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ON THE STRUCTURE OF THE TIME-EVOLUTION PROCESS IN MANY-BODY SYSTEMS

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Synopsis

The results of the Prigogine–George–Henin theory of "subdynamics" are extended to cover more general systems, such as spatially inhomogeneous systems or relativistic systems. The theory is presented in an abstract form, from which any particular case can be obtained by using an appropriate realization of the mathematical symbols. A number of new results are obtained in this way. The internal symmetry of the theory is clearly emphasized in the present formalism.

1. *Introduction*. In recent years a very significant advance has occurred in non-equilibrium statistical mechanics[1–3]. Going definitely beyond the traditional "simple" problems (weak coupling, low density, *etc.*) this work sheds a new light on the deeper, intrinsic structure of the law of evolution of many-body systems. One of the most important results achieved in this work is the separation of the distribution function into two components having remarkable invariance properties. The evolution of the system is such that when thermal equilibrium is approached, one of the components tends to zero, whereas the other is sufficient for the complete description of the thermodynamic behaviour of the system. It has also been shown[4]) that the "thermodynamic component" completely characterizes as well the stationary transport coefficients.

In the present paper we come back to this problem. Our motivation is the following. In the first place, we generalize the results of Prigogine, George and Henin* in such a way that they are no longer bound to the usual Fourier representation of the distribution function. The formalism adopted here is a continuation of the "unified" approach developed by one of us in

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* These results, contained in refs. 1–3, will be henceforth referred to as the PGH theory.

ref. 5. The flexibility achieved in this way makes the theory especially well suited for application to spatially inhomogeneous systems. Particular applications will, however, not be treated here, but will be the subject of a separate paper[6]). We prefer to keep the results here on a rather abstract level. We therefore use the algebraic approach developed in ref. 5, which enhances the *structural* aspects of the theory. The various particular systems (classical or quantum systems, homogeneous or inhomogeneous systems) appear just as particular realizations of the various symbols of the formalism.

We are thus able to show that practically all features of the PGH theory go over without any change to the more general situations. Such are: the concept of the separation into mutually independent Π and $\hat{\Pi}$ subspaces (sec. 2), the projection operator property of Π, its commutation relation with the liouvillian (sec. 6), the properties of the equilibrium distribution (sec. 7), *etc.*

Other features have to be slightly generalized. For instance, the generalized kinetic operator $V\Gamma V$ has to be defined in a more general way, but reduces to the PGH form $\Omega\Psi$ in the case of homogeneous systems (sec. 4).

Last, but not least, the use of our formalism enabled us to discover new results and new relationships. For instance, we can write the explicit expression of the kinetic operator $V\Gamma V$, in the form of a series whose general term has a very regular structure (sec. 4). This operator was previously known only through a set of implicit recurrence relations. Related operators, such as C, D, $\hat{\Gamma}$, *etc.* can be expressed in a similar way.

Another new and important result is the extension of the semi-group $\Sigma(t)$ into a full group (sec. 5). This result clarifies the structure of the evolution process and will be important in a relativistic extension of the theory, to be presented in a forthcoming paper.

Finally, we derive a number of new properties of the operator $\hat{\Sigma}(t)$, which enhance the symmetry and the complementarity of this operator and $\Sigma(t)$ (sec. 6).

A strong overlapping with the paper by Prigogine, George and Henin[7]) was unavoidable†. We preferred, however, to give a full presentation of the theory, rather than breaking the continuity by numerous and fastidious references to other papers. Moreover, we show that in the present formalism, many of the known and important properties of the PGH theory can be proved in a way much simpler than before. Typical examples of such new proofs have been sketched in the appendices.

† In a private communication, Prof. Prigogine told us that some of the results of the present paper will also be found in a forthcoming paper by himself, L. Rosenfeld and C. George.

2. *Definition of the asymptotic evolution operator.* We consider a many-body system described by a distribution function $\mathfrak{f}(t)$ obeying the Liouville equation

$$\partial_t \mathfrak{f}(t) = \mathscr{L}\mathfrak{f}(t). \tag{2.1}$$

At this very first step it is important to make a remark about the interpretation of all the equations of this paper. All equations, such as (2.1) are considered as *abstract algebraic equations*, involving objects, like $\mathfrak{f}$, $\mathscr{L}$, whose nature is *not specified*. These equations can be *concretely realized* in various ways. The simplest realization consists in identifying $\mathfrak{f}$ with the ordinary distribution function $F(q, p)$ and $\mathscr{L}$ with the ordinary Liouville operator. Alternatively, $\mathfrak{f}$ can be considered as the "vector" whose components are the various Fourier components of F; $\mathscr{L}$ is then a matrix operator coupling these components with each other. In a quite different approach, $\mathfrak{f}$ is considered as the collection of all reduced distribution functions f_s derived from F. The vector equation (2.1) is then equivalent to the BBGKY hierarchy and $\mathscr{L}$ is a matrix connecting reduced distributions of various orders. This realization (which can be further refined) is particularly important in the study of inhomogeneous systems. It is developed in ref. 6 (see also ref. 7). Finally, the same structure occurs in quantum mechanics, where $\mathfrak{f}$ can be realized as the density matrix or the Wigner distribution function.

We now assume that $\mathscr{L}$ can be decomposed into two terms:

$$\mathscr{L} = \mathscr{L}^0 + \mathscr{L}', \tag{2.2}$$

where $\mathscr{L}^0$ describes the free motion of independent degrees of freedom and $\mathscr{L}'$ describes interactions.

We further introduce a decomposition of the distribution function $\mathfrak{f}$ into two components, called the vacuum component $V\mathfrak{f}$, and the correlation component $C\mathfrak{f}$,

$$\mathfrak{f}(t) = V\mathfrak{f}(t) + C\mathfrak{f}(t). \tag{2.3}$$

The separation can be performed by introducing two formal "projection operators"†, as was done in ref. 5. These operators have to satisfy the relations:

$$V + C = I, \tag{2.4}$$

$$V^2 = V, \qquad C^2 = C, \tag{2.5}$$

$$VC = CV = 0. \tag{2.6}$$

† The reader is warned not to associate with V and C any of the detailed properties of projection operators in Hilbert spaces, such as orthogonality with respect to a scalar product, hermiticity, *etc.* No Hilbert space structure has been defined here, and such concepts are just meaningless at the present stage.

We, moreover, express the physical idea that correlations cannot be created or destroyed by the free motion, and require therefore that

$$\mathscr{L}^0 V = V \mathscr{L}^0, \qquad \mathscr{L}^0 C = C \mathscr{L}^0. \tag{2.7}$$

We do not discuss here the various realizations of the projection operators. We mention, however, that in all interesting cases it is possible to construct a linear realization of the operators V and C†.

From (2.4) it follows that every operator $\mathscr{P}$ acting on the distribution functions can be decomposed as follows

$$\mathscr{P} = V\mathscr{P}V + V\mathscr{P}C + C\mathscr{P}V + C\mathscr{P}C. \tag{2.8}$$

Applying this decomposition to the operator $\mathscr{L}^0$, it follows from (2.5)–(2.7) that

$$\mathscr{L}^0 = V\mathscr{L}^0 V + C\mathscr{L}^0 C = V\mathscr{L}^0 + C\mathscr{L}^0. \tag{2.9}$$

The operator $V\mathscr{L}^0$ plays a special role: it describes the change of the vacuum component under the free flow. Similarly, the operator $V\mathscr{L}'V$ represents that part of the interactions which can be described by an average field through which the "particles" move independently[5]) (in a plasma, this term is the well-known Vlasov operator). None of these two effects exists in a *spatially homogeneous system*. We may therefore *define* such a system by the conditions

$$V\mathscr{L}^0 V = 0, \qquad V\mathscr{L}'V = 0 \qquad \text{(homogeneous system)}. \tag{2.10}$$

The solution of the initial value problem of the Liouville equation is conveniently expressed in terms of the *propagator* $\mathscr{U}(t)$:

$$\mathfrak{f}(t) = \mathscr{U}(t)\, \mathfrak{f}(0). \tag{2.11}$$

The operator $\mathscr{U}(t)$ can be formally written as

$$\mathscr{U}(t) = \exp(t\mathscr{L}). \tag{2.12}$$

The main purpose of this paper boils down to a study of the properties of this operator.

Alternatively, one often uses instead of the propagator $\mathscr{U}(t)$ its Laplace transform $\mathscr{R}(z)$; the relation between these operators is.

$$\mathscr{U}(t) = (2\pi)^{-1} \int_C \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt}\, \mathscr{R}(z). \tag{2.13}$$

† This statement corrects a claim made in ref. 5, where a nonlinear realization of V was used to describe inhomogeneous systems. The latter realization, however, leads to mathematical difficulties. It is shown in ref. 6 that even for inhomogeneous systems a linear realization of V can be explicitly constructed.

where C is a parallel to the real axis lying above all singularities of the integrand. $\mathscr{R}(z)$ can be expressed formally as

$$\mathscr{R}(z) = (-\mathscr{L} - \mathrm{i}z)^{-1}. \tag{2.14}$$

The corresponding operators for non-interacting systems will be denoted by a superscript ^{0}:

$$\mathscr{U}^0(t) = \exp(t\mathscr{L}^0) \tag{2.15}$$

and

$$\mathscr{R}^0(z) = (-\mathscr{L}^0 - \mathrm{i}z)^{-1}. \tag{2.16}$$

Let us note the following useful relation following from (2.16):

$$\mathscr{L}^0\mathscr{R}^0(z) = -1 - \mathrm{i}z\mathscr{R}^0(z). \tag{2.17}$$

It was shown in ref. 5 that the resolvent can be conveniently represented as follows†:

$$\begin{aligned}\mathscr{R}(z) &= \sum_{n=0}^{\infty} \mathscr{R}^0(z)[\mathscr{E}(z)\,\mathscr{R}^0(z)\,V]^n\,[V + V\mathscr{E}(z)\,\mathscr{R}^0(z)\,C] \\ &\equiv \sum_{n=0}^{\infty} (V + C\mathscr{R}^0\mathscr{E}V)\,\mathscr{R}^0\,(V\mathscr{E}\mathscr{R}^0V)^n\,(V + V\mathscr{E}\mathscr{R}^0C) \\ &+ C\mathscr{R}^0 + C\mathscr{R}^0\mathscr{E}\mathscr{R}^0C,\end{aligned} \tag{2.18}$$

where the irreducible evolution operator $\mathscr{E}(z)$ is defined by either of the following equations

$$\mathscr{E}(z) = \mathscr{L}' + \mathscr{L}'C\mathscr{R}^0(z)\,\mathscr{E}(z), \tag{2.19}$$

$$\mathscr{E}(z) = \mathscr{L}' + \mathscr{E}(z)\,\mathscr{R}^0(z)\,C\mathscr{L}'. \tag{2.20}$$

These equations can be solved by successive iterations. It is also convenient [5] to introduce a special notation for the following important operator:

$$\begin{aligned}\mathscr{G}(z) &\equiv \mathscr{E}(z) - \mathscr{L}' \\ &= \mathscr{L}'C\mathscr{R}^0(z)\,\mathscr{E}(z) = \mathscr{E}(z)\,\mathscr{R}^0(z)\,C\mathscr{L}'.\end{aligned} \tag{2.21}$$

The vacuum-to-vacuum component $V\mathscr{G}(z)\,V$ reduces (in the Fourier representation) to the well-known operator $\Psi(z)$ of PGH.

We note that the right-hand side of eq. (2.18) can be rearranged and combined with eqs. (2.11) and (2.13) as follows:

$$\begin{aligned}\mathfrak{f}(t) = (2\pi)^{-1} \int_C \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}zt}\,[&C\mathscr{R}^0 + C\mathscr{R}^0\mathscr{E}\mathscr{R}^0C \\ &+ \sum_{n=0}^{\infty} (-\mathrm{i}z)^{-n-1}\,(V + C\mathscr{R}^0\mathscr{E}V)(V\mathscr{L}^0 + V\mathscr{E}V)^n \\ &\times (V + V\mathscr{E}\mathscr{R}^0C)]\,\mathfrak{f}(0).\end{aligned} \tag{2.22}$$

† The argument z will often not be written out explicitly.

[It is easily verified that this function indeed satisfies the Liouville equation (2.1)]. This form has the advantage of exhibiting clearly the same kind of singularities as the corresponding expression for the homogeneous systems; the only difference is in the replacement of $\Psi(z) \equiv V\mathscr{G}(z)\, V$ by $V\mathscr{L}^0 + V\mathscr{E}(z)V$ in the right-hand side. We can therefore proceed just as in PGH to define an asymptotic evolution operator $\Sigma(t)$ †.

We *assume that the operators*

$$V\mathscr{E}(z)\, V, \qquad V\mathscr{E}(z)\, \mathscr{R}^0(z)\, C, \qquad C\mathscr{R}^0(z)\, \mathscr{E}(z)\, V,$$

$$C\mathscr{R}^0(z), \qquad C\mathscr{R}^0(z)\, \mathscr{E}(z)\, \mathscr{R}^0(z)\, C$$

are regular functions of z in the neighbourhood of $z = 0$. (2.23)

These basic assumptions have been discussed extensively in the literature. They represent a restriction both on the interaction potential (or $\mathscr{L}'$) and on the initial condition $\mathfrak{f}(0)$ on which these operators ultimately act. Although no simple general criterion can be found at present, it has been checked on simple but important examples that these properties are indeed satisfied in the problems of interest in statistical mechanics. We shall not further comment on this subject.

We now *define* the "asymptotic" part of the distribution function, $\bar{\mathfrak{f}}(t)$, and the "asymptotic" evolution operator $\Sigma(t)$, through the residue at $z = 0$ of the integrand in eq. (2.22) ‡:

$$\bar{\mathfrak{f}}(t) = \Sigma(t)\, \mathfrak{f}(0), \tag{2.24}$$

with

$$\begin{aligned} \Sigma(t) = \lim_{z\to 0} \sum_{n=0}^{\infty} (n!)^{-1}\, (t + \partial)^n\, [V + C\mathscr{R}^0(z)\, \mathscr{E}(z)\, V] \\ \times [V\mathscr{L}^0 + V\mathscr{E}(z)\, V]^n\, [V + V\mathscr{E}(z)\, \mathscr{R}^0(z)\, C]. \end{aligned} \tag{2.25}$$

In the limit $z \to 0$ (here and in all subsequent equations), z is supposed to approach the real axis from above. The abbreviation ∂ represents the operator:

$$\partial \equiv \mathrm{i}(\partial/\partial z). \tag{2.26}$$

In order to illustrate the use of the projection operators we give the explicit decomposition of the operator $\Sigma(t)$ according to the pattern (2.8), as deduced from eq. (2.25)

† We prefer to use the more compact notations $\Sigma(t)$, $\hat{\Sigma}(t)$, $\bar{\mathfrak{f}}(t)$, $\hat{\mathfrak{f}}(t)$, $\boldsymbol{\Pi}$, $\hat{\boldsymbol{\Pi}}$ instead of the corresponding notations $\overset{(0)}{\Sigma}(t)$, $\overset{(\wedge)}{\Sigma}(t)$, $\overset{(0)}{\mathfrak{f}}$, $\overset{(\wedge)}{\mathfrak{f}}$ $\overset{(0)}{\boldsymbol{\Pi}}$, $\overset{(\wedge)}{\boldsymbol{\Pi}}$ of PGH, which are very cumbersome in print.

‡ This definition implies the permutation of the n summation with the z integration.

$$V\Sigma(t)\,V = \lim_{z\to 0} \sum_{n=0}^{\infty} (n!)^{-1}\,(t+\partial)^n\,(V\mathscr{L}^0 + V\mathscr{E}V)^n, \tag{2.27}$$

$$V\Sigma(t)\,C = \lim_{z\to 0} \sum_{n=0}^{\infty} (n!)^{-1}\,(t+\partial)^n\,(V\mathscr{L}^0 + V\mathscr{E}V)^n\,V\mathscr{E}\mathscr{R}^0 C, \tag{2.28}$$

$$C\Sigma(t)\,V = \lim_{z\to 0} \sum_{n=0}^{\infty} (n!)^{-1}\,(t+\partial)^n\,C\mathscr{R}^0\mathscr{E}V\,(V\mathscr{L}^0 + V\mathscr{E}V)^n, \tag{2.29}$$

$$C\Sigma(t)\,C = \lim_{z\to 0} \sum_{n=0}^{\infty} (n!)^{-1}\,(t+\partial)^n\,C\mathscr{R}^0\mathscr{E}V\,(V\mathscr{L}^0 + V\mathscr{E}V)^n\,V\mathscr{E}\mathscr{R}^0 C. \tag{2.30}$$

These expressions correspond to the four operators arranged in a 2×2 matrix form in PGH. The single expression (2.25) is therefore precisely equivalent to the latter matrix.

We now define the "incoherent" part $\hat{\mathfrak{f}}(t)$ simply as the remainder after subtraction of $\bar{\mathfrak{f}}(t)$ from the complete distribution function:

$$\hat{\mathfrak{f}}(t) = \hat{\Sigma}(t)\,\mathfrak{f}(0) = [\mathscr{U}(t) - \Sigma(t)]\,\mathfrak{f}(0) \tag{2.31}$$

The main result of the present paragraph is in the decomposition of the propagator $\mathscr{U}(t)$ of the Liouville equation,

$$\mathscr{U}(t) = \Sigma(t) + \hat{\Sigma}(t), \tag{2.32}$$

and in the corresponding decomposition of $\mathfrak{f}(t)$,

$$\mathfrak{f}(t) = \bar{\mathfrak{f}}(t) + \hat{\mathfrak{f}}(t). \tag{2.33}$$

Let us conclude this paragraph by noting that, in the present derivation, extensive use was made of the (one-sided) Laplace-transformation technique. This implies that $\Sigma(t)$ and $\hat{\Sigma}(t)$ have been obtained for positive values of the time. From here on we shall *extend* these expressions and *define* $\Sigma(t)$ [resp. $\hat{\Sigma}(t)$] as being the right-hand side of eq. (2.25) [resp. (2.31)] *for all values of the time*, positive, zero or negative, $-\infty < t < +\infty$. This extension is a very important step in the theory, which allows us to fully investigate the properties of these operators.

3. *Correlation and vacuum parts of* $\Sigma(t)$. The first, very important property of $\Sigma(t)$ is the simple relation existing between its vacuum and correlation parts. That such a link should exist could be expected already from the similarity of the right sides of eqs. (2.27)–(2.30).

The explicit form, however, does not look very simple because of the presence of the differential operators $\partial/\partial z$. On the contrary, the relation becomes extremely transparent if we go over to a representation as a function of time (rather than of the Laplace variable z). It turns out that many expressions take a much simpler form in the time formalism.

It is a simple matter to show (see appendix 1) that:

$$C\Sigma(t) = \int_0^\infty \mathrm{d}\tau\, C\mathscr{C}(\tau)\, V\Sigma(t-\tau), \tag{3.1}$$

where the "*creation operator*" $C\mathscr{C}(\tau)\, V$ is defined as

$$C\mathscr{C}(\tau)\, V = \int_0^\tau \mathrm{d}\tau'\, C\mathscr{U}^0(\tau-\tau')\, \mathscr{E}(\tau')\, V, \tag{3.2}$$

with†

$$\mathscr{E}(t) = (2\pi)^{-1} \int_c \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}zt}\, \mathscr{E}(z). \tag{3.3}$$

Equation (3.1) is extremely simple and compact‡. It shows that the correlation part $C\Sigma(t)$ can be related to the vacuum part $V\Sigma(t)$ through a "kind of" convolution product with the creation operator* [a true convolution product would involve t rather than ∞ as the upper bound of integration in (3.1)]. One should actually think of (3.1) as a *couple* of equations, involving $C\Sigma(t)\, V$ and $C\Sigma(t)\, C$:

$$C\Sigma(t)\, V = \int_0^\infty \mathrm{d}\tau\, C\mathscr{C}(\tau)\, V\Sigma(t-\tau)\, V, \tag{3.4}$$

$$C\Sigma(t)\, C = \int_0^\infty \mathrm{d}\tau\, C\mathscr{C}(\tau)\, V\Sigma(t-\tau)\, C. \tag{3.5}$$

In this form, we see that two of the four components in the decomposition (2.8) of $\Sigma(t)$ can be expressed in terms of the two others. Let us finally note that, combining eqs. (3.1) and (2.24) we obtain:

$$C\bar{\mathfrak{f}}(t) = \int_0^\infty \mathrm{d}\tau\, C\mathscr{C}(\tau)\, V\bar{\mathfrak{f}}(t-\tau). \tag{3.6}$$

This fundamental equation shows that the correlation component is functionally related to the vacuum. Hence, the $\bar{\mathfrak{f}}$ component of the distribution function automatically satisfies a generalized form of the Bogoliubov ansatz[9].

It is easy to prove the existence of another, independent relation, symmetrical to (3.1):

$$\Sigma(t)\, C = \int_0^\infty \mathrm{d}\tau\, \Sigma(t-\tau)\, V\mathscr{D}(\tau)\, C, \tag{3.7}$$

† No confusion should arise from the use of the same symbol $\mathscr{E}$ for $\mathscr{E}(t)$ and its Laplace transform $\mathscr{E}(z)$. The particular variable involved will either be explicitly indicated, or will be clear from the context.

‡ Eq. (3.1) does not appear in PGH; however, a similar relation was derived by Baus[8].

* We note that $C\mathscr{C}(\tau)\, V$ is identical with the Laplace transform of the operator $\mathscr{C}(z)$ of ref. 3. It should not be mistaken with the operator C of ref. 3, which is much more complicated. The latter operator will appear below [see eq. (4.20)].

where the "*destruction fragment*" $V\mathscr{D}(\tau)\, C$ is defined as

$$V\mathscr{D}(\tau)\, C = \int_0^\tau \mathrm{d}\tau'\, V\mathscr{E}(\tau')\, \mathscr{U}^0(\tau - \tau')\, C. \tag{3.8}$$

Equation (3.7) must again be interpreted as a couple of equations, like (3.4)–(3.5). Combining these four equations (of which only three are independent) we immediately see that the only independent component in the decomposition (2.8) of $\Sigma(t)$ is $V\Sigma(t)\, V$. The three other components can be related functionally to the former.

It is now easily seen [the derivation is quite parallel to that of eq. (3.1)] that this unique independent component obeys a remarkable differential equation:

$$\partial_t V\Sigma(t)\, V = V(\mathscr{L}^0 + \mathscr{L}')\, V\Sigma(t)\, V + \int_0^\infty \mathrm{d}\tau\, V\mathscr{G}(\tau)\, V\Sigma(t-\tau)\, V, \tag{3.9}$$

where $\mathscr{G}(\tau)$ is the inverse Laplace transform of the operator $\mathscr{G}(z)$ defined in (2.21).

It follows from the previous discussion that the *solution of this equation, with the initial condition*

$$V\Sigma(0)\, V = \lim_{z\to 0} \sum_{n=0}^\infty (n!)^{-1}\, \partial^n\, (V\mathscr{L}^0 + V\mathscr{E}(z)\, V)^n, \tag{3.10}$$

completely determines the operator $\Sigma(t)$.

As a consequence of (3.9), (2.24) and (3.7) we derive the equation obeyed by the vacuum component of $\bar{\mathfrak{f}}(t)$:

$$\partial_t V\bar{\mathfrak{f}}(t) = V(\mathscr{L}^0 + \mathscr{L}')\, V\bar{\mathfrak{f}}(t) + \int_0^\infty \mathrm{d}\tau\, V\mathscr{G}(\tau)\, V\bar{\mathfrak{f}}(t-\tau). \tag{3.11}$$

This equation has to be solved with the initial condition:

$$V\bar{\mathfrak{f}}(0) = [V\Sigma(0)\, V + V\Sigma(0)\, C]\, \mathfrak{f}(0). \tag{3.12}$$

The solution of this problem completely determines [through eq. (3.6)] the $\bar{\mathfrak{f}}(t)$ component of the distribution function.

Equation (3.11) is identical to the well-known asymptotic evolution equation derived by Prigogine and Résibois[10]) in the homogeneous case and by Severne[11]) in the inhomogeneous case; it appeared in the general form (3.11) in ref. 5. However, the philosophy of the present approach (and of PGH) is different from the one of refs. 5, 10, 11. Instead of looking at eq. (3.11) as an *approximation* valid under limited circumstances, it now appears as an *exact* equation obeyed by a *part* of the distribution function, *viz.* $\bar{\mathfrak{f}}(t)$. The complementary part, $\hat{\mathfrak{f}}(t)$, is no longer neglected, but rather studied separately.

Let us finally note that the operator $V\Sigma(t)\,V$ also obeys another equation symmetric to (3.9):

$$\partial_t V\Sigma(t)\,V = V\Sigma(t)\,V(\mathscr{L}^0 + \mathscr{L}')\,V + \int_0^\infty \mathrm{d}\tau\, V\Sigma(t-\tau)\,V\mathscr{G}(\tau)\,V. \qquad (3.13)$$

The proof is the same as for (3.9). This equation will be useful in the next section.

4. *Reduction of the kinetic equation.* In PGH it was shown (for homogeneous systems) that the integro-differential equation (3.9) has a remarkable exponential solution. We shall denote it as follows:

$$V\Sigma(t)\,V = [\exp(V\boldsymbol{\Gamma}Vt)]\,V\Sigma(0)\,V, \qquad (4.1)$$

where $V\boldsymbol{\Gamma}V$ is a time-independent operator, having only a V–V component in the representation (2.8)†. It then follows that $V\Sigma(t)\,V$ obeys the differential equation (in time):

$$\partial_t V\Sigma(t)\,V = V\boldsymbol{\Gamma}V\Sigma(t)\,V, \qquad (4.2)$$

and correspondingly, the vacuum part of $\mathfrak{f}(t)$ obeys

$$\partial_t V\mathfrak{f}(t) = V\boldsymbol{\Gamma}V\mathfrak{f}(t). \qquad (4.3)$$

This is the basic *general kinetic equation* obeyed by $V\mathfrak{f}(t)$. Simple particular cases are the Boltzmann equation for dilute gases, the weak-coupling Landau equation, *etc.* Some of the simple examples are worked out in ref. 6.

In order to determine the form of the operator $V\boldsymbol{\Gamma}V$, we substitute (4.1) into (3.9) and obtain the following integral equation[5,8]:

$$V\boldsymbol{\Gamma}V = V(\mathscr{L}^0 + \mathscr{L}')\,V + \int_0^\infty \mathrm{d}\tau\, V\mathscr{G}(\tau)\,V\exp(-\tau V\boldsymbol{\Gamma}V). \qquad (4.4)$$

In order to solve this nonlinear equation, some iterative procedure must be used. By expanding the exponential in the integrand of (4.4), as a power series in τ, Baus[8] obtained the functional relation

$$V\boldsymbol{\Gamma}V = \lim_{z\to 0}\sum_{n=0}^{\infty}(n!)^{-1}\,[\partial^n(V\mathscr{L}^0 + V\mathscr{E}V)](V\boldsymbol{\Gamma}V)^n. \qquad (4.5)$$

Baus showed that eq. (4.5) gives the same result as a more complicated recurrence relation derived earlier by Prigogine and Résibois[10] (in the homogeneous case) and by Severne[11] for inhomogeneous systems. The solution of eq. (4.5) cannot be written in a general form: each successive

† The operator $V\boldsymbol{\Gamma}V$ is denoted by $\Omega\Psi$ in PGH, and by K in ref. 5. We prefer not to use the PGH notation $\Omega\Psi$ because in the general inhomogeneous case this operator is not necessarily factorizable.

iteration must be worked out separately. Introducing the abbreviation $\Psi(z) \equiv V\mathscr{L}^0 + V\mathscr{E}(z)\,V$ the first few terms appear to be:

$$\begin{aligned} V\Gamma V &\equiv \Omega\Psi(0) \\ &= [1 + \Psi'(0) + \tfrac{1}{2}\Psi''(0)\,\Psi(0) + \Psi'(0)\,\Psi'(0) + \ldots]\,\Psi(0), \end{aligned} \tag{4.6}$$

(where $\Psi' \equiv \partial\Psi$, ...). Besides the impossibility of writing out the general term, this equation has another, major disadvantage when applied to inhomogeneous systems ($V\mathscr{L}^0 \neq 0$). If we want to write down the simplest weak-coupling equation, we expand $\Psi(z)$ in powers of the coupling parameter through order 2. Then

$$\Psi^{[2]}(z) = V\mathscr{L}^0 + V\mathscr{L}'V + V\mathscr{G}^{[2]}(z)\,V,$$

and, substituting into (4.6) we obtain, to second order:

$$\begin{aligned} V\Gamma^{[2]}V &= V\mathscr{L}^0 + V\mathscr{L}'V + V\mathscr{G}^{[2]}(0)\,V \\ &\quad + V\mathscr{G}^{[2]\prime}(0)\,V\mathscr{L}^0 + \tfrac{1}{2}V\mathscr{G}^{[2]\prime\prime}(0)\,V\mathscr{L}^0V\mathscr{L}^0 + \ldots. \end{aligned} \tag{4.7}$$

Hence, even for a weakly coupled gas, the kinetic operator is expressed as an *infinite series* of terms of the same order. What is worse is that we do not even know the form of the general term in this series, so we cannot sum it. [Note that this difficulty does not exist for homogeneous systems [see (2.10)]; only the third term in the right side of eq. (4.7) subsists in that case]. Hence eq. (4.6) cannot be consistently applied to the derivation of kinetic equations for inhomogeneous systems.

We now show that a different iteration procedure can be applied for the solution of eq. (4.4). In this way we shall be able to express $V\Gamma V$ in a form which has none of the two previous disadvantages:

a) the general term in the series can be written down explicitly;

b) only a finite number of terms appears in any order of an expansion in powers of the coupling parameter.

The zeroth-order approximation is taken as

$$V\Gamma^{(0)}V = V(\mathscr{L}^0 + \mathscr{L}')\,V. \tag{4.8}$$

The corresponding exponentiated operator is then clearly the *propagation of the Vlassov equation*, which we denote by $V\tilde{\mathscr{U}}(t)\,V$:

$$\exp[tV(\mathscr{L}^0 + \mathscr{L}')\,V] \equiv V\tilde{\mathscr{U}}(t)\,V. \tag{4.9}$$

Substituting this expression into the integral of eq. (4.4) we obtain the first approximation:

$$V\Gamma^{(1)}V = V(\mathscr{L}^0 + \mathscr{L}')\,V + \int_0^\infty \mathrm{d}\tau_2 V\mathscr{G}(\tau_2)\,V\tilde{\mathscr{U}}(-\tau_2)\,V. \tag{4.10}$$

To go further, we calculate the exponentiated form of this operator, using the well-known identity

$$e^{t(A+B)} = e^{tA} + \int_0^t d\theta \, e^{(t-\theta)A} B \, e^{\theta(A+B)}. \tag{4.11}$$

Calling A and B respectively the first and second term of the right-hand side of (4.10), we get, to first order (one single $\mathscr{G}$ factor):

$$[\exp(tV\boldsymbol{\Gamma}V)]^{(1)} = V\tilde{\mathscr{U}}(t)\,V + \int_0^t d\theta V\tilde{\mathscr{U}}(t-\theta)\,V\int_0^\infty d\tau_2 V\mathscr{G}(\tau_2)\,V\tilde{\mathscr{U}}(-\tau_2)\,V\tilde{\mathscr{U}}(\theta)\,V. \tag{4.12}$$

With this result the second iteration of (4.4) takes the form

$$V\boldsymbol{\Gamma}^{(2)}V = V(\mathscr{L}^0 + \mathscr{L}')\,V + \int_0^\infty d\tau_2 V\mathscr{G}(\tau_2)\,V\tilde{\mathscr{U}}(-\tau_2)\,V + \int_0^\infty d\tau_2 \int_0^\infty d\tau_4 \int_0^{-\tau_2} d\tau_3 V\mathscr{G}(\tau_2)\,V\tilde{\mathscr{U}}(-\tau_2-\tau_3)\,V\mathscr{G}(\tau_4)\,V\tilde{\mathscr{U}}(\tau_3-\tau_4). \tag{4.13}$$

This iteration process can be continued systematically in the same way. One will soon be convinced that the successive terms have a very regular structure. The complete formal solution can therefore be written as:

$$V\boldsymbol{\Gamma}V = V(\mathscr{L}^0 + \mathscr{L}')\,V + \sum_{n=1}^{\infty} V\boldsymbol{\Gamma}_n V, \tag{4.14}$$

with

$$\begin{aligned} V\boldsymbol{\Gamma}_n V = {} & \int_0^\infty d\tau_2 \int_0^\infty d\tau_4 \ldots \int_0^\infty d\tau_{2n} \int_0^{\tau_1-\tau_2} d\tau_3 \int_0^{\tau_3-\tau_4} d\tau_5 \ldots \int_0^{\tau_{2n-3}-\tau_{2n-2}} d\tau_{2n-1} \\ & \times V\mathscr{G}(\tau_2)\,V\tilde{\mathscr{U}}(\tau_1-\tau_2-\tau_3)\,V\mathscr{G}(\tau_4)\,V\tilde{\mathscr{U}}(\tau_3-\tau_4-\tau_5) \\ & \times V\mathscr{G}(\tau_6)\,V \ldots V\tilde{\mathscr{U}}(\tau_{2n-3}-\tau_{2n-2}-\tau_{2n-1}) \\ & \times V\mathscr{G}(\tau_{2n})\,V\tilde{\mathscr{U}}(\tau_{2n-1}-\tau_{2n}-\tau_{2n+1})\,V, \end{aligned} \tag{4.15}$$

where one has to set $\tau_1 = \tau_{2n+1} = 0$. We preferred to write these two spurious variables in (4.16) in order to enhance the basic regularity in the limits of integration of the odd τ_i's and in the order of succession of the arguments of the $\tilde{\mathscr{U}}$ factors. We have thus achieved our first goal of deriving a fully explicit expression for $V\boldsymbol{\Gamma}V$.

If we now consider again the example of a weakly coupled system, as in (4.7), we obtain from (4.14)–(4.15), to second order in λ:

$$V\boldsymbol{\Gamma}^{[2]}V = V(\mathscr{L}^0 + \mathscr{L}')\,V + \int_0^\infty d\tau_2 V\mathscr{G}^{[2]}(\tau_2)\,V\mathscr{U}^0(-\tau_2)\,V. \tag{4.16}$$

Hence the infinite series in eq. (4.7) appears here automatically summed in closed form. This is an appreciable advantage, especially in higher orders.

To show that our equations (4.14)–(4.15) reduce to the PGH expression (4.6) in the homogeneous case (2.10), we note that for such systems

$$V\tilde{\mathscr{U}}(t)\,V = V, \qquad \text{(homogeneous systems).} \tag{4.17}$$

Hence, the first few terms of (4.15) are

$$\begin{aligned}
V\Gamma V &= \int_0^\infty \mathrm{d}\tau_2 V\mathscr{G}(\tau_2)\,V \\
&+ \int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_4 \int_0^{-\tau_2} \mathrm{d}\tau_3\, V\mathscr{G}(\tau_2)\,V\mathscr{G}(\tau_4)\,V \\
&+ \int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_4 \int_0^\infty \mathrm{d}\tau_6 \int_0^{-\tau_2} \mathrm{d}\tau_3 \int_0^{\tau_3-\tau_4} \mathrm{d}\tau_5\, V\mathscr{G}(\tau_2)\,V\mathscr{G}(\tau_4)\,V\mathscr{G}(\tau_6) + \dots \\
&= \int_0^\infty \mathrm{d}\tau_2\, V\mathscr{G}(\tau_2)\,V + \int_0^\infty \mathrm{d}\tau_2\,(-\tau_2)\,V\mathscr{G}(\tau_2)\,V \int_0^\infty \mathrm{d}\tau_4 V\mathscr{G}(\tau_4)\,V \\
&+ \int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_4 (\tfrac{1}{2}\tau_2^2 + \tau_2\tau_4)\,V\mathscr{G}(\tau_2)\,V\mathscr{G}(\tau_4)\,V \\
&\times \int_0^\infty \mathrm{d}\tau_6 V\mathscr{G}(\tau_6)\,V + \dots = \Psi(0) + \Psi'(0)\,\Psi(0) \\
&+ \tfrac{1}{2}\Psi''(0)\,\Psi(0)\,\Psi(0) + \Psi'(0)\,\Psi'(0)\,\Psi(0) + \dots,
\end{aligned} \tag{4.18}$$

in agreement with (4.7). [In the last step, use was made of eq. (A.1.3) of appendix 1 and of the fact, already mentioned after eq. (2.21), that

$$V\mathscr{G}(z)\,V = \Psi(z)$$

in the homogeneous case].

It is now easy to see that the representation (4.1) of operator $V\Sigma(t)\,V$ allows us to transform the correlation component (3.1) into a different, remarkable form†:

$$C\Sigma(t) = \int_0^\infty \mathrm{d}\tau C\mathscr{C}(\tau)\,V[\exp(t-\tau)\,V\Gamma V]\,V\Sigma(0)$$

and hence

$$C\Sigma(t) = C\mathbf{C}V\Sigma(t), \tag{4.19}$$

with

$$C\mathbf{C}V = \int_0^\infty \mathrm{d}\tau C\mathscr{C}(\tau)\,V\exp(-\tau V\Gamma V). \tag{4.20}$$

Eq. (4.19) relates the correlation components $C\Sigma(t)$ to the vacuum components $V\Sigma(t)$ evaluated at the *same time* through the action of the time-independent operator $C\mathbf{C}V$. It is the generalization to inhomogeneous

† It is not difficult to show that eq. (4.1) also holds for the component $V\Sigma(t)C$.

systems of a relation due to PGH. Equation (4.20) is, however, much more compact than the corresponding expression derived by C. George[3]). The former reduces to the latter for homogeneous systems, upon expansion of the exponential as a power series in τ, and use of eq. (A.1.3).

Actually, by using the expansion of the exponential in the same way as above, we can get an explicit series expansion for $C\mathsf{C}V$, which is quite useful in practical calculations.

$$C\mathsf{C}V = \sum_{n=1}^{\infty} C\mathsf{C}_n V, \tag{4.21}$$

with

$$\begin{aligned} C\mathsf{C}_n V = & \int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_4 \dots \int_0^\infty \mathrm{d}\tau_{2n} \int_0^{\tau_1-\tau_2} \mathrm{d}\tau_3 \int_0^{\tau_3-\tau_4} \mathrm{d}\tau_5 \dots \int_0^{\tau_{2n-3}-\tau_{2n-2}} \mathrm{d}\tau_{2n-1} \\ & C\mathscr{C}(\tau_2)\, V\tilde{\mathscr{U}}(\tau_1-\tau_2-\tau_3)\, V\mathscr{G}(\tau_4)\, V\tilde{\mathscr{U}}\,(\tau_3-\tau_4-\tau_5) \\ & \times V\mathscr{G}(\tau_6)\, V \dots V\tilde{\mathscr{U}}(\tau_{2n-3}-\tau_{2n-2}-\tau_{2n-1}) \\ & \times V\mathscr{G}(\tau_{2n})\, V\tilde{\mathscr{U}}(\tau_{2n-1}-\tau_{2n}-\tau_{2n+1})\, V; \\ & \tau_1 = \tau_{2n+1} \equiv 0. \end{aligned} \tag{4.22}$$

This expansion is closely similar to (4.15); the only difference is in the replacement of the first factor, $V\mathscr{G}(\tau_2)\,V$, in the integrand by the factor $C\mathscr{C}(\tau_2)\,V$.

We now note that the operator $V\Sigma(t)\,V$ also admits a representation slightly different from (4.1);

$$V\Sigma(t)\,V = V\Sigma(0)\,V\exp(tV\varDelta V). \tag{4.23}$$

The operator $V\varDelta V$ obeys an equation derived by substituting (4.23) into eq. (3.13):

$$V\varDelta V = V(\mathscr{L}^0 + \mathscr{L}')\,V + \int_0^\infty \mathrm{d}\tau \exp(-\tau V\varDelta V)\, V\mathscr{G}(\tau)\,V. \tag{4.24}$$

Going through the same derivation as before, we obtain the following explicit solution

$$V\varDelta V = V(\mathscr{L}^0 + \mathscr{L}')\,V + \sum_{n=1}^{\infty} V\varDelta_n V, \tag{4.25}$$

with

$$\begin{aligned} V\varDelta_n V = & \int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_4 \dots \int_0^\infty \mathrm{d}\tau_{2n} \int_0^{\tau_1-\tau_2} \mathrm{d}\tau_3 \int_0^{\tau_3-\tau_4} \mathrm{d}\tau_5 \dots \int_0^{\tau_{2n-3}-\tau_{2n-2}} \mathrm{d}\tau_{2n-1} \\ & V\tilde{\mathscr{U}}(-\tau_{2n+1}-\tau_{2n}+\tau_{2n-1})\, V\mathscr{G}(\tau_{2n})\, V \\ & \times \tilde{\mathscr{U}}(-\tau_{2n-1}-\tau_{2n-2}+\tau_{2n-3})\, V\mathscr{G}(\tau_{2n-2})\, V \\ & \dots V\mathscr{G}(\tau_6)\, V\tilde{\mathscr{U}}(-\tau_5-\tau_4+\tau_3)\, V\mathscr{G}(\tau_4)\, V \\ & \times \tilde{\mathscr{U}}(-\tau_3-\tau_2+\tau_1)\, V\mathscr{G}(\tau_2)\, V, \end{aligned} \tag{4.26}$$

where one sets again $\tau_1 = \tau_{2n+1} = 0$. One immediately notes the nice symmetry existing between the operators $\boldsymbol{\Gamma}$ and $\boldsymbol{\Delta}$.

Combining now eq. (4.23) with (3.7) we obtain

$$\Sigma(t)\, C = \int_0^\infty \mathrm{d}\tau\, \Sigma(0) \exp[(t-\tau)\, V\boldsymbol{\Delta}V]\, V\mathscr{D}(\tau)\, C$$

and hence

$$\Sigma(t)\, C = \Sigma(t)\, V\mathsf{D}C, \tag{4.27}$$

with

$$V\mathsf{D}C = \int_0^\infty \mathrm{d}\tau \exp(-\tau V\boldsymbol{\Delta}V)\, V\mathscr{D}(\tau)\, C. \tag{4.28}$$

Again, the operator $V\mathsf{D}C$ can be expanded in the same way as the other operators:

$$V\mathsf{D}C = \sum_{n=1}^{\infty} V\mathsf{D}_n C. \tag{4.29}$$

The general term $V\mathsf{D}_n C$ looks exactly like $V\boldsymbol{\Delta}_n C$, eq. (4.26), the only difference being the replacement of the last factor $V\mathscr{G}(\tau_2)\, V$ by $V\mathscr{D}(\tau_2)\, C$.

Eq. (4.27) completes the final reduction of the four components of $\Sigma(t)$ to the single component $V\Sigma(t)\, V$. This reduction is summarized by the following relations:

$$V\Sigma(t)\, C = V\Sigma(t)\, V\mathsf{D}C, \tag{4.30}$$

$$C\Sigma(t)\, V = C\mathsf{C}V\Sigma(t)\, V, \tag{4.31}$$

$$C\Sigma(t)\, C = C\mathsf{C}V\Sigma(t)\, V\mathsf{D}C, \tag{4.32}$$

$$V\Sigma(t)\, V = \exp(tV\boldsymbol{\Gamma}V)\, V\Sigma(0)\, V = V\Sigma(0)\, V \exp(tV\boldsymbol{\Delta}V). \tag{4.33}$$

It also follows that the correlation component $C\bar{\mathfrak{f}}(t)$ is related to the vacuum component $V\bar{\mathfrak{f}}(t)$ through a time-independent functional:

$$C\bar{\mathfrak{f}}(t) = C\mathsf{C}V\bar{\mathfrak{f}}(t). \tag{4.34}$$

These relations completely characterize the $\Sigma(t)$ part of the propagator and the $\bar{\mathfrak{f}}(t)$ part of the distribution function. They may be taken as a *definition* of these concepts.

5. *Group properties of $\Sigma(t)$ and of $\hat{\Sigma}(t)$.* We now establish a certain number of global properties of the operator $\Sigma(t)$. In this context a very special role is played by the operators $\Sigma(0)$ and $\hat{\Sigma}(0)$; we shall therefore denote them by particular symbols:

$$\Sigma(0) = \boldsymbol{\Pi}, \qquad \hat{\Sigma}(0) = \hat{\boldsymbol{\Pi}}. \tag{5.1}$$

Two fundamental theorems are at the basis of this theory. The first one states that $\boldsymbol{\Pi}$ *is an idempotent operator*:

$$\boldsymbol{\Pi}^2 = \boldsymbol{\Pi}. \tag{5.2}$$

This property was conjectured by Haubold, then proved by a quite involved perturbational method by Turner[12]). Recently, a very straightforward and simple proof has been given by George[13]). His proof only needs trivial changes to be carried over to the general case treated here (see appendix 2).

The second fundamental theorem gives the *intrinsic relation between the Liouville propagator* $\mathscr{U}(t)$ *and the operator* $\Sigma(t)$:

$$\Sigma(t) = \mathscr{U}(t)\, \boldsymbol{\Pi} = \boldsymbol{\Pi}\mathscr{U}(t). \tag{5.3}$$

As a particular case, we obtain from a consideration of infinitesimal time translations:

$$\mathscr{L}\boldsymbol{\Pi} = \boldsymbol{\Pi}\mathscr{L}. \tag{5.4}$$

This theorem was conjectured by Turner and Clavin, and then proved in ref. 15. A simpler and more general proof is given in appendix 3.

We now develop the consequences of these two theorems. The first important property is the following: $\Sigma(t)$ *is a representation of the one-parameter continuous group of translations, i.e.*

$$\Sigma(t_1)\, \Sigma(t_2) = \Sigma(t_1 + t_2), \qquad \forall\, t_1, t_2 \gtrless 0, \tag{5.5}$$

$$\Sigma(t)\, \boldsymbol{\Pi} = \boldsymbol{\Pi}\Sigma(t) = \Sigma(t), \qquad \forall\, t, \tag{5.6}$$

$$\Sigma(t)\, \Sigma(-t) = \Sigma(-t)\, \Sigma(t) = \boldsymbol{\Pi}. \tag{5.7}$$

The group properties (5.5)–(5.7) appear now as almost trivial consequences of the relations (5.3) and (5.4). Indeed, we know that $\mathscr{U}(t)$ is a representation of the group of translations, in which the unit element is the identity transformation: $\mathscr{U}(0) = I$, *i.e.*

$$\mathscr{U}(t_1)\, \mathscr{U}(t_2) = \mathscr{U}(t_1 + t_2), \tag{5.8}$$

$$\mathscr{U}(t)\, I = I\mathscr{U}(t) = \mathscr{U}(t), \tag{5.9}$$

$$\mathscr{U}(t)\, \mathscr{U}(-t) = \mathscr{U}(-t)\, \mathscr{U}(t) = I. \tag{5.10}$$

Applying now eqs. (5.2), (5.3) we easily obtain:

$$\Sigma(t_1)\, \Sigma(t_2) = \mathscr{U}(t_1)\, \boldsymbol{\Pi}\mathscr{U}(t_2)\, \boldsymbol{\Pi} = \mathscr{U}(t_1)\, \mathscr{U}(t_2)\, \boldsymbol{\Pi}^2$$
$$= \mathscr{U}(t_1 + t_2)\, \boldsymbol{\Pi} = \Sigma(t_1 + t_2).$$

Eqs. (5.6) and (5.7) are proved in the same elementary way.

In the PGH theory, $\Sigma(t)$ was only shown to have the *semi-group property*, *i.e.* properties (5.5) and (5.6) for *positive* values of t only. The reason for this apparent restriction is the extensive use made in the PGH theory of the

Laplace transformation method. In our formulation (which reduces to PGH for homogeneous systems) it is clear from the very first steps, that an extension of the definition (2.22) to $t < 0$ is not only natural but absolutely necessary. Actually, we may adopt eqs. (4.30)–(4.33) as an equivalent definition of $\Sigma(t)$. This definition, completed by eqs. (4.15) and (4.16) makes no more any use of the Laplace transformation. It clearly has a meaning only if all the operators are defined for all values of t, positive and negative. It is therefore quite gratifying that the PGH semi-group can be naturally extended into a group if these definitions are used. This extension is particularly important in a relativistic theory.

The peculiar feature of the representation $\Sigma(t)$ of the group of translations is in the fact that the unit element $\boldsymbol{\Pi}$ of the group does *not* correspond to the identity transformation $\boldsymbol{I}$ [as happens in the representation $\mathscr{U}(t)$]. It then follows from the group axioms that $\boldsymbol{\Pi}$ must be an *idempotent operator, i.e.* it must satisfy eq. (5.2). For this reason $\boldsymbol{\Pi}$ can be called a *projector* (although the warning in the footnote of p. 479 applies to $\boldsymbol{\Pi}$ as well). Combining eqs. (2.32) (for $t = 0$) and (5.2) we can easily see that the operators $\boldsymbol{\Pi}$, $\hat{\boldsymbol{\Pi}}$ provide a decomposition of the identity, having the following properties:

$$\boldsymbol{\Pi} + \hat{\boldsymbol{\Pi}} = \boldsymbol{I} \tag{5.11}$$

$$\boldsymbol{\Pi}^2 = \boldsymbol{\Pi}, \qquad \hat{\boldsymbol{\Pi}}^2 = \hat{\boldsymbol{\Pi}}, \tag{5.12}$$

$$\boldsymbol{\Pi}\hat{\boldsymbol{\Pi}} = \hat{\boldsymbol{\Pi}}\boldsymbol{\Pi} = 0. \tag{5.13}$$

The analogy between these equations and (2.4)–(2.6) is striking. The analogy goes even further if we compare eqs. (5.4) and (2.7). We have introduced at this stage two different systems of projectors acting on the distribution function space. It is interesting to note that these two systems of projectors merge into each other as the interaction strength goes to zero:

$$\boldsymbol{\Pi} \to V, \qquad \hat{\boldsymbol{\Pi}} \to C \qquad \text{(no interactions)}. \tag{5.14}$$

This is easily seen from eq. (2.25), noting that $\mathscr{E}(z) \equiv 0$ when $\mathscr{L}' = 0$ [see eq. (2.19)].

It is now an elementary matter to show that $\hat{\Sigma}(t)$ has quite similar group properties as $\Sigma(t)$. Indeed, from eqs. (2.32), (5.3) and (5.11) we obtain

$$\hat{\Sigma}(t) = \mathscr{U}(t)\,\hat{\boldsymbol{\Pi}} = \hat{\boldsymbol{\Pi}}\mathscr{U}(t). \tag{5.15}$$

This relation, together with (5.12) then yields another representation of the one-parameter group of translations:

$$\hat{\Sigma}(t_1)\,\hat{\Sigma}(t_2) = \hat{\Sigma}(t_1 + t_2), \tag{5.16}$$

$$\hat{\Sigma}(t)\,\hat{\boldsymbol{\Pi}} = \hat{\boldsymbol{\Pi}}\hat{\Sigma}(t) = \hat{\Sigma}(t), \tag{5.17}$$

$$\hat{\Sigma}(t)\,\hat{\Sigma}(-t) = \hat{\boldsymbol{\Pi}}. \tag{5.18}$$

Finally, combining eqs. (5.3), (5.15) and (5.13) we obtain

$$\Sigma(t_1)\,\hat{\Sigma}(t_2) = \hat{\Sigma}(t_1)\,\Sigma(t_2) = 0. \tag{5.19}$$

The decomposition (5.11) defines a separation of the space of distribution functions into two subspaces. Any element of this space can therefore be split into two components by means of the projections $\boldsymbol{\Pi}$, $\hat{\boldsymbol{\Pi}}$. Let us consider an arbitrary element $\mathfrak{f}(0)$ of this space, representing the state of the system at some fixed time which may be called $t = 0$. We then have the decomposition

$$\mathfrak{f}(0) = \boldsymbol{\Pi}\mathfrak{f}(0) + \hat{\boldsymbol{\Pi}}\mathfrak{f}(0). \tag{5.20}$$

Under the effect of the motion, the element $\mathfrak{f}(0)$ is transformed into another element of the functional space, $\mathfrak{f}(t) = \mathscr{U}(t)\,\mathfrak{f}(0)$. In other words $\mathscr{U}(t)$ is a group of automorphisms, transforming the space of distribution functions into itself. Splitting now $\mathfrak{f}(t)$ into $\boldsymbol{\Pi}$ and $\hat{\boldsymbol{\Pi}}$ components, we obtain

$$\mathfrak{f}(t) = \boldsymbol{\Pi}\mathfrak{f}(t) + \hat{\boldsymbol{\Pi}}\mathfrak{f}(t) = \boldsymbol{\Pi}\mathscr{U}(t)\,\mathfrak{f}(0) + \hat{\boldsymbol{\Pi}}\mathscr{U}(t)\,\mathfrak{f}(0). \tag{5.21}$$

It then first follows from (5.3) and (5.15) that in the decomposition (2.33), we may identify

$$\bar{\mathfrak{f}}(t) = \boldsymbol{\Pi}\mathfrak{f}(t), \qquad \hat{\mathfrak{f}}(t) = \hat{\boldsymbol{\Pi}}\mathfrak{f}(t). \tag{5.22}$$

In other words, we may now define the components $\bar{\mathfrak{f}}(t)$, $\hat{\mathfrak{f}}(t)$ by means of a *time-independent projection operator* acting on the distribution function evaluated at the *same time t*. This is to be contrasted to our initial definitions (2.24), (2.31) in terms of time-dependent operators $\Sigma(t)$, $\hat{\Sigma}(t)$ acting on the function $\mathfrak{f}$ at time 0. The new point of view (5.22) gives an intrinsic geometrical meaning to the decomposition.

More important still is the following other consequence of (5.3) and (5.15):

$$\boldsymbol{\Pi}\mathfrak{f}(t) = \mathscr{U}(t)\,\boldsymbol{\Pi}\mathfrak{f}(0), \tag{5.23}$$

$$\hat{\boldsymbol{\Pi}}\mathfrak{f}(t) = \mathscr{U}(t)\,\hat{\boldsymbol{\Pi}}\mathfrak{f}(0). \tag{5.24}$$

These equations mean that the function $\boldsymbol{\Pi}\mathfrak{f}(t)$ is the result of the exact time evolution of $\boldsymbol{\Pi}\mathfrak{f}(0)$ [a similar statement holds for $\hat{\boldsymbol{\Pi}}\mathfrak{f}(t)$]. As time proceeds, there is no mixing between the two components; they evolve independently of each other. In other words, the subspaces $\{\boldsymbol{\Pi}\mathfrak{f}\}$ and $\{\hat{\boldsymbol{\Pi}}\mathfrak{f}\}$ are *invariant manifolds* under the motion; they transform into themselves under the automorphisms $\mathscr{U}(t)$ of the complete space $\{\mathfrak{f}\}$. We can formulate this property in the language of group theory. From the existence of two invariant manifolds in the "carrier space" $\{\mathfrak{f}\}$ of the representation $\mathscr{U}(t)$ of the group of time translations we may infer that *the representation $\mathscr{U}(t)$ is reducible*. This reducibility property of $\mathscr{U}(t)$, valid even for the case of infinite systems of interacting degrees of freedom, is perhaps the most important and highly nontrivial feature of the PGH theory. It could not be predicted by any elementary or semi-intuitive argument. It throws a new light on the deep structure of the evolution process.

Let us now concentrate upon the component $\boldsymbol{\Pi}\mathfrak{f}(t)$. (The component $\hat{\boldsymbol{\Pi}}\mathfrak{f}(t)$ will be considered in sec. 6.) It follows from (5.23) that it obeys the following equation

$$\partial_t \boldsymbol{\Pi}\mathfrak{f}(t) = \mathscr{L}\boldsymbol{\Pi}\mathfrak{f}(t). \tag{5.25}$$

Hence $\boldsymbol{\Pi}\mathfrak{f}(t) \equiv \bar{\mathfrak{f}}(t)$ is a solution of the Liouville equation. However, there is an important difference between (5.25) and the initial Liouville equation (2.1). In the complete space $\{\mathfrak{f}\}$ the vacuum and the correlation component of $\mathfrak{f}$ are quite independent functions. This is not so in the $\{\boldsymbol{\Pi}\mathfrak{f}\}$ subspace, as we know that the correlation component $C\boldsymbol{\Pi}\mathfrak{f}(t)$ must be functionally related to the vacuum $V\boldsymbol{\Pi}\mathfrak{f}(t)$ through eq. (4.34) *at all times*. Hence eq. (4.34) is a *constraint* which has to be added to eq. (5.25) in order to define the motion completely in the subspace $\{\boldsymbol{\Pi}\mathfrak{f}\}$. The only independent part being the vacuum component of $\boldsymbol{\Pi}\mathfrak{f}$, we now derive an equation for the latter, taking the constraint explicitly into account:

$$\begin{aligned}\partial_t V\boldsymbol{\Pi}\mathfrak{f}(t) &= V\mathscr{L}\boldsymbol{\Pi}\mathfrak{f}(t) = V\mathscr{L}V\boldsymbol{\Pi}\mathfrak{f}(t) + V\mathscr{L}C\boldsymbol{\Pi}\mathfrak{f}(t)\\ &= V\mathscr{L}V\boldsymbol{\Pi}\mathfrak{f}(t) + V\mathscr{L}C\mathsf{C}V\boldsymbol{\Pi}\mathfrak{f}(t),\end{aligned}$$

or, using also eq. (2.7),

$$\partial_t V\boldsymbol{\Pi}\mathfrak{f}(t) = V(\mathscr{L}^0 + \mathscr{L}')\, V\boldsymbol{\Pi}\mathfrak{f}(t) + V\mathscr{L}'C\mathsf{C}V\boldsymbol{\Pi}\mathfrak{f}(t).$$

We therefore obtained a closed equation for the component $V\boldsymbol{\Pi}\mathfrak{f}(t)$. This equation must be identical with eq. (4.3); hence we have a new compact expression for the kinetic operator $V\boldsymbol{\Gamma}V$:

$$V\boldsymbol{\Gamma}V = V(\mathscr{L}^0 + \mathscr{L}')\, V + V\mathscr{L}'C\mathsf{C}V. \tag{5.26}$$

One should not forget however that $C\mathsf{C}V$ is itself expressed in terms of $\boldsymbol{\Gamma}$; hence (5.26) is really an implicit non-linear equation for $\boldsymbol{\Gamma}$. By using eqs. (4.20). (3.2) and (2.21) one easily sees that

$$V\mathscr{L}'C\mathsf{C}V = \int_0^\infty \mathrm{d}\tau\, V\mathscr{G}(\tau)\, V \exp(-\tau V\boldsymbol{\Gamma}V), \tag{5.27}$$

which establishes the equivalence of eqs. (5.26) and (4.4).

6. *Properties of* $\hat{\Sigma}(t)$. We now turn our attention to the second part of the propagator $\mathscr{U}(t)$ and show that it has properties analogous, though complementary, to those of $\Sigma(t)$.

Using eqs. (5.13) and (4.30) we obtain

$$0 = V\boldsymbol{\Pi}\hat{\boldsymbol{\Pi}} = (V\boldsymbol{\Pi}V + V\boldsymbol{\Pi}C)\,\hat{\boldsymbol{\Pi}} = V\boldsymbol{\Pi}V(V + V\mathsf{D}C)\,\hat{\boldsymbol{\Pi}}$$

and therefore (as $V\boldsymbol{\Pi}V \neq 0$):

$$V\hat{\boldsymbol{\Pi}} = -V\mathsf{D}C\hat{\boldsymbol{\Pi}}, \tag{6.1}$$

similarly

$$\hat{\boldsymbol{\Pi}}V = -\hat{\boldsymbol{\Pi}}C\mathsf{C}V. \tag{6.2}$$

Using these equations together with (5.15) we obtain three relations analogous to (4.30)–(4.32):

$$V\hat{\Sigma}(t)\, C = -V\mathsf{D}C\hat{\Sigma}(t)\, C, \tag{6.3}$$

$$C\hat{\Sigma}(t)\, V = -C\hat{\Sigma}(t)\, C\mathsf{C}V, \tag{6.4}$$

$$V\hat{\Sigma}(t)\, V = V\mathsf{D}C\hat{\Sigma}(t)\, C\mathsf{C}V. \tag{6.5}$$

Hence, in the decomposition (2.8) of $\hat{\Sigma}(t)$ we can again eliminate three of the components in terms of the fourth. However, unlike $\Sigma(t)$, the independent component is now the correlation–correlation part, $C\hat{\Sigma}(t)\, C$. It also follows that the vacuum part of $\hat{\mathfrak{f}}(t)$ is a functional of the correlation, a situation quite opposite to $\bar{\mathfrak{f}}(t)$:

$$V\hat{\mathfrak{f}}(t) = -V\mathsf{D}C\hat{\mathfrak{f}}(t). \tag{6.6}$$

This relation was derived (for homogeneous systems) in PGH, but eqs. (6.3)–(6.5) do not appear in the latter theory.

To complete the picture, we now show that the independent component can be represented as

$$C\hat{\Sigma}(t)\, C = \exp(tC\hat{\boldsymbol{\Lambda}}C)\, C\hat{\boldsymbol{\Pi}}C = C\hat{\boldsymbol{\Pi}}C\exp(tC\hat{\boldsymbol{\Gamma}}C), \tag{6.7}$$

in complete analogy with (4.33). To derive an expression for $C\hat{\boldsymbol{\Lambda}}C$, we start from

$$\begin{aligned}\partial_t C\hat{\Sigma}(t)\, C &= \partial_t C\mathscr{U}(t)\, \hat{\boldsymbol{\Pi}}C = C\mathscr{L}\hat{\Sigma}(t)\, C \\ &= C\mathscr{L}V\hat{\Sigma}(t)\, C + C\mathscr{L}C\hat{\Sigma}(t)\, C \\ &= -C\mathscr{L}V\mathsf{D}C\hat{\Sigma}(t)\, C + C\mathscr{L}C\hat{\Sigma}(t)\, C.\end{aligned}$$

This equation can therefore be rewritten as

$$\partial_t C\hat{\Sigma}(t)\, C = C\hat{\boldsymbol{\Lambda}}C\hat{\Sigma}(t)\, C \tag{6.8}$$

and provides the definition of $C\hat{\boldsymbol{\Lambda}}C$ as:

$$C\hat{\boldsymbol{\Lambda}}C = C(\mathscr{L}^0 + \mathscr{L}')\, C - C\mathscr{L}'V\mathsf{D}C. \tag{6.9}$$

This is the generalization of the corresponding equation (4.13) of ref. 1, where $\hat{\boldsymbol{\Lambda}}$ is denoted by the letter $\mathscr{L}$. Eq. (6.9) looks quite similar to (5.25); however, it is intrinsically much simpler. Indeed, this relation explicitly expresses $C\hat{\boldsymbol{\Lambda}}C$ in terms of the operator $V\boldsymbol{\Lambda}V$, which has been determined earlier; another equivalent form, analogous to eq. (4.4) is:

$$C\hat{\Delta}C = C(\mathscr{L}^0 + \mathscr{L}')\,C - \int_0^\infty \mathrm{d}\tau\, C\mathscr{L}'V \exp(-\tau V\Delta V)\, V\mathscr{D}(\tau)\, C. \qquad (6.10)$$

Using the methods of sec. 4, it is a very easy matter to expand the exponential in this equation and obtain an explicit expansion for the operator $C\hat{\Delta}C$:

$$C\hat{\Delta}C = C(\mathscr{L}^0 + \mathscr{L}')\,C - \sum_{n=1}^{\infty} C\hat{\Delta}_n C, \qquad (6.11)$$

with

$$\begin{aligned} C\hat{\Delta}_n C = &\int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_4 \ldots \int_0^\infty \mathrm{d}\tau_{2n} \int_0^{\tau_1-\tau_2} \mathrm{d}\tau_3 \int_0^{\tau_3-\tau_4} \mathrm{d}\tau_5 \ldots \int_0^{\tau_{2n-3}-\tau_{2n-2}} \mathrm{d}\tau_{2n-1} \\ & C\mathscr{L}'V\tilde{\mathscr{U}}(-\tau_{2n+1} - \tau_{2n} + \tau_{2n-1})\, V\mathscr{G}(\tau_{2n})\, V \\ & \times \tilde{\mathscr{U}}(-\tau_{2n-1} - \tau_{2n-2} + \tau_{2n-3})\, V\mathscr{G}(\tau_{2n-2})\, V \ldots V\mathscr{G}(\tau_6)\, V \\ & \times \tilde{\mathscr{U}}(-\tau_5 - \tau_4 + \tau_3)\, V\mathscr{G}(\tau_4)\, V\tilde{\mathscr{U}}(-\tau_3 - \tau_2 - \tau_1)\, V\mathscr{D}(\tau_2)\, C. \end{aligned} \qquad (6.12)$$

In an exactly analogous fashion it is shown that the right-hand operator $C\hat{\Gamma}C$ of eq. (6.7) is related to the operator $V\Gamma V$ through the equation

$$C\hat{\Gamma}C = C(\mathscr{L}^0 + \mathscr{L}')\,C - \int_0^\infty \mathrm{d}\tau C\mathscr{C}(\tau)\, V \exp(-\tau V\Gamma V)\, V\mathscr{L}'C, \qquad (6.13)$$

which has the solution

$$C\hat{\Gamma}C = C(\mathscr{L}^0 + \mathscr{L}')\,C - \sum_{n=1}^{\infty} C\hat{\Gamma}_n C, \qquad (6.14)$$

with

$$\begin{aligned} C\hat{\Gamma}_n C = &\int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_4 \\ & \ldots \int_0^\infty \mathrm{d}\tau_{2n} \int_0^{\tau_1-\tau_2} \mathrm{d}\tau_3 \int_0^{\tau_3-\tau_4} \mathrm{d}\tau_5 \ldots \int_0^{\tau_{2n-3}-\tau_{2n-2}} \mathrm{d}\tau_{2n-1} \\ & C\mathscr{C}(\tau_2)\, V\tilde{\mathscr{U}}(\tau_1 - \tau_2 - \tau_3)\, V\mathscr{G}(\tau_4)\, V\tilde{\mathscr{U}}(\tau_3 - \tau_4 - \tau_5) \\ & \times V\mathscr{G}(\tau_6)\, V \ldots V\tilde{\mathscr{U}}(\tau_{2n-3} - \tau_{2n-2} - \tau_{2n-1}) \\ & \times V\mathscr{G}(\tau_{2n})\, V\tilde{\mathscr{U}}(\tau_{2n-1} - \tau_{2n} - \tau_{2n+1})\, V\mathscr{L}'C. \end{aligned} \qquad (6.15)$$

It is hardly necessary to stress the close similarity between eqs. (6.12) and (6.15) and the corresponding eqs. (4.26) and (4.15). The only difference between corresponding equations is in the terminal factors of the integrand: replacement of a factor $V\mathscr{G}(\tau_2)\, V$ by a factor $V\mathscr{D}(\tau_2)\, C$ (resp. $C\mathscr{C}(\tau_2)\, V$) and adjunction of a factor $C\mathscr{L}'V$ (resp. $V\mathscr{L}'C$) at the other end. Everything else is identical. This beautiful symmetry between the $\boldsymbol{\Pi}$ and $\hat{\boldsymbol{\Pi}}$ subspaces is clearly enhanced in the present formulation.

7. *Equilibrium properties.* We now show that the equilibrium distribution function has very remarkable properties in connection with the $\boldsymbol{\Pi}$–$\hat{\boldsymbol{\Pi}}$ concept.

An *equilibrium distribution function* $\mathfrak{f}_e$ will be very broadly defined as a stationary solution of the Liouville equation (2.1):

$$\mathscr{L}\mathfrak{f}_e = 0. \tag{7.1}$$

This equation can also be written [see (2.2) and (2.4)] as

$$\mathscr{L}^0\mathfrak{f}_e + \mathscr{L}'V\mathfrak{f}_e + \mathscr{L}'C\mathfrak{f}_e = 0. \tag{7.2}$$

We now evaluate the function $\mathscr{E}(z)\,V\mathfrak{f}_e$; using eq. (2.20), then (7.2) and finally (2.17) we obtain

$$\begin{aligned}\mathscr{E}V\mathfrak{f}_e &= \mathscr{L}'V\mathfrak{f}_e + \mathscr{E}C\mathscr{R}^0\mathscr{L}'V\mathfrak{f}_e\\ &= -\mathscr{L}^0\mathfrak{f}_e - \mathscr{L}'C\mathfrak{f}_e - \mathscr{E}C\mathscr{R}^0\mathscr{L}^0\mathfrak{f}_e - \mathscr{E}C\mathscr{R}^0\mathscr{L}'C\mathfrak{f}_e\\ &= -\mathscr{L}^0\mathfrak{f}_e - \mathscr{L}'C\mathfrak{f}_e + \mathscr{E}C\mathfrak{f}_e + \mathrm{i}z\mathscr{E}\mathscr{R}^0C\mathfrak{f}_e - \mathscr{E}C\mathfrak{f}_e + \mathscr{L}'C\mathfrak{f}_e,\end{aligned}$$

hence

$$\mathscr{E}(z)\,V\mathfrak{f}_e = -\mathscr{L}^0\mathfrak{f}_e + \mathrm{i}z\mathscr{E}(z)\,\mathscr{R}^0(z)\,C\mathfrak{f}_e. \tag{7.3}$$

From this equation we obtain a first important result by projecting both sides on the vacuum and taking the limit $z \to 0$:

$$\begin{aligned}&\lim_{z\to 0} V\mathscr{E}(z)\,V\mathfrak{f}_e\\ &\quad = -V\mathscr{L}^0\mathfrak{f}_e + \lim_{z\to 0}(\mathrm{i}z)\,V\mathscr{E}(z)\,\mathscr{R}^0(z)\,C\mathfrak{f}_e = -V\mathscr{L}^0\mathfrak{f}_e,\end{aligned}$$

where the regularity assumption (2.23) has been used. We thus obtain

$$[V\mathscr{L}^0 + \lim_{z\to 0} V\mathscr{E}(z)\,V]\,V\mathfrak{f}_e = 0. \tag{7.4}$$

This is the generalized form of the PGH result: $\Psi(0)\,\rho_0(H_0) = 0$. It is translated into time-dependent operators as follows [see eq. (A.1.3)]

$$V(\mathscr{L}^0 + \mathscr{L}')\,V\mathfrak{f}_e + \int_0^\infty \mathrm{d}\tau\, V\mathscr{G}(\tau)\,V\mathfrak{f}_e = 0. \tag{7.5}$$

A second important lemma is obtained from (7.2) by multiplying both sides by $C\mathscr{R}^0(z)$:

$$\begin{aligned}C\mathscr{R}^0\mathscr{E}V\mathfrak{f}_e &= -C\mathscr{R}^0\mathscr{L}^0\mathfrak{f}_e + \mathrm{i}z\,C\mathscr{R}^0\mathscr{E}C\mathscr{R}^0\mathfrak{f}_e\\ &= C\mathfrak{f}_e + \mathrm{i}z\,C\mathscr{R}^0\mathfrak{f}_e + \mathrm{i}z\,C\mathscr{R}^0\mathscr{E}\mathscr{R}^0C\mathfrak{f}_e,\end{aligned}$$

where we used (2.17). From (2.23), it follows that

$$\lim_{z\to 0} C\mathscr{R}^0(z)\,\mathscr{E}(z)\,V\mathfrak{f}_e = C\mathfrak{f}_e. \tag{7.6}$$

This equation is equivalent [see (3.2)] to

$$C\mathfrak{f}_e = \int_0^\infty d\tau\, C\mathscr{C}(\tau)\, V\mathfrak{f}_e. \tag{7.7}$$

Recalling that $\mathfrak{f}_e$ is a time-independent function, we see that the correlation and vacuum parts of $\mathfrak{f}_e$ are related to each other as the corresponding components of an $\bar{\mathfrak{f}}$ function, see eq. (3.6). We may derive a couple of other results which strengthen this statement. We note first the property

$$V\boldsymbol{\Gamma}V\mathfrak{f}_e = 0. \tag{7.8}$$

This property is very easily proved by using eq. (4.4):

$$V\boldsymbol{\Gamma}V\mathfrak{f}_e = V(\mathscr{L}^0 + \mathscr{L}')\, V\mathfrak{f}_e + \int_0^\infty d\tau\, V\mathscr{G}(\tau)\, V \exp(-\tau V\boldsymbol{\Gamma}V)\, \mathfrak{f}_e.$$

By substituting (7.8) as an ansatz, this equation reduces to (7.5) which has been proved above. From eqs. (7.8) and (4.21) follows then

$$C\boldsymbol{C}V\mathfrak{f}_e = \int_0^\infty d\tau\, C\mathscr{C}(\tau)\, V \exp(-\tau V\boldsymbol{\Gamma}V)\, \mathfrak{f}_e = \int_0^\infty d\tau\, C\mathscr{C}(\tau)\, V\mathfrak{f}_e,$$

and hence, from (7.7):

$$C\mathfrak{f}_e = C\boldsymbol{C}V\mathfrak{f}_e. \tag{7.9}$$

This is the characteristic relation defining the subspace $\boldsymbol{\Pi}$, as we saw in eq. (4.34). We may therefore also write this equation as

$$\boldsymbol{\Pi}\mathfrak{f}_e = \mathfrak{f}_e, \tag{7.10}$$

$$\hat{\boldsymbol{\Pi}}\mathfrak{f}_e = 0. \tag{7.11}$$

The set of stationary distribution functions lies entirely in the $\boldsymbol{\Pi}$ subspace.

We may now summarize all the results of our discussion as follows. Starting from an arbitrary initial state, the $\boldsymbol{\Pi}$ and $\hat{\boldsymbol{\Pi}}$ parts of the distribution function evolve quite independently. The $\boldsymbol{\Pi}$ component obeys a generalized kinetic equation (4.2) which drives that component to equilibrium; on the other hand the $\hat{\boldsymbol{\Pi}}$ component has an evolution which could be compared to a phase-mixing process. In the end stage, the $\hat{\boldsymbol{\Pi}}$ component vanishes. We refer the reader to refs. 1 and 2 for a further discussion of these concepts. We may, however, note that a detailed understanding of the evolution process will require further work. The results of the PGH theory and of the present paper clearly show the way for these future developments.

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APPENDIX 1

Proof of eq. (3.1). From eq. (2.25) we obtain

$$C\Sigma(t) = \lim_{z\to 0} \sum_{n=0}^{\infty} (n!)^{-1} (t+\partial)^n\, C\mathscr{R}^0\mathscr{E}V\,(V\mathscr{L}^0 + V\mathscr{E}V)^n\,(V + V\mathscr{E}\mathscr{R}^0C)$$

$$= \lim_{z\to 0} \sum_{n=0}^{\infty} \sum_{p=0}^{n} \sum_{q=0}^{n-p} [t^p/p!q!\,(n-p-q)!](\partial^q C\mathscr{R}^0\mathscr{E}V)$$

$$[\partial^{n-p-q}(V\mathscr{L}^0 + V\mathscr{E}V)^n\,(V + V\mathscr{E}\mathscr{R}^0C)]. \tag{A.1.1}$$

On the other hand

$$\int_0^\infty \mathrm{d}\tau\, C\mathscr{C}(\tau)\, V\Sigma(t-\tau) = \int_0^\infty \mathrm{d}\tau\, C\mathscr{C}(\tau)\, V$$

$$\times \lim_{z\to 0} \sum_{n=0}^{\infty} (n!)^{-1}\,(t-\tau+\partial)^n\,(V\mathscr{L}^0 + V\mathscr{E}V)^n\,(V + V\mathscr{E}\mathscr{R}^0C)$$

$$= \lim_{z\to 0} \int_0^\infty \mathrm{d}\tau\, C\mathscr{C}(\tau)\, V \sum_{n=0}^{\infty} \sum_{p=0}^{n} \sum_{q=0}^{n-p} [t^p/p!q!\,(n-p-q)!\,]$$

$$\times\ (-\tau)^q\,[\partial^{n-p-q}\,(V\mathscr{L}^0 + V\mathscr{E}V)^n\,(V + V\mathscr{E}\mathscr{R}^0C)]. \tag{A.1.2}$$

We now use the following relation connecting a pair of Laplace transforms $A(z)$ and $A(t)$:

$$\left(\mathrm{i}\frac{\partial}{\partial z}\right)^n A(z) = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}z\tau}\,(-\tau)^n\, A(\tau). \tag{A.1.3}$$

It follows that

$$\int_0^\infty \mathrm{d}\tau\, C\mathscr{C}(\tau)\, V(-\tau)^q = [\partial^q C\mathscr{C}(z)\, V]_{z=0}$$

$$= [\partial^q C\mathscr{R}^0(z)\,\mathscr{E}(z)\, V]_{z=0}.$$

Substituting this result into the right-hand side of (A.1.2) we obtain the right-hand side of (A.1.1).

Equations (3.7), (3.9) and (3.13) are derived in a quite similar way.

APPENDIX 2

On the proof of $\boldsymbol{\Pi}^2 = \boldsymbol{\Pi}$. In ref. 13, George gave a complete and elegant direct proof of a theorem which, in his notations, is stated as

$$A^2 + \sum_k AD_kC_kA = A. \tag{A.2.1}$$

Translated into our notation,

$$A = V\boldsymbol{\Pi}V, \qquad D_k = V\mathsf{D}C, \qquad C_k = C\mathsf{C}V.$$

Hence his theorem is equivalent to

$$V\boldsymbol{\Pi}V\boldsymbol{\Pi}V + V\boldsymbol{\Pi}V\mathsf{D}C\mathsf{C}V\boldsymbol{\Pi}V \equiv V\boldsymbol{\Pi}V\boldsymbol{\Pi}V + V\boldsymbol{\Pi}C\boldsymbol{\Pi}V = V\boldsymbol{\Pi}V, \quad \text{(A.2.2)}$$

or

$$V\boldsymbol{\Pi}^2V = V\boldsymbol{\Pi}V. \quad \text{(A.2.3)}$$

There is practically no change necessary in order to generalize his proof to the inhomogeneous systems considered here. One simple has to translate his symbol $\Psi(z)$ by $(V\mathscr{L}^0 + V\mathscr{E}(z)\,V)$.

It should be clear, on the other hand, that eq. (A.2.2), together with the relations (4.30)–(4.32) ensure the validity of all four component equations of $\boldsymbol{\Pi}^2 = \boldsymbol{\Pi}$. Indeed, consider for instance:

$$V\boldsymbol{\Pi}^2C = V\boldsymbol{\Pi}^2V\mathsf{D}C = V\boldsymbol{\Pi}V\mathsf{D}C = V\boldsymbol{\Pi}C,$$

and similarly for the C–V and C–C components.

APPENDIX 3

Proof of $\Sigma(t) = \boldsymbol{\Pi}\mathscr{U}(t)$. We note again that it is sufficient, because of eq. (4.19), to prove:

$$V\boldsymbol{\Pi}\mathscr{U}(t) = V\Sigma(t). \quad \text{(A.3.1)}$$

Using eqs. (2.13) and (2.22) (for $t = 0$), the left side is

$$V\boldsymbol{\Pi}\mathscr{U}(t)$$
$$= \lim_{z\to 0} \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} (t^r/p!\,r!)\, \partial^p(V\mathscr{L}^0 + V\mathscr{E}V)^p\,(V + V\mathscr{E}\mathscr{R}^0C)\,\mathscr{L}^r.$$

On the other side, from (2.25) and the binomial expansion of $(t+\partial)^n$:

$$V\Sigma(t) = \lim_{z\to 0} \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} (t^r/r!\,p!)\, \partial^p\,(V\mathscr{L}^0 + V\mathscr{E}V)^{p+r}\,(V + V\mathscr{E}\mathscr{R}^0C).$$

It is therefore sufficient to prove the following relation

$$\lim_{z\to 0} \sum_{p=0}^{\infty} (p!)^{-1}\,\partial^p\,(V\mathscr{L}^0 + V\mathscr{E}V)^p\,(V + V\mathscr{E}\mathscr{R}^0C)\,\mathscr{L}^r$$
$$= \lim_{z\to 0} \sum_{p=0}^{\infty} (p!)^{-1}\,\partial^p(V\mathscr{L}^0 + V\mathscr{E}V)^{p+r}\,(V + V\mathscr{E}\mathscr{R}^0C),$$
$$r = 0, 1, 2, \ldots \quad \text{(A.3.2)}$$

The proof is by induction. The theorem is trivially true for $r = 0$.

Assume now that (A.3.2) is true for given r and consider

$$\lim_{z\to 0} \sum_{p=0}^{\infty} (p!)^{-1}\,\partial^p(V\mathscr{L}^0 + V\mathscr{E}V)^p\,(V + V\mathscr{E}\mathscr{R}^0C)\,\mathscr{L}^{r+1}$$
$$= \lim_{z\to 0} \sum_{p=0}^{\infty} (p!)^{-1}\,\partial^p(V\mathscr{L}^0 + V\mathscr{E}V)^{p+r}\,(V + V\mathscr{E}\mathscr{R}^0C)\,\mathscr{L}. \quad \text{(A.3.2)}$$

Expanding $\mathscr{L}$ in the form (2.8) and using (2.2), we can write:

$$(V + V\mathscr{E}\mathscr{R}^0 C)\,\mathscr{L} = V(\mathscr{L}^0 + \mathscr{L}' + \mathscr{E}\mathscr{R}^0 C\mathscr{L}')\,V + V(\mathscr{L}' + \mathscr{E}\mathscr{R}^0 C\mathscr{L}' + \mathscr{E}\mathscr{R}^0\mathscr{L}^0)\,C. \tag{A.3.3}$$

Making use of eqs. (2.20) and (2.17), we can simplify (A.3.3) to

$$(V + V\mathscr{E}\mathscr{R}^0 C)\,\mathscr{L} = (V\mathscr{L}^0 + V\mathscr{E}V) = \mathrm{i}zV\mathscr{E}\mathscr{R}^0 C. \tag{A.3.4}$$

Substituting this result into the right-hand side of eq. (A.3.2) we get

$$\begin{aligned}\lim_{z\to 0}\sum_{p=0}^{\infty}(p!)^{-1}\,\partial^p\,(V\mathscr{L}^0 + V\mathscr{E}V)^p\,(V + V\mathscr{E}\mathscr{R}^0 C)\,\mathscr{L}^{r+1} &= \lim_{z\to 0}\sum_{p=0}^{\infty}(p!)^{-1}\,\partial^p(V\mathscr{L}^0 + V\mathscr{E}V)^{p+r}\,(V\mathscr{L}^0 + V\mathscr{E}V) \\ &+ \lim_{z\to 0}\sum_{q=0}^{\infty}(q!)^{-1}\,\partial^q\,[(-\mathrm{i}z)(V\mathscr{L}^0 + V\mathscr{E}V)^{q+r}\,V\mathscr{E}\mathscr{R}^0 C].\end{aligned} \tag{A.3.5}$$

Using the Leibnitz formula, the second term can be written as

$$\begin{aligned}&\lim_{z\to 0}\sum_{q=0}^{\infty}\sum_{n=0}^{q}[n!\,(q-n)!]^{-1}\,[\partial^n\,(-\mathrm{i}z)] \\ &\quad\times[\partial^{q-n}\,(V\mathscr{L}^0 + V\mathscr{E}V)^{q+r}V\mathscr{E}\mathscr{R}^0 C] \\ &= \lim_{z\to 0}\sum_{q=1}^{\infty}[(q-1)!]^{-1}\,\partial^{q-1}\,(V\mathscr{L}^0 + V\mathscr{E}V)^{q+r}\,V\mathscr{E}\mathscr{R}^0 C.\end{aligned}$$

The last step is obtained by noting that the first bracketed factor differs from zero only for $n = 1$ and $n = 0$. Moreover, it follows from the regularity assumptions (2.23) that the contribution of the term $n = 0$ vanishes in the limit $z \to 0$; we are therefore left with the contribution of $n = 1$. Changing the summation index q to $p = q - 1$ and substituting the result into the right side of (A.3.5) we get

$$\begin{aligned}&\lim_{z\to 0}\sum_{p=0}^{\infty}(p!)^{-1}\,\partial^p\,(V\mathscr{L}^0 + V\mathscr{E}V)^p\,(V + V\mathscr{E}\mathscr{R}^0 C)\,\mathscr{L}^{r+1} \\ &= \lim_{z\to 0}\sum_{p=0}^{\infty}(p!)^{-1}\,\partial^p\,(V\mathscr{L}^0 + V\mathscr{E}V)^{p+r+1}\,(V + V\mathscr{E}\mathscr{R}^0 C).\end{aligned} \tag{A.3.6}$$

The induction hypothesis (A.3.2) is thus extended to the value $r + 1$, and hence the theorem (A.3.1) is proved. The proof of the relation $\Sigma(t) = \mathscr{U}(t)\,\boldsymbol{\Pi}$ is quite analogous to the present one. It is important to note that the *sign of t is irrelevant* in this proof; the theorem holds for any $t \gtrless 0$. This remark is important in connection with the extension of the PGH semi-group property of $\Sigma(t)$ into the full-group property derived in eqs. (5.5)–(5.7).

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