(Note: In the original image, there are additional arrows from $F'_A/F^\times \cdot N(U)$ to G_v/G'_v labeled ϕ_v and ψ_v)

PROBLEM. *How large is the kernel of the homomorphism ι_v of the diagram? Does the degree $[K:F]$ divide $[F'_A \cap U: F^\times \cdot N(U)]$?*

Let O_F be the maximal order of F . For a prime divisor \mathfrak{p} of F , let $F_\mathfrak{p}$ be the \mathfrak{p} -adic completion of F , and $O_\mathfrak{p}$ the closure of O_F in $F_\mathfrak{p}$. Put $U_F = F^\times \cdot F_\infty^\times \cdot \prod_\mathfrak{p} O_\mathfrak{p}^\times$ where $\prod_\mathfrak{p}$ is the direct product over all the non-Archimedean prime divisors of F . Then F'_A/U_F is canonically isomorphic to the absolute ideal class group of F . Define U_K for K in the same way. Suppose that K is an unramified abelian extension of F . Then $F^\times \cdot N(U_K) = U_F$. Furthermore, the subgroup $(F'_A \cap U_K)/U_F$ of F'_A/U_F is canonically isomorphic to the subgroup of the absolute ideal class group of F consisting of the classes whose ideals become principal in K . Therefore if, moreover, K is a cyclic extension of F , then $[K:F]$ certainly divides $[F'_A \cap U_K: F^\times \cdot N(U_K)]$, which is just Hilbert's Theorem 94. Adachi questioned in [1] if this would be true for any unramified abelian extension K of F .

3. The subgroup $X_{K/F}(U)$ of $\text{Hom}(F'_A/F^\times \cdot N(U), K'_A/U \cdot N^{-1}(F^\times))$. Put $H_v = K'_A/U$. This is a normal abelian subgroup of G_v , and is naturally regarded as a G_v -module. By Proposition 2, we have $K'_A/U \cdot N^{-1}(F^\times) = H_v/G'_v$. Note that the norm map N gives an isomorphism of this group onto the subgroup $F^\times \cdot N(K'_A)/F^\times \cdot N(U)$ of $F'_A/F^\times \cdot N(U)$.

PROPOSITION 4. *There is a canonical isomorphism of the abelian group $\text{Hom}(F'_A/F^\times \cdot N(U), K'_A/U \cdot N^{-1}(F^\times))$ onto the cohomology group $H^1(G_v, H_v/G'_v)$.*

PROOF. Obviously G_v acts on H_v/G'_v trivially. Therefore we have

$H^1(G_U, H_U/G'_U) = \text{Hom}(G_U, H_U/G'_U)$. Since H_U/G'_U is abelian, we may identify these groups with $\text{Hom}(G_U/G'_U, H_U/G'_U)$. The isomorphism ψ_U of Proposition 3 now establishes the proposition.

Composing the isomorphism of Proposition 4 with the natural homomorphism of $H^1(G_U, H_U)$ to $H^1(G_U, H_U/G'_U)$, we have

$$\pi_U: H^1(G_U, H_U) \rightarrow \text{Hom}(F_A^\times/F^\times \cdot N(U), K_A^\times/U \cdot N^{-1}(F^\times)).$$

We put

$$X_{K/F}(U) = \pi_U(H^1(G_U, H_U)).$$

For $f \in X_{K/F}(U)$, put

$$d(f) = |\text{Coker}(f)| = [K_A^\times/U \cdot N^{-1}(F^\times) : \text{Im}(f)].$$

THEOREM 1. *Let K be a finite abelian extension of an algebraic number field F . Let U be an open subgroup of K_A^\times which contains $K^\times \cdot K_{\infty+}^\times$ and satisfies (*), and $X_{K/F}(U)$ as above. Then for $f \in X_{K/F}(U)$,*

$$\{a^{d(f)} \mid a \in F_A^\times, a \text{ mod } (F^\times \cdot N(U)) \in \text{Ker}(f)\} \subset F_A^\times \cap U.$$

In this section, we reduce the theorem to Proposition 6 in §5.

Put $G = G_U$ and $H = H_U = K_A^\times/U$. Let $V_{G \rightarrow H}: G \rightarrow H$ be the transfer of G to H . Then it induces a homomorphism $\bar{V}: G/G' \rightarrow H$ where G' is the commutator subgroup of G . By Proposition 3, we have a commutative diagram

$$\begin{array}{ccc} K_A^\times & \xrightarrow{\quad} & H = K_A^\times/U & \xleftarrow{\quad \bar{V} \quad} & G/G' \\ \uparrow \iota & & \uparrow \iota_U & \nearrow \psi = \psi_U & \\ F_A^\times & \xrightarrow{\quad} & F_A^\times/F^\times \cdot N(U) & & \end{array}$$

Therefore, an element x of F_A^\times belongs to U if and only if $\psi(x \text{ mod } (F^\times \cdot N(U))) \in \text{Ker}(\bar{V})$.

Let π be the homomorphism of $H^1(G, H)$ to $\text{Hom}(G/G', H/G') \cong H^1(G, H/G')$. Then

$$X_{K/F}(U) = \{\tilde{f} \circ \psi \mid \tilde{f} \in \text{Im}(\pi)\}.$$

Take $f = \tilde{f} \circ \psi \in X_{K/F}(U)$. For $a \in F_A^\times$, put

$$z = \psi(a \text{ mod } (F^\times \cdot N(U))).$$

Then it is sufficient to show that $\tilde{f}(z) = 1 \Rightarrow z^{d(f)} \in \text{Ker}(\bar{V})$. We have $d(f) = |\text{Coker}(f)| = |\text{Coker}(\tilde{f})| = [H/G' : \text{Im}(\tilde{f})]$ since ψ is an isomorphism. Theorem 1 now follows from Proposition 6 in §5 immediately.

By the corollary to Proposition 6, we see at once:

COROLLARY. *Let the notation and the assumptions be as in the theorem. If there exists an f in $X_{K/F}(U)$ with $d(f) = 1$, then the degree $[K:F]$ divides the index $[F_A^\times \cap U : F^\times \cdot N(U)]$.*

4. A generalization of Terada's theorem. Let G be a finite group, and G' the commutator subgroup. Let $\text{End}(G)$ be the set of all the endomorphisms of G . For $\phi \in \text{End}(G)$, put

$$H(\phi) = \langle g^{-1} \cdot \phi(g) \mid g \in G \rangle \cdot G'.$$

This is the subgroup of G generated by the elements of the form $g^{-1} \cdot \phi(g)$ with $g \in G$, and by the commutators of G . Denote the transfer of G to $H(\phi)$ by V_ϕ . (If ϕ is the trivial endomorphism, then $H(\phi) = G$. In this case, $V_\phi: G \rightarrow G/G'$ is the natural projection.) We generalize Terada's theorem [5] as follows:

PROPOSITION 5. *Let ϕ be an element of $\text{End}(G)$. Then*

$$\{g \in G \mid g^{-1} \cdot \phi(g) \in G'\} \subset \text{Ker}(V_\phi).$$

PROOF. We may assume that $H(\phi)$ is abelian. In fact: The commutator subgroup $H(\phi)'$ of $H(\phi)$ is normal in G . Put $\bar{G} = G/H(\phi)'$. It is obvious that ϕ induces an endomorphism $\bar{\phi}$ of \bar{G} . Then $H(\bar{\phi}) = \langle \bar{g}^{-1} \cdot \bar{\phi}(\bar{g}) \mid \bar{g} \in \bar{G} \rangle \cdot \bar{G}'$ is equal to $H(\phi)/H(\phi)'$. As for V_ϕ , it is a homomorphism of G to the abelian group $H(\phi)/H(\phi)'$. Since $H(\phi)' \subset G' \subset \text{Ker}(V_\phi)$, V_ϕ induces a homomorphism of \bar{G} to $H(\bar{\phi})$, which coincides with the transfer $V_{\bar{\phi}}$ of \bar{G} to $H(\bar{\phi})$ as is easily seen. Obviously,

$$\{g \in G \mid g^{-1} \cdot \phi(g) \in G'\} / H(\phi)' = \{\bar{g} \in \bar{G} \mid \bar{g}^{-1} \cdot \bar{\phi}(\bar{g}) \in \bar{G}'\}.$$

Thus we may replace G and $\phi \in \text{End}(G)$ by \bar{G} and $\bar{\phi} \in \text{End}(\bar{G})$ to show the theorem. We now assume that $H(\phi)$ is abelian. (Then G has to be metabelian.)

Terada [5] showed the theorem in the case that ϕ is an automorphism of G . A clear and fairly simple proof is obtained by Terada [6] (with the assistance of Adachi). It should be noted that Terada's setting in [6] might seem rather special. But the proof of Reduction 1 of [6] is applicable to show our Proposition 5 for an automorphism ϕ of G , putting $\mathfrak{G} = G \cdot \langle \phi \rangle$, the semi-direct product of G and the cyclic subgroup $\langle \phi \rangle$ of $\text{Aut}(G)$.

Now we reduce the case of $\phi \in \text{End}(G)$ to the case of an automorphism of a certain subgroup of G . Put $H = H(\phi)$. Since $\text{Ker}(\phi)$ is contained in H , we have $[G:H] = [\phi(G):\phi(H)]$. Put $G_1 = \phi(G)$, $H_1 = \phi(H)$

and $\phi_1 = \phi|_{G_1}$. Then ϕ_1 is an endomorphism of G_1 . Since $G'_1 = \phi(G')$, we have

$$H_1 = H(\phi_1) = \langle g^{-1} \cdot \phi_1(g) \mid g \in G_1 \rangle \cdot G'_1.$$

Put $d = [G : H]$, and let $R = \{x_1, \dots, x_d\}$ be a set of representatives of the cosets of G/H . Since $\phi(g) = g \cdot (g^{-1} \cdot \phi(g)) \equiv g \pmod{H}$, $\phi(R) = \{\phi(x_1), \dots, \phi(x_d)\}$ is a set of representatives not only of G_1/H_1 but also of G/H . Let V and V_1 be the transfers of G to H and of G_1 to H_1 , respectively. Then we have

$$V(\phi(g)) = V_1(\phi(g)) \quad \text{for } g \in G$$

at once if we express them according to the definition using the set of representatives $\phi(R)$.

Let g be an element of G such that $g^{-1} \cdot \phi(g)$ belongs to G' . Then $V(g) = V(\phi(g))$ because G' is contained in $\text{Ker}(V)$. Therefore, we have $V(g) = V_1(\phi(g))$. Put $g_1 = \phi(g)$. Then we have $V(g) = V_1(g_1)$ and $g_1^{-1} \cdot \phi_1(g_1) \in G'_1$. Define G_n , ϕ_n , H_n and g_n for $n \geq 2$ inductively by

$$G_n = \phi_{n-1}(G_{n-1}), \quad \phi_n = \phi_{n-1}|_{G_n}, \quad H_n = \phi_{n-1}(H_{n-1}) \quad \text{and} \quad g_n = \phi_{n-1}(g_{n-1}).$$

Then $\phi_n \in \text{End}(G_n)$, $G'_n = \phi_{n-1}(G'_{n-1})$, $H_n = \langle g^{-1} \cdot \phi_n(g) \mid g \in G_n \rangle \cdot G'_n$ and $g_n^{-1} \cdot \phi_n(g_n) \in G'_n$. Let V_n be the transfer of G_n to H_n . Then we also have $V_{n-1}(g_{n-1}) = V_n(g_n)$. Since G is finite, the series $G \supset G_1 \supset G_2 \supset \dots \supset G_n \supset \dots$ become stable. That is, there exists an integer m such that $G_m = G_{m+1} = \phi_m(G_m)$. Then ϕ_m is an automorphism of G_m . Therefore, by Terada's theorem, we have $V_m(g_m) = 1$. Since $V(g) = V_1(g_1) = \dots = V_m(g_m)$, the proof of Proposition 5 is now completed.

COROLLARY. For $\phi \in \text{End}(G)$, the index $[G : H(\phi)]$ divides the index $[\text{Ker}(V_{G \rightarrow H(\phi)}) : G']$.

PROOF. Let $f: G \rightarrow G/G'$ be the map defined by $f(g) = g^{-1} \cdot \phi(g) \pmod{G'}$. Then this is a homomorphism. Since $\text{Im}(f) = H(\phi)/G'$, we have $[H(\phi) : G'] = [G : \text{Ker}(f)]$. Therefore, $[G : H(\phi)] = [G : G'] \cdot [H(\phi) : G']^{-1} = [G : G'] \cdot [G : \text{Ker}(f)]^{-1} = [\text{Ker}(f) : G']$. This divides $[\text{Ker}(V_{G \rightarrow H(\phi)}) : G']$ because $\text{Ker}(f)$ is a subgroup of $\text{Ker}(V_{G \rightarrow H(\phi)})$. q.e.d.

5. Cohomological interpretation. Let G be a finite metabelian group, and H an abelian subgroup containing the commutator subgroup G' of G . Since H is a normal subgroup, G acts on H through the inner automorphisms of G . The action of G induced on H/G' is trivial. Therefore

$$H^1(G, H/G') = \text{Hom}(G, H/G') = \text{Hom}(G/G', H/G').$$

Let π be the homomorphism

$$\pi: H^1(G, H) \rightarrow H^1(G, H/G') = \text{Hom}(G/G', H/G')$$

induced by the natural map of H onto H/G' . For $f \in \text{Im}(\pi)$, put

$$d(f) = |\text{Coker}(f)| = [H/G' : \text{Im}(f)].$$

PROPOSITION 6. *Let G be a finite metabelian group, and H an abelian subgroup containing G' . Let $V_{G \rightarrow H}$ be the transfer of G to H . Then for $f \in \text{Im}(\pi)$,*

$$\{x^{d(f)} \mid x \in G, f(x \text{ mod } (G')) = 1\} \subset \text{Ker}(V_{G \rightarrow H}).$$

PROOF. Let ζ be a cocycle in $Z^1(G, H)$. Composing ζ with the natural projection of H to H/G' , we have an element f of $\text{Hom}(G, H/G')$, which is regarded as an element of $\text{Hom}(G/G', H/G')$. Put $\phi(x) = x \cdot \zeta(x)$ for $x \in G$. Then this is an endomorphism of G . It is obvious that $H \supset H(\phi)$, and that $\text{Im}(f) = H(\phi)/G'$. Therefore, we have $d(f) = [H : H(\phi)]$. Let x be an element of G such that

$$f(x \text{ mod } (G')) = \zeta(x) \text{ mod } (G') = 1.$$

Then $x^{-1} \cdot \phi(x)$ belongs to G' . Therefore by Proposition 5, we have $V_{G \rightarrow H(\phi)}(x) = 1$. As is well known,

$$V_{G \rightarrow H(\phi)}(x) = V_{H \rightarrow H(\phi)}(V_{G \rightarrow H}(x)).$$

Because H is abelian, we have

$$V_{H \rightarrow H(\phi)}(V_{G \rightarrow H}(x)) = V_{G \rightarrow H}(x)^{[H : H(\phi)]}.$$

Therefore

$$V_{G \rightarrow H}(x^{d(f)}) = V_{G \rightarrow H}(x^{[H : H(\phi)]}) = V_{G \rightarrow H(\phi)}(x) = 1.$$

The proposition is proved.

COROLLARY. *Let the notation and the assumptions be as in Proposition 6. If there exists an f in $\text{Im}(\pi)$ such that $d(f) = 1$, then $[G : H]$ divides $[\text{Ker}(V_{G \rightarrow H}) : G']$.*

PROOF. Suppose that $d(f) = 1$ for $f \in \text{Im}(\pi)$. Corresponding to f , take a cocycle $\zeta \in Z^1(G, H)$, and define $\phi \in \text{End}(G)$ as in the above proof. Then $H = H(\phi)$ because $d(f) = 1$. By the corollary to Proposition 5, we have the desired result.

6. The subgroup $X_{K/F}^0(U)$ of $X_{K/F}(U)$. As was in §3, let K be a finite abelian extension of F , $\mathfrak{g} = \text{Gal}(K/F)$ and U an open subgroup of K_A^\times which contains $K^\times \cdot K_{\infty+}^\times$ and satisfies the condition (*) of Proposition 1.

Let $H_U^g = (K_A^\times/U)^g$ be the subgroup of H_U consisting of the elements fixed by g , i.e.,

$$H_U^g = \{x \in H_U \mid x^\sigma = x \text{ for } \forall \sigma \in g\}.$$

Composing with the natural map of K_A^\times/U onto $K_A^\times/U \cdot N^{-1}(F^\times)$, we have a homomorphism,

$$\text{Hom}(F_A^\times/F^\times \cdot N(U), (K_A^\times/U)^g) \rightarrow \text{Hom}(F_A^\times/F^\times \cdot N(U), K_A^\times/U \cdot N^{-1}(F^\times)).$$

Denote the image of this homomorphism by $X_{K/F}^g(U)$. This is a subgroup of $X_{K/F}(U)$ defined in §3. In fact: $H^1(G_U, H_U^g)$ is equal to $\text{Hom}(G_U, H_U^g) = \text{Hom}(G_U/G'_U, H_U^g)$, since G_U acts on H_U^g trivially. Therefore, the isomorphism $\psi_U: F_A^\times/F^\times \cdot N(U) \rightarrow G_U/G'_U$ induces an isomorphism of $H^1(G_U, H_U^g)$ onto $\text{Hom}(F_A^\times/F^\times \cdot N(U), H_U^g)$. If we naturally map $H^1(G_U, H_U^g)$ onto a subgroup of $H^1(G_U, H_U)$, then we see at once that it is mapped by π_U onto $X_{K/F}^g(U)$, which is, therefore, certainly a subgroup of $X_{K/F}(U)$.

7. The case of ι_U . In this section, we investigate the case of the homomorphism

$$\iota_U: F_A^\times/F^\times \cdot N(U) \rightarrow K_A^\times/U,$$

which is induced by the inclusion map $\iota: F_A^\times \hookrightarrow K_A^\times$. Obviously, ι_U belongs to $\text{Hom}(F_A^\times/F^\times \cdot N(U), (K_A^\times/U)^g)$. Applying Theorem 1 to the image of ι_U in $X_{K/F}^g(U)$, we have

THEOREM 2. *Let K be a finite abelian extension of F . Let U be an open subgroup of K_A^\times which contains $K^\times \cdot K_{\infty+}^\times$ and satisfies (*). Put $d(U) = [K_A^\times : F_A^\times \cdot U \cdot N^{-1}(F^\times)]$. Then*

$$\{a^{d(U)} \mid a \in F_A^\times \cap (U \cdot N^{-1}(F^\times))\} \subset F_A^\times \cap U.$$

We extend this theorem to the case of a Galois extension of F . Let L be a finite Galois extension of F , and K the maximal abelian extension of F in L . We specify the norm maps of the extensions L/F , L/K and K/F as $N_{L/F}$, $N_{L/K}$ and $N_{K/F}$, respectively. Because K is the maximal abelian extension of F in L , we have

$$F^\times \cdot N_{L/F}(L_A^\times) = F^\times \cdot N_{K/F}(K_A^\times).$$

Therefore

$$(1) \quad K_A^\times = N_{L/K}(L_A^\times) \cdot N_{K/F}^{-1}(F^\times).$$

Let V be an open subgroup of L_A^\times which contains $L^\times \cdot L_{\infty+}^\times$ and satisfies

$$(*) \quad V^\sigma = V \quad \text{for any } \sigma \in \text{Gal}(L/F).$$

Put $U = K^\times \cdot N_{L/K}(V)$. Then this is an open subgroup of K_A^\times which contains $K^\times \cdot K_{\infty+}^\times$ and satisfies (*) for U . Put

$$\begin{aligned} d &= [L : K], \\ d(V) &= [L_A^\times : F_A^\times \cdot V \cdot N_{L/F}^{-1}(F^\times)], \\ d(U) &= [K_A^\times : F_A^\times \cdot U \cdot N_{K/F}^{-1}(F^\times)]. \end{aligned}$$

Note that $U \subset V$ and $N_{K/F}^{-1}(F^\times) \subset N_{L/F}^{-1}(F^\times)$. The norm map $N_{L/K}$ induces an injective homomorphism of $L_A^\times / N_{L/F}^{-1}(F^\times)$ to $K_A^\times / N_{K/F}^{-1}(F^\times)$. Therefore, we have

$$d(V) = [N_{L/K}(L_A^\times) \cdot N_{K/F}^{-1}(F^\times) : F_A^{\times d} \cdot U \cdot N_{K/F}^{-1}(F^\times)].$$

Here $F_A^{\times d} = \{a^d \mid a \in F_A^\times\} = N_{L/K}(F_A^\times)$. Then by (1), we have

$$d(V) = [K_A^\times : F_A^{\times d} \cdot U \cdot N_{K/F}^{-1}(F^\times)].$$

Put $W = U \cdot N_{K/F}^{-1}(F^\times)$ for the simplicity, and let $e(V)$ be the exponent of the finite abelian group $F_A^\times \cdot W / F_A^{\times d} \cdot W$. Then $e(V)$ divides d . Put $d = m \cdot e(V)$. Since $e(V)$ divides $[F_A^\times \cdot W : F_A^{\times d} \cdot W]$, we see that $d(U) \cdot e(V)$ divides $d(V)$. By the choice of $e(V)$, we have

$$F_A^{\times e(V)} \subset F_A^{\times d} \cdot W = (F_A^{\times e(V)})^m \cdot W.$$

Therefore, as is easily seen, $[F_A^{\times e(V)} : F_A^{\times e(V)} \cap W]$ is relatively prime to m .

If a is an element of $F_A^\times \cap V \cdot N_{L/F}^{-1}(F^\times)$, then $(a^{e(V)})^m = a^d = N_{L/K}(a)$ is an element of W . Therefore $a^{e(V)}$ belongs to W since m is relatively prime to $[F_A^{\times e(V)} : F_A^{\times e(V)} \cap W]$. By Theorem 2, then, we see that $(a^{e(V)})^{d(U)}$ belongs to U . We have shown the following generalization of Theorem 2.

THEOREM 3. *Let L be a finite Galois extension of F . Let V be an open subgroup of L_A^\times which contains $L^\times \cdot L_{\infty+}^\times$ and satisfies (*), and put $d(V) = [L_A^\times : F_A^\times \cdot V \cdot N_{L/F}^{-1}(F^\times)]$. Then we have*

$$\{\alpha^{d(V)} \mid \alpha \in F_A^\times \cap V \cdot N_{L/F}^{-1}(F^\times)\} \subset F_A^\times \cap V.$$

More precisely, let K be the maximal abelian extension of F in L , and put $U = K^\times \cdot N_{L/K}(V)$ and $d(U) = [K_A^\times : F_A^\times \cdot U \cdot N_{K/F}^{-1}(F^\times)]$. Let $e(V)$ be the exponent of

$$F_A^\times \cdot U \cdot N_{K/F}^{-1}(F^\times) / F_A^{\times [L:K]} \cdot U \cdot N_{K/F}^{-1}(F^\times).$$

Then $d(U) \cdot e(V)$ divides $d(V)$, and

$$\{\alpha^{d(U)e(V)} \mid \alpha \in F_A^\times \cap V \cdot N_{L/F}^{-1}(F^\times)\} \subset F_A^\times \cap U.$$

Moreover, $[L : K] \cdot e(V)^{-1}$ is relatively prime to the index

$$[F_A^{\times e(V)} : F_A^{\times e(V)} \cap U \cdot N_{K/F}^{-1}(F^\times)].$$

The case of $d(V) = 1$ of Theorem 3 is worth being pointed out, by which [4, Theorem 2] is obtained at once as a special case.

THEOREM 4. *Let L be a finite Galois extension of F , and K the maximal abelian extension of F in L . Let V be an open subgroup of L_A^\times which contains $L^\times \cdot L_{\infty+}^\times$, and suppose that V satisfies (*) above and the following (**):*

$$(**) \quad L_A^\times = F_A^\times \cdot V \cdot N_{L/F}^{-1}(F^\times).$$

Then the open subgroup $U = K^\times \cdot N_{L/K}(V)$ of K_A^\times satisfies

$$K_A^\times = F_A^\times \cdot U \cdot N_{K/F}^{-1}(F^\times).$$

Furthermore, we have

$$F_A^\times \cap V \cdot N_{L/F}^{-1}(F^\times) = F_A^\times \cap U \cdot N_{K/F}^{-1}(F^\times) = F_A^\times \cap V = F_A^\times \cap U.$$

The degree $[L : K]$ is relatively prime to the index $[F_A^\times : F_A^\times \cap U]$. The degree $[K : F]$ divides the indices

$$[F_A^\times \cap V : F^\times \cdot N_{L/F}(V)] = [F_A^\times \cap U : F^\times \cdot N_{K/F}(U)].$$

One can easily see the theorem by Theorem 3 except the last assertion, which is also easily seen by Corollary to Theorem 1.

8. On capitulation of the ideals. In this final section, we point out some consequences of our results obtained above, on capitulation of the ideals of F .

Let K be an unramified abelian extension of F . Let C_F be the absolute ideal class group of F , and $H_F(K)$ the subgroup of C_F corresponding to K , which consists of the classes containing norms of the ideals of K . Let $P_F(K)$ be the subgroup of C_F consisting of the classes whose ideals become principal in K . If we take the open subgroups U_F of F_A^\times and U_K of K_A^\times defined in the last paragraph of §2, then we may canonically identify C_F with F_A^\times/U_F , $H_F(K)$ with $F^\times \cdot N(K_A^\times)/U_F$ and $P_F(K)$ with $(F_A^\times \cap U_K)/U_F$.

Let \tilde{K} be the absolute class field of K . This is the abelian extension K_{U_K} of K corresponding to U_K . Therefore, G_{U_K} is isomorphic to $\mathfrak{G} = \text{Gal}(\tilde{K}/F)$. The maximal abelian extension K'_{U_K} of F in K_{U_K} is the absolute class field \tilde{F} of F . Since K is unramified over F , we have

$$F^\times \cdot N(U_K) = F^\times \cdot N(U_K \cdot N^{-1}(F^\times)) = U_F.$$

Identifying $F_A^\times/F^\times \cdot N(U_K)$ with C_F , we see that the norm map N maps $K_A^\times/U_K \cdot N^{-1}(F^\times)$ isomorphically onto $H_K(K)$. The homomorphism π_{U_K} in §3 induces the homomorphism

$$\pi_{K/F} : H^1(\text{Gal}(\tilde{K}/F), C_K) \rightarrow \text{Hom}(C_F, H_F(K))$$

where C_K is the absolute ideal class group of K . We use

$$\tilde{X}_{K/F} = \pi_{K/F}(H^1(\text{Gal}(\tilde{K}/F), C_K))$$

in place of $X_{K/F}(U_K)$, which are naturally isomorphic. For $f \in \tilde{X}_{K/F}$, put

$$d(f) = |\text{Coker}(f)| = [H_F(K) : \text{Im}(f)].$$

By Theorem 1 and its corollary, we have:

THEOREM 5. *Let K be an unramified abelian extension of F , and the notation as above. Then for any $f \in \tilde{X}_{K/F}$,*

$$\{x^{d(f)} \mid x \in \text{Ker}(f)\} \subset P_F(K).$$

If there exists an f in $\tilde{X}_{K/F}$ such that $d(f) = 1$, then the degree $[K:F]$ divides $|P_F(K)|$.

Put $n = [K:F]$, and $C_F(n) = \{x \in C_F \mid x^n = 1\}$. Then n divides $|C_F(n)|$. Put $m = |C_F(n)|/n$, and

$$C_F(n)^m = \{x^m \mid x \in C_F(n)\}.$$

THEOREM 6. *The notation and the assumption being as above,*

$$C_F(n)^m \subset P_F(K) \subset C_F(n).$$

PROOF. We use Theorem 2 for U_K . Put $C_F^n = \{x^n \mid x \in C_F\}$. By the norm map N , $K_A^\times/U_K \cdot N^{-1}(F^\times)$ is isomorphic to $H_F(K)$, and the subgroup $F_A^\times \cdot U_K \cdot N^{-1}(F^\times)/U_K \cdot N^{-1}(F^\times)$ is isomorphically mapped onto C_F^n . Therefore $d(U_K) = [H_F(K) : C_F^n]$. The endomorphism $x \mapsto x^n$ of C_F gives an exact sequence

$$1 \rightarrow C_F(n) \rightarrow C_F \rightarrow C_F^n \rightarrow 1.$$

Therefore $|C_F(n)| = [C_F : C_F^n]$. Since $n = [K:F] = [C_F : H_F(K)]$, we have $m = d(U_K)$. It is obvious that, for $a \in F_A^\times$, a belongs to $F_A^\times \cap U_K \cdot N^{-1}(F^\times)$ if and only if a^n belongs to U_F . Therefore, we may canonically identify $F_A^\times \cap U_K \cdot N^{-1}(F^\times)/U_F$ with $C_F(n)$, and $\{a^{d(U_K)} \mid a \in F_A^\times \cap (U_K \cdot N^{-1}(F^\times))\}/U_F$ with $C_F(n)^m$. The theorem now follows from Theorem 2 at once.

REMARK. If K is the absolute class field \tilde{F} of F , then $n = [K:F] = C_F$, $C_F(n) = C_F$ and $m = 1$. Therefore, Theorem 6 becomes the principal ideal theorem in this case.

Instead of the cohomological formulation as in Theorem 5, we can apply Proposition 5 directly as follows:

Let $\tilde{\tilde{F}}$ be the second class field of F , that is, the absolute class field of \tilde{F} . Put $\mathfrak{G} = \text{Gal}(\tilde{\tilde{F}}/F)$. For $\phi \in \text{End}(\mathfrak{G})$, put

$$\mathfrak{G}(\phi) = \langle g^{-1} \cdot \phi(g) \mid g \in \mathfrak{G} \rangle \cdot \mathfrak{G}',$$

and let $K(\phi)$ be the subfield of \tilde{F} corresponding to $\mathfrak{G}(\phi)$. Then $K(\phi)$ is contained in \tilde{F} . Denote the endomorphism of $\text{Gal}(\tilde{F}/F) = \mathfrak{G}/\mathfrak{G}'$ induced by ϕ by $\bar{\phi}$. Then for $g \in \mathfrak{G}$, we have $g^{-1} \cdot \phi(g) \in \mathfrak{G}'$ if and only if $\bar{\phi}(g \bmod (\mathfrak{G}')) = g \bmod (\mathfrak{G}')$. Let $[\cdot, \tilde{F}/F]$ be the Artin map.

THEOREM 7. *Let α be an ideal of F . If $\bar{\phi}([\alpha, \tilde{F}/F]) = [\alpha, \tilde{F}/F]$ for $\phi \in \text{End}(\mathfrak{G})$, then α becomes principal in $K(\phi)$.*

REMARK. Let α be an automorphism of the field F , and k the subfield of F fixed by α . Take an automorphism τ of \tilde{F} such that $\tau|_F = \alpha$. Then τ induces an automorphism ϕ of $\mathfrak{G} = \text{Gal}(\tilde{F}/F)$ through the inner automorphism of $\text{Gal}(\tilde{F}/k)$. Since the inner automorphisms of \mathfrak{G} act on $\text{Gal}(\tilde{F}/F) = \mathfrak{G}/\mathfrak{G}'$ trivially, the effect of $\bar{\phi} \in \text{End}(\text{Gal}(\tilde{F}/F))$ is independent of the choice of τ , and determined by α . It is easy to see that $\mathfrak{G}(\phi)$ is the commutator subgroup of $\text{Gal}(\tilde{F}/k)$. Therefore, $K(\phi)$ is the maximal abelian extension of k unramified over F . In other words, $K(\phi)$ is the genus field of the cyclic extension F over k . By [4, Theorem 1], one can now easily see that Theorem 7 gives Terada's principal ideal theorem in the genus field of [6] in this case.

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