ON THE SUBGROUP STRUCTURE
OF EXCEPTIONAL GROUPS OF LIE TYPE

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ABSTRACT. We study finite subgroups of exceptional groups of Lie type, in particular maximal subgroups. Reduction theorems allow us to concentrate on almost simple subgroups, the main case being those with socle $X(q)$ of Lie type in the natural characteristic. Our approach is to show that for sufficiently large q (usually $q > 9$ suffices), $X(q)$ is contained in a subgroup of positive dimension in the corresponding exceptional algebraic group, stabilizing the same subspaces of the Lie algebra. Applications are given to the study of maximal subgroups of finite exceptional groups. For example, we show that all maximal subgroups of sufficiently large order arise as fixed point groups of maximal closed subgroups of positive dimension.

INTRODUCTION

In this paper we establish results aimed at the study of the subgroup structure of the finite exceptional groups of Lie type, in particular of their maximal subgroups. Let $G(r)$ be an exceptional group of Lie type over a finite field $\mathbb{F}_r$ of order r, where r is a power of the prime p. Write $G(r) = O^{p'}(G_\sigma)$, where G is a simple algebraic group over $\bar{\mathbb{F}}_r$ and σ is a Frobenius morphism of G.

The reduction theorem [LS1, Theorem 2] determines all maximal subgroups of $G(r)$ which are not almost simple. Thus we concentrate on almost simple maximal subgroups of $G(r)$; let M be such a subgroup. Of particular interest is the so-called "generic case", in which $F^*(M) = X(q)$, a group of Lie type over a field $\mathbb{F}_q$, also of characteristic p.

Our approach to the generic case is to attempt to show that the embedding $X(q) < G(r)$ lifts to an embedding $\bar{X} \leq G$, where $\bar{X}$ is a simple closed connected subgroup of G of the same type as $X(q)$. Once this is achieved, we are in a position to apply results from [Se2, LS1] on the subgroup structure of algebraic groups.

The first lifting result for exceptional groups appeared in [ST1], and showed that the finite embedding can be lifted under certain hypotheses, one of the hypotheses being that the characteristic p be suitably large. In this paper we establish lifting results in which there are hypotheses on the size of the ground field $\mathbb{F}_q$, but no assumptions on the characteristic. Consequently, the results hold for all but finitely many possibilities for the group $X(q)$. For example, Theorem 6 below shows that for q sufficiently large, either $X(q)$ is the group of fixed points of an automorphism of

$G(r)$, or $X(q) = O^{p'}(\bar{X}_\sigma)$ with $\bar{X}$ maximal among closed connected $M\langle\sigma\rangle$-invariant subgroups of G. (In most cases, "sufficiently large" means that $q > 9$.)

Once the lifting has been achieved we can apply the results of [Se2, LS1], which determine those subgroups $\bar{X}$ of G as in the last sentence of the previous paragraph, under some mild characteristic restrictions. These restrictions are much weaker than those required for the result in [ST1]; in particular, $p = 0$ or $p > 7$ covers all the restrictions. However, in view of the results obtained in this paper, it becomes a high priority issue to extend the analysis in [Se2, LS1] so as to remove the characteristic restrictions. Such an analysis is under way.

The preceding discussion has been couched in terms of maximal subgroups, but our results apply much more generally. We establish lifting results for all subgroups $X(q)$ of $G(r)$, with q sufficiently large. Namely, in Theorem 1 we show that any such subgroup is contained in a closed connected subgroup $\bar{X}$ of G stabilizing precisely the same subspaces of the Lie algebra $L(G)$ as $X(q)$. We also show (Theorem 4 and Corollary 5) that this connected group $\bar{X}$ is almost always proper in G, and that, given mild characteristic restrictions, $X(q) = O^{p'}(\bar{X}_\delta)$ for some Frobenius morphism δ (Theorem 10). At this point the results of [LS2] determine the conjugacy class of $X(q)$ in $G(r)$.

Finally, we use our lifting results to clear up a nagging problem on classical groups. In Theorem 11, we show that an embedding $X < Y < C$ of finite groups of Lie type in the same characteristic, with C classical and X absolutely irreducible on the usual module for C, can usually be lifted to a corresponding embedding of simple algebraic groups; such embeddings of algebraic groups are known [Se1]. See [Se3] for further discussion of this problem.

We now state our results in detail, beginning with our main lifting results. As indicated above, these require no assumption on the characteristic p, but do need an assumption on q, where $X = X(q)$ is as above. In order to specify this assumption we need the following definition.

Definition. Let G be a simple adjoint algebraic group, and let $\Sigma = \Sigma(G)$ be the root system of G. For a subgroup L of the lattice $\mathbb{Z}\Sigma$, let $t(L)$ be the exponent of the torsion subgroup of $\mathbb{Z}\Sigma/L$. For $\alpha, \beta \in \Sigma$, call the element $\alpha - \beta$ of $\mathbb{Z}\Sigma$ a *root difference*. Define

$$t(\Sigma(G)) = \max\{t(L) : L \text{ a subgroup of } \mathbb{Z}\Sigma \text{ generated by root differences}\}.$$

Theorem 1. *Let $X = X(q)$ be a quasisimple group of Lie type in characteristic p, and suppose that $X < G$, where G is a simple adjoint algebraic group of exceptional type, also in characteristic p. Assume that*

$$\begin{aligned} &q > t(\Sigma(G)).(2, p-1) && \text{if } X = A_1(q),\ ^2B_2(q)\ \text{or}\ ^2G_2(q), \\ &q > 9 \text{ and } X \neq A_2^\epsilon(16) && \text{otherwise.} \end{aligned}$$

Then the following hold:

(i) *there is a closed connected subgroup $\bar{X}$ of G containing X, such that every X-invariant subspace of the Lie algebra $L(G)$ is also $\bar{X}$-invariant;*

(ii) *if also $X \leq G_\sigma$ where σ is a Frobenius morphism, then there is a σ-stable and $N_G(X)$-stable subgroup $\bar{X}$ containing X such that each X-invariant subspace of each G-composition factor of $L(G)$ is also $\bar{X}$-invariant.*

Theorem 4 below will show that the subgroup $\bar{X}$ in the conclusion is proper in G, unless X has the same type as G.

Remark. If p is a good prime for G, then $L(G)$ is irreducible as a G-module, so the subspace invariance conditions in conclusions (i) and (ii) are equivalent; however, for bad primes $L(G)$ can be reducible (see Proposition 1.10 below), in which case (i) is stronger than (ii).

In view of Theorem 1, it is of interest to know the values of $t(\Sigma(G))$, particularly for groups G of exceptional type. Dr Ross Lawther has informed us that he has verified (using a computer) that $t(G_2) = 12$, $t(F_4) = 68$, $t(E_6) = 124$ and $t(E_7) = 388$, while $t(E_8)$ is unknown at present.

The bounds in Theorem 1 for groups of type A_1, ${}^2B_2$, ${}^2G_2$ are relatively large, and follow from Corollary 3 below, which we shall deduce from the following general proposition. It should be possible to improve these bounds with additional work.

Proposition 2. *Let G be a simple adjoint algebraic group, and let x be a semisimple element of G of finite order greater than $mt(\Sigma(G))$, for some positive integer m. Then there is an infinite closed subgroup S of G such that the following both hold:*

(i) *$x \in S$, and some nontrivial power of x, of order greater than m, lies in S^0;*

(ii) *every x-invariant subspace of $L(G)$ is also S-invariant.*

Corollary 3. *Let $X = X(q)$ be a quasisimple group of Lie type in characteristic p, and suppose that $X < G$, where G is a simple adjoint algebraic group, also in characteristic p. Assume that $q > t(\Sigma(G))m(X)$, where $m(X)$ is the order of the Schur multiplier of $X/Z(X)$. Then conclusions (i) and (ii) of Theorem 1 hold.*

Note that Proposition 2 and Corollary 3 apply to all types of simple algebraic group, not just exceptional types.

The proofs of Proposition 2 and Corollary 3 are rather short, and are given in §2. Reducing the bounds on q to those in Theorem 1 requires a great deal more effort. After some preliminary results in §3, this is carried out for $p \neq 2$ in §§4,5 and in §6 for $p = 2$. With some further effort, our proof could no doubt be extended to improve the $q > 9$ bound for subgroups X of rank larger than 2; indeed, when X has rank greater than half the rank of G, [LST, Theorem 2] gives a stronger conclusion assuming only that $q > 2$.

In order to make use of Theorem 1 (or Corollary 3), we need to know that the subgroup $\bar{X}$ is proper in G. For this it suffices to show that $\bar{X}$ acts reducibly on some G-composition factor of $L(G)$. This information is provided by the next result.

Definition. We define X and G to be *of the same type* if $X \cong G_\sigma^{(\infty)}$ for some Frobenius morphism σ; we also say that X is a group *of type G*.

Theorem 4. *Let G be a simple algebraic group in characteristic p, and let $X = X(q)$ be a subgroup of G, where $q = p^e$ and X is quasisimple. Suppose that X is not of the same type as G, and that X acts irreducibly on every G-composition factor of $L(G)$.*

Then $(G, p) = (C_n, 2)$ or $(B_n, 2)$, and either $X = D_n^\epsilon(q)$ or $n = 3$ and $X = G_2(q)$; further, X lies in a simple connected subgroup D_n or G_2 of G.

Combining Theorems 1 and 4, we immediately have

Corollary 5. *Let G be a simple adjoint algebraic group of exceptional type, and suppose that $X = X(q) < G$, where q is as in the hypotheses of Theorem 1. Then*

either X is of the same type as G, or $X < \bar{X} < G$, where $\bar{X}$ is closed and connected, and leaves invariant every X-invariant subspace of $L(G)$.

Using this result we shall derive Theorem 6 below, which concerns the maximal subgroups of the finite exceptional groups of Lie type. In the statement we refer to [LS1, Theorem 1]; in this result, certain positive integers $N(\bar{X}, G)$ are defined for pairs $(\bar{X}, G)$ of simple algebraic groups, as follows:

	$G = E_8$	E_7	E_6	F_4	G_2
$X = A_1$	7	7	5	3	3
A_2	5	5	3	3	
B_2, G_2	5	3	3	2	
B_3	2	2	2		
A_3, C_3, B_4	2				

For example, $N(A_2, E_7) = 5$, and so on. For $(\bar{X}, G)$ not in the table, set $N(\bar{X}, G) = 1$. We define $N(X(q), G)$ to be equal to $N(\bar{X}, G)$, where $\bar{X}$ is a simple algebraic group of the same type as $X(q)$.

The statement of Theorem 6 and its corollaries involves groups G, L and L_1 such that G is a simple adjoint algebraic group of exceptional type in characteristic p, σ is a Frobenius morphism such that $L = O^{p'}(G_\sigma)$ is a finite simple group of exceptional Lie type, and L_1 is a finite group with socle L (i.e. $L \leq L_1 \leq \operatorname{Aut} L$).

Theorem 6. *Let $L = O^{p'}(G_\sigma)$ and $L \leq L_1 \leq \operatorname{Aut} L$, as above. Suppose that H is an almost simple maximal subgroup of L_1, and $F^*(H) = X(q)$ with q satisfying the hypotheses of Theorem 1. Then one of the following holds:*

(i) $X(q)$ has the same type as G;

(ii) $X(q) = O^{p'}(\bar{X}_\sigma)$ for some maximal closed connected reductive $H\langle\sigma\rangle$-stable subgroup $\bar{X}$ of G. (The subgroup $\bar{X}$ is given by [LS1, Theorem 1] if $\bar{X}$ is non-simple or if $p > N(X(q), G)$.)

Remarks. 1. There is no assumption on p in the theorem.

2. The subgroups satisfying (i) are uniquely determined up to G_σ-conjugacy by [LS2, 5.1].

3. Our proof of Theorem 6 shows, more generally, that if H is any (not necessarily maximal) subgroup of L_1 such that $F^*(H) = X(q)$ with q as in Theorem 1, then $H \leq N(\bar{X}_\sigma)$ for some $\bar{X}$ as in conclusion (ii).

4. In our proof of Theorem 6 we do not require the full force of the hypothesis on q; indeed, in the proof we only require the existence of a subgroup $\bar{X}$ satisfying the conclusions of Theorem 1.

We recall one further definition before stating the next result. If G, L, L_1 are as above, and D is a σ-stable connected reductive subgroup of G of maximal rank (i.e. containing a maximal torus), then we call $N_{L_1}(D_\sigma)$ a *subgroup of maximal rank* of L_1. The subgroups of maximal rank which are maximal in L_1 are determined in [LSS].

Corollary 7. *Let $L = O^{p'}(G_\sigma)$ and $L \leq L_1 \leq \operatorname{Aut} L$, as above. Let H be an almost simple maximal subgroup of L_1, with $F^*(H) = X(q)$ and q as in Theorem 1, and let $G_\sigma = G(q_1)$. Then one of the following holds:*

(i) H is a subgroup of maximal rank;

(ii) $X(q)$ has the same type as G;

(iii) $q = q_1$ and $X(q) = O^{p'}(\bar{X}_\sigma)$, where $\bar{X}$ is a simple maximal connected σ-stable subgroup of G not containing a maximal torus. Moreover, assuming that $p > N(X(q), G)$, the possibilities are:

$G = G_2 :$ $X(q) = A_1(q)\,(p \geq 7)$,
$G = F_4 :$ $X(q) = A_1(q)\,(p \geq 13)$ or $G_2(q)\,(p = 7)$,
$G = E_6 :$ $X(q) = A_2^\epsilon(q)\,(\epsilon = \pm, p \geq 5)$, $G_2(q)\,(p \geq 5, p \neq 7)$, $F_4(q)$ or $C_4(q)\,(p \neq 2)$,
$G = E_7 :$ $X(q) = A_1(q)\,(2$ classes, $p \geq 17, 19)$ or $A_2^\epsilon(q)\,(\epsilon = \pm, p \geq 7)$,
$G = E_8 :$ $X(q) = A_1(q)\,(3$ classes, $p \geq 23, 29, 31)$ or $B_2(q)\,(p \geq 7)$.

Assuming $p > N(X(q), G)$, there is just one $\mathrm{Aut}(G)$-class of subgroups $\bar{X}$ of each isomorphism type occurring, and each such $\mathrm{Aut}(G)$-class gives a unique $\mathrm{Aut}(G_\sigma)$-class of subgroups $X(q)$.

Corollary 8. Let $L = O^{p'}(G_\sigma)$ and $L \leq L_1 \leq \mathrm{Aut}\, L$, as above. There is a constant c (independent of G, L, L_1), such that if H is a maximal subgroup of L_1 with $|H| > c$, then either

(i) H is the normalizer of a subgroup of the same type as G, or

(ii) $H = N_{L_1}(\bar{X}_\sigma)$ for some maximal closed connected $H\langle\sigma\rangle$-stable subgroup $\bar{X}$ of G.

Next we state our results on embeddings of arbitrary subgroups $X(q)$ of an exceptional adjoint algebraic group G. Theorem 9 is the analogue of [LS2, Theorem 1] for finite groups. In the statement we use numbers $N'(X, G)$ as defined in [LS2]; these are roughly the same as the numbers $N(X, G)$, with a few differences. The precise definition of $N'(X, G)$ is given by the following table:

	$G = E_8$	E_7	E_6	F_4	G_2
X of type A_1	7	7	5	3	3
A_2	5	5	3	3	
B_2	5	3	3	2	
G_2	7	7	3	2	
B_3	2	2	2	2	
C_3	3	2	2	2	
A_3, B_4, C_4, D_4	2	2	2		

Theorem 9. Let $X = X(q) < G$, with G of exceptional type and q as in Theorem 1, and assume that $p > N'(X, G)$. If X lies in a parabolic subgroup QL of G, with unipotent radical Q and Levi subgroup L, then X lies in a Q-conjugate of L. Further, if also $X < G_\sigma$ (where σ is a Frobenius morphism), and QL is σ-stable, then X lies in a σ-stable Q-conjugate of L.

Theorem 10. Let $X = X(q) < G$, with G of exceptional type and q as in Theorem 1, and assume that $p > N'(X, G)$. Then

(i) X lies in a closed connected simple subgroup $\tilde{X}$ of G of the same type as X, and $C_G(X)^0 = C_G(\tilde{X})^0$;

(ii) if also $X < G_\sigma$ with σ a Frobenius morphism of G, then X lies in a closed connected semisimple σ-stable subgroup $\bar{X} = X_1 \ldots X_t$ of G, where each factor X_i is simple of the same type as X; moreover, $C_G(X)^0 = C_G(\bar{X})^0$.

The embeddings of the subgroups $\tilde{X}$ and $\bar{X}$ in the conclusion of Theorem 10 are given by [LS2, Theorems 5 and 7], so Theorem 10 determines the embeddings of $X(q)$ in G and G_σ.

For $X = X(q)$ of rank more than $\frac{1}{2}\mathrm{rank}(G)$, the conclusion of Theorem 10 is obtained in [LST, Theorem 2] with no assumption on p, and assuming only that $q > 2$.

Finally, we give a consequence of Theorem 1 to the study of subgroups of finite classical groups.

Theorem 11. *Let C be a finite classical simple group in characteristic p, with usual module V. Suppose that $X = X(q)$ and Y are finite simple groups of Lie type in characteristic p, not of the same type, such that $X < Y < C$ and X is absolutely irreducible on V; when Y is of exceptional type, assume also that q is as in Theorem 1 (relative to Y).*

Then the embedding $X < Y < C$ lifts to an embedding of closed subgroups $\bar{X} < \bar{Y} \leq \bar{C}$, where $\bar{X}, \bar{Y}, \bar{C}$ are connected simple algebraic groups of the same types as X, Y, C.

Remark. Assuming $\bar{Y}, \bar{C}$ are not both classical groups on the same natural module, all possibilities for the triple $(\bar{X}, \bar{Y}, \bar{C})$ in the conclusion are given by [Se1, Te1].

The layout of the paper is as follows. After the preliminary first section, we give in §2 the proofs of Proposition 2 and Corollary 3. The following four sections are devoted to the proof of Theorem 1. In §3 we describe the strategy and give some preliminary lemmas. The proof is then carried out for q odd in §§4,5, and for q even in §6. In §7 we prove Theorem 4, and §8 contains proofs of Theorem 6 and its corollaries. Theorems 9 and 10 are proved in §9, and Theorem 11 in §10.

We use the following notation throughout the paper. If Y is a connected simple algebraic group over the algebraically closed field K, or a corresponding finite group of Lie type, and λ is a dominant weight, then $V_Y(\lambda)$ denotes the rational irreducible KY-module with high weight λ, and $W_Y(\lambda)$ denotes the corresponding Weyl module. If $V_1, \dots, V_k$ are modules, then $V_1/\dots/V_k$ denotes a module having the same composition factors as $V_1 \oplus \dots \oplus V_k$. We often abbreviate this notation as follows: if $\mu_1, \dots, \mu_k$ are dominant weights for the group Y, and $c_1, \dots, c_k$ are positive integers, then

$$\mu_1^{c_1}/\dots/\mu_k^{c_k}$$

always denotes a KY-module having the same composition factors as the module $V_Y(\mu_1)^{c_1} \oplus \dots \oplus V_Y(\mu_k)^{c_k}$.

When Y is a subgroup of G and V is a KG-module, we use $V \downarrow Y$ for the restriction of V to Y. If Y is connected and simple in characteristic $p > 0$, and q is a power of p, then σ_q denotes a standard Frobenius morphism of Y; that is, a morphism inducing the map $x_\alpha(t) \to x_\alpha(t^q)$ on root groups. If V is a KY-module, we write $V^{(q)}$ for the KY-module obtained from V by twisting the action of Y by σ_q (i.e. changing the action from $v \to vy$ to $v \to vy^{\sigma_q}$).

We use T_i to denote a torus of rank i, and $W(G)$ for the Weyl group of G. The fundamental roots in a fundamental system for G are denoted $\alpha_1, \dots, \alpha_l$, and the corresponding fundamental dominant weights are $\lambda_1, \dots, \lambda_l$. The Dynkin diagram of G is labelled as in [Bo, p.250].

For $\epsilon = \pm$, $A_n^\epsilon(q)$ denotes the group $A_n(q)$ if $\epsilon = +$ and $^2A_n(q)$ if $\epsilon = -$; we use similar notation for types D_n^ϵ, E_6^ϵ, B_2^ϵ, G_2^ϵ, F_4^ϵ, and also for D_4^ϵ we allow $\epsilon = 3$ to denote $^3D_4$. Finally, if X is a finite group of Lie type, then $\mathrm{rk}(X)$ denotes the rank of the simple algebraic group corresponding to X.

1. Preliminaries

In this section we present some preliminary results which will be used throughout the paper.

The first few results concern representations of groups of type A_1. For the group $G = SL_2(K)$ over the algebraically closed field K of characteristic $p > 0$, and for $0 \leq i \leq p - 1$, we denote simply by i (or sometimes $V(i)$) the rational irreducible KG-module with high weight $i\lambda_1$, of dimension $i + 1$; these are the restricted KG-modules. By Steinberg's tensor product theorem, every finite-dimensional rational irreducible KG-module is a tensor product $i_0 \otimes i_1^{(p)} \otimes \ldots \otimes i_k^{(p^k)}$ of field twists of restricted modules $i_0, \ldots, i_k$, and such a module has high weight $\sum i_r p^r$.

Using [St, Lemma 79], we have the following well known result.

Proposition 1.1. *Let $V(i)$ denote the restricted module for $G = SL_2(K)$ of high weight i. If i is odd, then the induced group*

$$G^{V(i)} = SL_2(K) \leq Sp(V(i));$$

and if i is even, then $G^{V(i)} = PSL_2(K) \leq SO(V(i))$.

The next result concerns extensions of modules for $SL_2(K)$ and its subgroups $SL_2(p^e)$, and is taken from [AJL, 3.9 and 4.5]. We thank Professor J-P. Serre for pointing out to us that [AJL, 4.5] has a small omission when $e = 2$, and giving us the corrected version below.

Proposition 1.2. *Let $G = SL_2(K)$, and for $e \geq 1$ denote by G_e a subgroup $SL_2(p^e)$ of G.*

(i) *Suppose $p > 2$ and let $0 \leq a, b \leq p - 1$. Then $\mathrm{Ext}^1_{G_1}(a, b) \neq 0$ if and only if $a + b = p - 3$ or $p - 1$.*

(ii) *Let $e \geq 2$, and suppose $\lambda, \mu < p^e$ have p-adic expansions $\lambda = \sum_0^{e-1} a_i p^i, \mu = \sum_0^{e-1} b_i p^i$ (where $0 \leq a_i, b_i \leq p-1$). Set $a_e = a_0$ and $b_e = b_0$. Then $\mathrm{Ext}^1_{G_e}(\lambda, \mu) \neq 0$ if and only if there exists $k \geq 0$ such that*

$$a_i = b_i \ \text{ for } \ i \notin \{k, k+1\}, \ a_k + b_k = p - 2, \ \text{ and } \ a_{k+1} - b_{k+1} = \pm 1.$$

Further, $\mathrm{Ext}^1_{G_e}(\lambda, \mu)$ has dimension at most 1, except if $e = 2$ and, writing $s = \frac{1}{2}(p - 1), t = \frac{1}{2}(p - 3)$, we have $(a_0, a_1, b_0, b_1) = (s, s, t, t), (t, t, s, s), (s, t, t, s)$ or (t, s, s, t) (in which case it has dimension 2).

(iii) *Let λ, μ be positive integers with p-adic expansions $\lambda = \sum a_i p^i, \mu = \sum b_i p^i$. Then $\mathrm{Ext}^1_G(\lambda, \mu) \neq 0$ if and only if there exists $k \geq \nu_p(\lambda + 1)$ (where $\nu_p(\lambda + 1) = \max\{i : p^i \text{ divides } \lambda + 1\}$), such that*

$$a_i = b_i \ \text{ for } \ i \notin \{k, k+1\}, \ a_k + b_k = p - 2, \ \text{ and } \ a_{k+1} - b_{k+1} = \pm 1.$$

Further, $\mathrm{Ext}^1_G(\lambda, \mu)$ has dimension at most 1.

Proposition 1.3. *Let $G = SL_2(K), G_e = SL_2(p^e)$, and let $0 \leq \lambda, \mu < p^e$.*

(i) *The restriction map $\mathrm{Ext}^1_G(\lambda, \mu) \to \mathrm{Ext}^1_{G_e}(\lambda, \mu)$ is injective; indeed, this map is an isomorphism unless either $e = 1$, or λ, μ are as in the exceptional cases at the end of 1.2(ii).*

(ii) *Suppose that $e \geq 2$, and, if $e = 2$, that λ, μ are not as in the exceptional cases at the end of 1.2(ii). Then there exist λ', μ' such that $\lambda' \downarrow G_e \cong \lambda \downarrow G_e$,*

$\mu' \downarrow G_e \cong \mu \downarrow G_e$ *and*

$$\text{Ext}_G^1(\lambda', \mu') \cong \text{Ext}_{G_e}^1(\lambda', \mu') \cong \text{Ext}_{G_e}^1(\lambda, \mu).$$

In particular, taking $\mu = 0$, we have $H^1(G, \lambda') \cong H^1(G_e, \lambda') \cong H^1(G_e, \lambda)$.

Proof. Part (i) is immediate from [CPSK, 7.4], and part (ii) follows from 1.2; note that in (ii) it may be necessary to choose $\lambda' \neq \lambda$ or $\mu' \neq \mu$, since by 1.2 there exist λ, μ such that $\text{Ext}_G^1(\lambda, \mu) = 0$ but $\text{Ext}_{G_e}^1(\lambda, \mu) \neq 0$ (the pairs λ, λ' and μ, μ' are related by a field morphism of G). $\square$

Proposition 1.4. *Let G be a connected simple algebraic group over K and let H be a finite subgroup of G. Suppose V is a finite-dimensional rational KG-module satisfying the following conditions:*

(i) *every G-composition factor of V is H-irreducible;*

(ii) *for any G-composition factors M, N of V, the restriction map $\text{Ext}_G^1(M, N) \to \text{Ext}_H^1(M, N)$ is injective;*

(iii) *for any G-composition factors M, N of V, if $M \downarrow H \cong N \downarrow H$ then $M \cong N$ (as G-modules).*

Then G and H fix exactly the same subspaces of V.

Proof. The proof goes by induction on $\dim V$. It is clearly sufficient to show that if W is an irreducible H-submodule of V, then G fixes W.

Thus let W be an irreducible H-submodule of V. Define $U = \langle W^G \rangle$, and suppose that $W \neq U$. If U is G-irreducible, then by (i) it is H-irreducible, so $U = W$, contrary to assumption. Therefore U is G-reducible; let W_0 be a G-irreducible submodule of U. Then $W_0 \neq W$.

Consider V/W_0. Now H fixes the subspace $(W + W_0)/W_0$ of this, which is H-irreducible. By induction, G fixes $W + W_0$. Hence $U = W + W_0$. As W, W_0 are H-irreducible, $U = W + W_0 = W \oplus W_0$. Thus U is not H-indecomposable, and so by (ii), U is also not G-indecomposable. Therefore there is a G-module W_1 such that

$$U = W \oplus W_0 = W_1 \oplus W_0.$$

Note that W is H-isomorphic to W_1. If W is not H-isomorphic to W_0, then W, W_0 are the only irreducible H-submodules of U, so $W_1 = W$ and G fixes W, a contradiction. Therefore W is H-isomorphic to W_0, and hence by (iii) these two spaces are also G-isomorphic. Now $W \subseteq W_1 \oplus W_0$, so $W = \{w + w\phi : w \in W_1\}$ for some H-isomorphism $\phi : W_1 \to W_0$. But ϕ is a G-isomorphism: for let α be any G-isomorphism from W_1 to W_0. Then $\alpha\phi^{-1} : W_1 \to W_1$ is an H-isomorphism, so by Schur's Lemma, $\alpha\phi^{-1} = \lambda.\text{id}$ for some $\lambda \in K^*$. Hence $\phi = \lambda^{-1}\alpha$, and so ϕ is a G-isomorphism, as claimed. It follows that W is fixed by G, contrary to our assumption that $W \neq U$. Therefore $W = U$, and this completes the proof. $\square$

The next result is a consequence of 1.4; it is actually a special case of [CPSK, 7.5].

Corollary 1.5. *Let $G = SL_2(K)$ and $G_e = SL_2(p^e) < G$. If V is a finite-dimensional rational KG-module such that every composition factor of $V \downarrow G$ has high weight less than p^e, then G and G_e fix exactly the same subspaces of V.*

Proof. Take $H = G_e$ in 1.4. Conditions (i) and (iii) of 1.4 are clearly satisfied, and condition (ii) follows from 1.3(i). $\square$

Proposition 1.6. *Let $G = SL_2(K)$ and $Y \cong SL_2(p^e)$, and let T, T_Y be Cartan subgroups of G, Y respectively. Let Q be a finite-dimensional rational KG-module, and regard the semidirect product QG as an algebraic group. Suppose that $Y < QG$, $T_Y < QT$, $C_Q(T) = C_Q(T_Y)$ and that $H^1(G, Q) \cong H^1(Y, Q)$. Then there is a closed complement to Q in QG which contains Y.*

Proof. We may assume that $T_Y < T$. For $x \in Y$, let $x = u_x r_x$ with $u_x \in Q, r_x \in G$. Then $X = \{r_x : x \in Y\}$ is a subgroup of G isomorphic to Y, and the hypotheses hold with X replacing Y. The map $\gamma : X \to Q$ given by $\gamma(r_x) = u_x$ is a 1-cocycle. The assumption that $C_Q(T) = C_Q(T_Y)$ implies that the restriction map $H^1(G, Q) \to H^1(X, Q)$ is injective, by [CPSK, 7.3]. Hence by hypothesis, the restriction map is an isomorphism, and γ extends to a rational 1-cocycle $\bar{\gamma} : G \to Q$. Then $\{\bar{\gamma}(g)g : g \in G\}$ is a closed complement to Q in QG containing Y. □

Proposition 1.7. *Let $G = SO_{2n}(K)$, where K is algebraically closed of characteristic $p > 2n$, and let u be an element of G of order p. Then there is a closed connected subgroup $A \cong A_1(K)$ of G containing u, such that every composition factor of $V_G(\lambda_1) \downarrow A$ is restricted.*

Proof. Let $V = V_G(\lambda_1)$. By [SS, IV, 2.14 and 2.15], u is determined up to G-conjugacy by the sizes of its Jordan blocks on V; moreover, the number of Jordan blocks of each even size is even. Let $r_1, \ldots, r_k$ be the sizes of the odd Jordan blocks of u, and $s_1, s_1, \ldots, s_l, s_l$ the sizes of the even Jordan blocks, listed in pairs. Then G has a subgroup $\prod_1^k SO_{r_i} \times \prod_1^l SL_{s_i}$. Each factor has an irreducible subgroup of type $A_1(K)$, and we define A to be a diagonal subgroup of the product of these A_1's such that the composition factors of $V \downarrow A$ have high weights $r_i - 1, s_i - 1$. Any nontrivial unipotent element of A has the same Jordan block sizes on V as u, hence is G-conjugate to u. This completes the proof. □

Next we state two results on modules for groups of type A_2. The first is taken from [LS2, 1.9, 1.11], and is a consequence of the sum formula in [An].

Proposition 1.8. *Let $G = A_2(K)$, and let $a, b \in \{0, 1, \ldots, p - 1\}$.*
   (i) If $a + b + 2 \leq p$, or if $b = 0$, then $W_G(ab)$ is irreducible.
   (ii) If $W_G(ab)$ is reducible, then it is indecomposable with two composition factors, $V_G(ab)$ and $V_G(p - b - 2, p - a - 2)$.
   (iii) $W_G(ab)$ has dimension $\frac{1}{2}(a + 1)(b + 1)(a + b + 2)$.

The next result is a straightforward application of 1.8.

Proposition 1.9. *Let $A = A_2^\epsilon(p)$ with $p = 11$ or 13, and let S be a subgroup $SL_2(p)$ of A centralizing an involution. Suppose $V = V_A(ab)$ is a restricted module for A satisfying*
   (i) $a \geq b$ and $a + b \geq 9$, and
   (ii) $\dim V \leq 124$ if $a \neq b$, and $\dim V \leq 248$ if $a = b$.
Then the possibilities for a, b and $V \downarrow S$ are as in the following table (in the last column, the superscripts denote the multiplicities of each composition factor).

ab	p	$\dim V$	$V \downarrow S$
$12, 0$	13	91	$12^1/11^1/10^1/....$
$11, 0$	13	78	$11^1/10^1/9^1/....$
$10, 0$	$11, 13$	66	$10^1/9^1/8^1/....$
90	$11, 13$	55	$9^1/8^1/7^1/....$
$11, 1$	13	102	$12^1/11^2/10^1/....$
91	13	120	$10^1/9^2/....$
	11	75	$10^1/9^2/8^2/....$
81	$11, 13$	99	$9/8^2/....$
$10, 2$	13	111	$12^1/11^2/10^2/....$
82	11	82	$10^1/9^2/8^2/7^1....$
93	13	118	$12^1/11^2/10^2/9^2/....$
73	11	87	$10^1/9^2/8^2/....$
84	13	123	$12^1/11^2/10^2/....$
66	13	127	$12^1/11^2/10^2/....$
64	11	90	$10^1/9^2/8^2/....$
55	13	216	$10^1/9^2/8^3/....$
	11	91	$10^1/9^2/8^2/7^2....$

Next we prove some results concerning the representation of a simple algebraic group G on its Lie algebra $L(G)$. The first is well known, but we give a proof for completeness.

Proposition 1.10. *The G-composition factors of $L(G)$ are as in the following table.*

G	p	comp. factors of $L(G)$	dimensions
A_n	$p \nmid n+1$	$\lambda_1 + \lambda_n$	$(n+1)^2 - 1$
	$p \mid n+1$	$\lambda_1 + \lambda_n/0$	$(n+1)^2 - 2, 1$
B_n	$p \neq 2$	λ_2	$2n^2 + n$
	$p = 2$, n odd	$\lambda_1/\lambda_2/0$	$2n, 2n^2 - n - 1, 1$
	$p = 2$, $n > 2$ even	$\lambda_1/\lambda_2/0^2$	$2n, 2n^2 - n - 2, 2$
	$p = 2, n = 2$	$\lambda_1/2\lambda_2/0^2$	$4, 4, 2$
C_n	$p \neq 2$	$2\lambda_1$	$2n^2 + n$
	$p = 2$, n odd	$2\lambda_1/\lambda_2/0$	$2n, 2n^2 - n - 1, 1$
	$p = 2$, n even	$2\lambda_1/\lambda_2/0^2$	$2n, 2n^2 - n - 2, 2$
D_n	$p \neq 2$	λ_2	$2n^2 - n$
	$p = 2$, n odd	$\lambda_2/0$	$2n^2 - n - 1, 1$
	$p = 2$, n even	$\lambda_2/0^2$	$2n^2 - n - 2, 2$
E_8	any	λ_8	248
E_7	$p \neq 2$	λ_1	133
	$p = 2$	$\lambda_1/0$	$132, 1$
E_6	$p \neq 3$	λ_2	78
	$p = 3$	$\lambda_2/0$	$77, 1$
F_4	$p \neq 2$	λ_1	52
	$p = 2$	λ_1/λ_4	$26, 26$
G_2	$p \neq 3$	λ_2	14
	$p = 3$	λ_1/λ_2	$7, 7$

Proof. For G of exceptional type this is proved in [Se2, 1.2]. Assume now that G is classical.

Consider first $G = A_n$. The highest root vector in $L(G)$ affords weight $\lambda_1 + \lambda_n$, so the composition factor $V_G(\lambda_1 + \lambda_n)$ of $L(G)$ has all $n^2 + n$ roots occuring as weights. By [Se1, 8.6], the 0-weight space of $V_G(\lambda_1 + \lambda_n)$ has dimension n if $p \nmid n + 1$, and dimension $n - 1$ if $p | n$. The conclusion follows for A_n.

Next consider $G = B_n, C_n, D_n$ with $p \neq 2$. The highest long root vector affords weight $\lambda_2, 2\lambda_1, \lambda_2$ respectively; hence $L(G)$ has a composition factor of this high weight. Also, if $V = V_G(\lambda_1)$ (of dimension $2n + 1, 2n, 2n$ respectively), then by [Se1, 8.1], the G-modules $\bigwedge^2 V, S^2 V, \bigwedge^2 V$ (respectively) are irreducible, and are isomorphic to $V_G(\lambda_2), V_G(2\lambda_1), V_G(\lambda_2)$. Since $\bigwedge^2 V, S^2 V, \bigwedge^2 V$ (respectively) have dimension equal to that of $L(G)$, it follows that $L(G)$ is irreducible of high weight $\lambda_2, 2\lambda_1, \lambda_2$ respectively, as in the conclusion.

In all the remaining cases we have $p = 2$. We begin with $G = D_n$. We may take $n \geq 4$ and $G = SO_{2n} = SO(V)$ (note that the composition factors of $L(G)$ are independent of the form of the group G). The highest root vector affords weight λ_2. We work out the dimension of the composition factor $V_G(\lambda_2)$, by considering the action of G on $\bigwedge^2 V$ (of which $V_G(\lambda_2)$ is a section). Observe first that the orbit of λ_2 under the Weyl group of G has size $2n^2 - 2n$ (which is the number of roots of D_n); so the dimension of $V_G(\lambda_2)$ will be determined once we calculate the dimension of the 0-weight space. Now G has a subgroup GL_n containing a maximal torus T. The Weyl group S_n of GL_n acts on the 0-weight space M of T on $\bigwedge^2 V$ as on the natural n-dimensional permutation module over K. Writing V_i for an irreducible S_n-module of dimension i, we have

$$M \downarrow S_n = \begin{array}{l} V_1 \oplus V_{n-1}, \text{ if } n \text{ is odd} \\ V_1/V_{n-2}/V_1 \text{ (indecomposable), if } n \text{ is even.} \end{array}$$

A subgroup GL_4 of G has a composition factor of high weight $\lambda_1 + \lambda_3$ in $V_G(\lambda_2)$. Hence by the A_n case above, we see that the 0-weight space of $V_G(\lambda_2)$ has dimension at least 2; from the S_n-action described above, the 0-weight space therefore has dimension at least $n - 1$ if n is odd, and at least $n - 2$ if n is even. Therefore $\dim V_G(\lambda_2)$ is at least $2n^2 - n - 1$ if n is odd, and at least $2n^2 - n - 2$ if n is even. As $G = SO(V) < Sp(V)$, we see that G fixes a 1-space $\langle w \rangle$ in $\bigwedge^2 V$. It follows that $\dim V_G(\lambda_2) = 2n^2 - n - 1$ if n is odd. Finally, $\dim V_G(\lambda_2) = 2n^2 - n - 2$ if n is even: for in this case the G-invariant 1-space $\langle w \rangle$ cannot be a direct summand (as $M \downarrow S_n$ is indecomposable), so the self-duality of $\bigwedge^2 V$ implies that there is a G-invariant subspace W of codimension 1 in $\bigwedge^2 V$, and $V_G(\lambda_2) \cong W/\langle w \rangle$.

Next consider $G = B_n$, with $p = 2$ and $n > 2$. The highest long root affords weight λ_2, and the highest short root affords λ_1. Also $V_G(\lambda_1)$ has dimension $2n$. Moreover, G fixes the ideal I of $L(G)$ generated by all e_α with α a short root; and $I \subseteq L(A_1^n)$, where A_1^n is a commuting product of short A_1's. Thus a consideration of weights implies that $I \downarrow G = \lambda_1/0^k$ for some k. Now the long roots form a subsystem D_n, so the composition factor $V_{D_n}(\lambda_2)$ of $L(D_n)$ is a section of $L(G)/I$. By [Se1, 4.1(iii)], $V_{D_n}(\lambda_2)$ has the same dimension as $V_{B_n}(\lambda_2)$, and by the previous paragraph, this dimension is $2n^2 - n - 1$ if n is odd, and $2n^2 - n - 2$ if n is even. The conclusion now follows. The case where $G = B_2, p = 2$ follows from the C_2 case, given below.

Finally, let $G = C_n$ with $p = 2$. The highest long root affords $2\lambda_1$, and the highest short root affords λ_2. Again G fixes the ideal of $L(G)$ generated by all e_α with α short, which is contained in $L(D_n)$. By [Se1, 4.1(iii)], $\dim V_{C_n}(\lambda_2) = \dim V_{D_n}(\lambda_2)$, and the conclusion follows as above. $\qquad\square$

Proposition 1.11. *Let G be a simple algebraic group in characteristic p, and let T be a maximal torus of G. Then $C_{L(G)}(L(T)) = L(T)$, unless $(G, p) = (C_n, 2)$ (or $(B_2, 2)$). In the exceptional cases, $C_{L(G)}(L(T)) = L(A_1^n)$ (or $L(A_1^2)$).*

Proof. Let $A = C_{L(G)}(L(T))$. Then A is T-invariant, hence is spanned by $L(T)$, together with those elements e_α of $L(G)$ which centralize $L(T)$. Note that if $e_\alpha \in A$ then $e_{-\alpha} \in A$. We may assume that $G \neq A_1$.

Suppose $A \neq L(T)$, so that $e_\alpha \in A$ for some root α. Conjugating by an element of the Weyl group, we may assume that α is a fundamental root. As h_α centralizes e_α, we have $p = 2$.

If there is a root β such that $(\alpha, \beta) = -1$, then h_β does not centralize e_α, a contradiction. Therefore α cannot lie in an A_2 subsystem of $\Sigma(G)$. Hence there must be more than one root length, and α must be a long root. It follows that $G = B_2$ or C_n. The B_2 case can be treated as C_2, so we take $G = C_n$. Now the proof of 1.10 shows that the weight α occurs within a composition factor $V_{C_n}(2\lambda_1)$ of $L(G)$. Because of the field twist, $L(T)$ acts trivially on this module. Moreover, the fundamental A_1's corresponding to long roots commute, so this completes the proof. $\qquad\square$

Proposition 1.12. *Let G be a simple algebraic group over K, and let $\phi : G \to G$ be a morphism which is an automorphism of abstract groups.*

(i) *Suppose that G_ϕ is not a finite Suzuki or Ree group, and let V be a G-composition factor of $L(G)$. If M is a subspace of V, then $(G_M)^\phi = G_{M'}$ for some subspace M' of V.*

(ii) *Suppose G_ϕ is a finite Suzuki or Ree group, and let V_1, V_2 be the two G-composition factors of $L(G)$. If M is a subspace of V_i $(i = 1, 2)$, then $(G_M)^\phi = G_{M'}$ for some subspace M' of V_{3-i}.*

(iii) *Let X be a ϕ-stable subgroup of G, and let $\mathcal{M}$ be the collection of all X-invariant subspaces of all G-composition factors of $L(G)$. Then the subgroup $\bigcap_{W \in \mathcal{M}} G_W$ of G is ϕ-stable.*

Proof. (i) Let the G-module V have high weight λ and correspond to the representation $\rho : G \to GL(V)$. We may write $\phi = y\tau\sigma$, where y, τ, σ are (possibly trivial) inner, graph, field automorphisms, respectively. By 1.10, the representations ρ and $\tau\rho$ of G are equivalent, as they both have high weight λ. Hence if σ is a q-power field automorphism (where $q = p^e \geq 1$), then the high weight of the representation $\phi\rho$ is $q\lambda$. There is therefore a q-power field automorphism ω of $GL(V)$ such that the representations $\phi\rho$ and $\rho\omega$ of G are equivalent. The automorphism ω is induced by a semilinear transformation $V \to V$ which we shall also denote by ω. Then $y^\omega = \omega^{-1}y\omega$ for $y \in GL(V)$. Thus, identifying each $g \in G$ with its image $g\rho \in GL(V)$, there exists $x \in GL(V)$ such that $g^\phi = g^{\omega x} = x^{-1}\omega^{-1}g\omega x$, for all $g \in G$. Writing $\delta = \omega x$, this gives $\delta g^\phi = g\delta$ for all $g \in G$, and we have

$$(v\delta)g^\phi = (vg)\delta$$

for all $v \in V, g \in G$. If M is a subspace of V, and $m \in M, g \in G_M$, then $(m\delta)g^\phi = (mg)\delta \in M\delta$, and hence $g^\phi \in G_{M\delta}$. Therefore $(G_M)^\phi \leq G_{M\delta}$. For the

reverse inclusion, write $g \in G_{M\delta}$ as $g = (g^{\phi^{-1}})^{\phi}$. Using the displayed equality, we see that $g^{\phi^{-1}} \in G_M$, as required. Part (i) is now established.

(ii) Here G_ϕ is a finite group of type $^2B_2$, $^2G_2$ or $^2F_4$. Let V_1, V_2 be the composition factors of $L(G)$, as in 1.10, and let $\rho_i : G \to GL(V_i)$ $(i = 1, 2)$ be the corresponding representations. As above, consideration of high weights shows that there are automorphisms ω_i of $GL(V_i)$ such that the representations $\phi\rho_i$ and $\rho_{3-i}\omega_{3-i}$ are equivalent for $i = 1, 2$. Thus there are invertible linear transformations $\delta : V_1 \to V_2$, $\gamma : V_2 \to V_1$ such that

$$(v_1 g^\phi)\delta = (v_1\delta)g^{\omega_2}, \quad (v_2 g^\phi)\gamma = (v_2\gamma)g^{\omega_1}$$

for all $v_i \in V_i, g \in G$ (where we write just $v_i g$ instead of $v_i(g\rho_i)$). A calculation similar to that in part (i) now shows that for subspaces M_i of V_i, we have

$$G_{M_1} = (G_{M_1\delta\omega_2^{-1}})^\phi, \quad G_{M_2} = (G_{M_2\gamma\omega_1^{-1}})^\phi,$$

giving (ii).

Finally, (iii) is immediate from (i) and (ii).   □

The next result is presumably well known, but we have been unable to find a reference for it.

Proposition 1.13. *Let G be a simple algebraic group over $K = \bar{\mathbb{F}}_p$, and let σ be a Frobenius morphism of G such that $O^{p'}(G_\sigma) = G(q)$, a finite group of Lie type over $\mathbb{F}_q$. Suppose H is a closed, connected, simple subgroup of G which is σ-stable. Then $O^{p'}(H_\sigma) = H(q)$, a group of Lie type over the same field $\mathbb{F}_q$.*

Proof. Assume first that G_σ is not a Suzuki or Ree group, so that σ is defined over $\mathbb{F}_q$. By [SS, I, 2.9], σ fixes a Borel subgroup B_H of H, and a maximal torus T_H therein. Let $U_H = R_u(B_H)$. Notice that since σ is defined over $\mathbb{F}_q$, H_σ is also not a Suzuki or Ree group; hence there exists a σ-stable long root subgroup U of U_H.

We claim that $|U_\sigma| = q$. To see this, observe first that since the action of σ on $U \cong K^+$ is defined over $\mathbb{F}_q$ (see [Bor1, AG 14.4]), we have $K[U] = K[x] \cong K \otimes \mathbb{F}_q[x]$. Hence the comorphism σ^* induces the q-power map $x \to x^q$ on $\mathbb{F}_q[x]$. There is a polynomial f such that $\sigma(u) = f(u)$ for all $u \in U$, and we have

$$f(u) = x(\sigma(u)) = \sigma^*(x)(u) = x^q(u) = u^q.$$

Hence $|U_\sigma| = q$, as claimed.

We have now shown that a long root subgroup of H_σ has order q, from which the conclusion follows.

Finally, assume that G_σ is a Suzuki or Ree group. Then σ^2 is a q-power field morphism of G, and from the above proof we have $|U_{\sigma^2}| = q$, where U is a σ^2-stable long root subgroup of H. As σ squares to σ^2, it follows that either σ induces a field morphism of H, or H_σ is a Suzuki or Ree group. In the latter case we have the result. The former leads to a contradiction, as here we can choose U to be σ-stable and obtain $|U_\sigma| = q^{1/2}$. But this is impossible as q is an odd power of p.   □

We conclude this section by stating a result concerning the centralizers of semisimple elements of prime order in simple algebraic groups; this result is taken from [GL, 14.1].

Proposition 1.14. *Let G be a simple algebraic group, and let $\alpha_1, \ldots, \alpha_l$ be a system of fundamental roots for the root system of G; let $\alpha_0 = \sum c_i \alpha_i$ be the highest root. Suppose $x \in G$ is a semisimple element of prime order r. Then $C_G(x)^0$ is a*

reductive subgroup of maximal rank in G, and if Δ is the Dynkin diagram of the root system of $(C_G(x)^0)'$, then one of the following holds:

(i) *Δ is obtained by deleting nodes from the Dynkin diagram of G;*

(ii) *Δ is obtained from the extended Dynkin diagram of G by deleting one node α_i, where $r = c_i$.*

2. Proof of Proposition 2 and Corollary 3

The proof of Proposition 2 relies upon the following result.

Proposition 2.1. *Let G be a simple algebraic group over the algebraically closed field K, let T be a maximal torus of G and let $X(T)$ be the character group of T. For any subgroup L of $X(T)$, define*

$$\mathrm{Ann}(L) = \{t \in T : \lambda(t) = 1 \text{ for all } \lambda \in L\}$$

Then the exponent of $\mathrm{Ann}(L)/(\mathrm{Ann}(L))^0$ divides $t(L)$ (the exponent of the torsion subgroup of $X(T)/L$).

Proof. Choose a $\mathbb{Z}$-basis $f_1, \ldots, f_n$ of $X(T)$ such that $n_1 f_1, \ldots, n_r f_r$ is a $\mathbb{Z}$-basis of L, for some positive integers $n_1, \ldots, n_r$. Define a morphism $h : T \to (K^*)^n$ by

$$h(t) = (f_1(t), \ldots, f_n(t)) \quad \text{for } t \in T.$$

Then h has comorphism h^* such that $h^*(\pi_i) = f_i$ (where π_i is the ith projection map on $(K^*)^n$). It follows that h^* is an isomorphism. The surjectivity of h^* implies that h is injective. Since the image of h is closed, dimension considerations imply that h is an isomorphism of algebraic groups. Thus, if

$$T_1 = h^{-1}\{(\alpha_1, \ldots, \alpha_r, 1, \ldots, 1) : \alpha_i \in K^*\},$$

$$T_2 = h^{-1}\{(1, \ldots, 1, \alpha_{r+1}, \ldots, \alpha_n) : \alpha_i \in K^*\},$$

then $T = T_1 \times T_2$, and both T_1 and T_2 are tori. Also, $X(T) = X(T_1) \times X(T_2)$ and $L \leq X(T_1)$.

Write $g_i = n_i f_i$ for $1 \leq i \leq r$, and define $d : T \to (K^*)^n$ by

$$d(t) = (g_1(t), \ldots, g_r(t), 1, \ldots, 1) \quad \text{for } t \in T.$$

Then $\mathrm{Ann}(L) = \ker d$, and $(\ker d)^0 = T_2$. Let $t \in \mathrm{Ann}(L)$ and $e = \exp(X(T_1)/L)$. As $X(T_1)/L$ is the torsion group of $X(T)/L$, we have $e = t(L)$. We need to show that $t^e \in T_2$. This is equivalent to $\chi(t^e) = e\chi(t) = 1$ for all $\chi \in X(T_1)$. However, if $\chi \in X(T_1)$, then $e\chi \in L$, so that $e\chi(t) = 1$, as required. □

Proof of Proposition 2. As in the hypothesis of Proposition 2, let G be a simple adjoint algebraic group, and let x be a semisimple element of G of finite order greater than $mt(\Sigma(G))$, for some positive integer m. We aim to find an infinite closed subgroup S of G satisfying conclusions (i) and (ii) of Proposition 2.

Choose a maximal torus T of G containing x, and let $X(T)$ denote the character group of T. Regard the root system $\Sigma(G)$ of G as a subset of $X(T)$ in the usual way. As G is adjoint, $\mathbb{Z}\Sigma = X(T)$. Define subgroups L, L' of $\mathbb{Z}\Sigma$ as follows:

$$L' = \langle \alpha - \beta : \alpha, \beta \in \Sigma(G) \text{ and } \alpha(x) = \beta(x) \rangle,$$

$$L = \langle \alpha - \beta, \gamma : \alpha, \beta, \gamma \in \Sigma(G), \alpha(x) = \beta(x) \text{ and } \gamma(x) = 1 \rangle.$$

Since $\gamma(x) = 1$ implies that $\gamma(x) = \gamma(x)^{-1}$, hence $2\gamma \in L'$, we see that $t(L)$ (the exponent of the torsion group of $\mathbb{Z}\Sigma/L$) divides $t(L')$. By the definition of $t(\Sigma(G))$, we have $t(L') \le t(\Sigma(G))$, and so $t(L) \le t(\Sigma(G))$.

Let $S = \text{Ann}(L)$, the annihilator of L in T. Clearly $x \in S$. By 2.1, the exponent of S/S^0 divides $t(L)$. Since x has order greater than $mt(\Sigma(G))$, this means that some nontrivial power of x, of order greater than m, lies in S^0. In particular $S^0 \ne 1$. Thus conclusion (i) of Proposition 2 holds.

To prove (ii) of Proposition 2, write

$$L(G) = L_0 \oplus \sum_{\alpha \in \Sigma(G)} L_\alpha,$$

where $L_0 = L(T)$, and for $\alpha \in \Sigma(G)$,

$$L_\alpha = \{v \in L(G) : t(v) = \alpha(t)v \text{ for all } t \in T\}.$$

Let W be an x-invariant subspace of $L(G)$. For a character χ of $\langle x \rangle$, set $W_\chi = \{v \in W : x(v) = \chi(x)v\}$; then

$$W = \sum_\chi W_\chi,$$

the sum being over all irreducible characters χ of $\langle x \rangle$. For nontrivial such χ, we have

$$W_\chi \subseteq L(\chi) = \sum \{L_\alpha : \alpha \in \Sigma(G), \alpha(x) = \chi(x)\}.$$

If $\alpha_1, \ldots, \alpha_k$ are the roots such that $\alpha_i(x) = \chi(x)$, then $\alpha_i - \alpha_j \in L$ for $i \ne j$; hence $\alpha_i(s) = \alpha_j(s)$ for all $s \in S$. It follows that each $s \in S$ acts on $L(\chi)$ as a scalar multiple of the identity, and hence S fixes W_χ. Finally, to see that S fixes $W_1 = C_W(x)$, observe that

$$W_1 \subseteq L_0 + L(1) = L_0 + \sum \{L_\alpha : \alpha \in \Sigma(G), \alpha(x) = 1\}.$$

If $\alpha(x) = 1$ then $\alpha \in L$, hence $\alpha(s) = 1$ for all $s \in S$; thus S acts as the identity on $L_0 + L(1)$, so S fixes W_1. We conclude that S stabilizes W, completing the proof of Proposition 2. $\qquad\square$

Proof of Corollary 3. Assume that G is a simple adjoint algebraic group in characteristic p, and that $X = X(q) < G$, with X quasisimple of Lie type over $\mathbb{F}_q$ ($q = p^e$). Assume also that $q > t(\Sigma(G))m(X)$, where $m(X)$ is the order of the Schur multiplier of $X/Z(X)$.

Observe that X contains a subgroup isomorphic to $SL_2(q)$ or ${}^2B_2(q)$, unless $X = L_2(q)$ or ${}^2G_2(q)$. Excluding the latter possibilities for X, we see that X has a semisimple element x of order at least $q + 1$, and by assumption this is greater than $t(\Sigma(G)).|Z(X)|$; and when $X = L_2(q)$ or ${}^2G_2(q)$, X has an element of order $(q + 1)/(2, p - 1)$ or $q + \sqrt{3q} + 1$ respectively, which is again greater than $t(\Sigma(G)).|Z(X)|$. Applying Proposition 2 with $m = |Z(X)|$, we see that there is an infinite closed subgroup S of G such that $x \in S$, $1 \ne x^n \in S^0 - Z(X)$ for some n, and every x-invariant subspace of $L(G)$ is S-invariant. Define

$$\bar{X} = \langle X, S \rangle^0.$$

Then $x^n \in (X \cap \bar{X}) - Z(X)$; hence, as $X \cap \bar{X}$ is normal in X and X is quasisimple, we have $X < \bar{X}$. Also every X-invariant subspace of $L(G)$ is certainly x-invariant,

hence is also $\langle X, S \rangle$-invariant, and therefore is $\bar{X}$-invariant. This completes the proof of part (i) of Corollary 3.

For part (ii) of Corollary 3, suppose that $X \leq G_\sigma$, where σ is a Frobenius morphism. Let $\mathcal{M}$ be the collection of all X-invariant subspaces of all G-composition factors of $L(G)$, and define

$$Y = \bigcap_{W \in \mathcal{M}} G_W.$$

By 1.12(iii), Y is σ-stable; also Y is clearly closed and contains the subgroup $\bar{X}$ defined above. Clearly Y fixes every X-invariant subspace of each G-composition factor of $L(G)$, so this completes the proof of part (ii) of Corollary 3. $\square$

3. Proof of Theorem 1: Strategy and lemmas

We embark upon the proof of Theorem 1. In this section, after some preliminary observations, we explain the strategy of our proof, which is based on the consideration of the centralizer of a suitably chosen element a of the subgroup X. We then present some general lemmas concerning this centralizer.

Assume that G is a simple adjoint algebraic group of exceptional type over the algebraically closed field K of characteristic p, and that $X = X(q)$ is a quasisimple group of Lie type over $\mathbb{F}_q$ ($q = p^e$), with $X < G$. The case where $X = A_1(q)$, ${}^2B_2(q)$ or ${}^2G_2(q)$ follows from Corollary 3, so we assume that X is not one of these groups. Suppose further that $q > 9$ and $X \neq A_2^\epsilon(16)$ (as in the hypothesis of Theorem 1). We aim to show that X lies in a connected subgroup $\bar{X}$ of G fixing the same subspaces of $L(G)$ as X. This is enough to complete the proof of Theorem 1, since part (ii) of Theorem 1 follows exactly as at the end of §2.

We begin with a lemma which restricts attention to the case where $G = E_8$.

Lemma 3.1. *If Theorem 1 holds for $G = E_8$, then it holds in all cases.*

Proof. Suppose that Theorem 1 holds for $G = E_8$. We deduce the result for $G = E_7$ as follows. Suppose $X = X(q) < E_7$, with q as in the hypothesis of Theorem 1. By the conclusion of Theorem 1 for E_8, there is a connected subgroup $\bar{X}$ of E_8 containing X which fixes exactly the same subspaces of $L(E_8)$ as X does. Since X fixes $L(E_7)$, so does $\bar{X}$, and therefore $\bar{X}$ lies in $N_{E_8}(L(E_7))$; this is equal to A_1E_7 (since it certainly contains A_1E_7, which is maximal in E_8). Passing to the actions of $X, \bar{X}$ on $L(E_7)$, and projecting to the adjoint group E_7, we have the conclusion of Theorem 1 for E_7. Theorem 1 for $G = E_6, F_4, G_2$ follows using the same argument. $\square$

In view of Lemma 3.1, we assume from now on that $G = E_8$.

Observe that X contains a subgroup A isomorphic to $A_2^\epsilon(q)$ ($\epsilon = \pm$) or $B_2(q)$. There is an element $a \in A$ such that $C_A(a)^{(\infty)} = S \cong SL_2(q)$ generated by long root subgroups of A, and such that a has order divisible by $(q - \epsilon)/(3, q - \epsilon)$ if $A \cong A_2^\epsilon(q)$, and divisible by $q + 1$ if $A \cong B_2(q)$. By [SS, II, 4.4], $C_G(a)$ is a connected reductive group; write

$$D = C_G(a)',$$

so that D is connected and semisimple, and $S < D$. Note that $o(a) \geq 6$, except when $X = L_3(13)$ or $U_3(11)$.

The strategy of our proof is based on the following elementary lemma.

Lemma 3.2. *Suppose that there is a subgroup Y of X and a closed connected subgroup Z of G such that*

(i) $X \cap Z \not\leq Z(X)$, *and*

(ii) *every Y-invariant subspace of $L(G)$ is Z-invariant.*

Then the conclusion of Theorem 1 holds.

Proof. Take $\bar{X} = \langle X, Z \rangle^0$. Then $X \cap \bar{X}$ contains $X \cap Z$, so $X \cap \bar{X} \not\leq Z(X)$, and hence $X < \bar{X}$. By (ii), X and $\bar{X}$ fix the same subspaces of $L(G)$. □

In the ensuing proof, we attempt to fulfil the hypotheses of 3.2 using two basic methods. The first method is to try to find a torus T_k such that $\langle a \rangle \cap T_k$ and T_k fix the same subspaces of $L(G)$. Then, assuming $\langle a \rangle \cap T_k \not\leq Z(X)$, the hypotheses of 3.2 hold, taking $Y = \langle a \rangle \cap T_k$, $Z = T_k$.

This method only works in relatively few cases (usually when $D = C_G(a)'$ has large rank). In other cases, we use the following approach. Observe that $S < D$, and with a couple of exceptions, D is a product of classical groups. By studying the possible embeddings of S in D, we attempt to find a closed subgroup $\bar{S} \cong SL_2(K)$ of D containing S, such that every composition factor of $\bar{S}$ on $L(G)$ has high weight less than q. Then by 1.5, S and $\bar{S}$ fix the same subspaces of $L(G)$, so the hypotheses of 3.2 hold with $Y = S, Z = \bar{S}$. It turns out (after much work) that this approach is successful in most cases; the remaining cases are usually dealt with using the representation theory of the subgroup A.

It is convenient to carry out the above strategy by contradiction; thus, in view of 3.2, we assume

(†)   there are no subgroups $Y \leq X$, $Z \leq G$ such that Z is closed and connected, $X \cap Z \not\leq Z(X)$ and every Y-invariant subspace of $L(G)$ is Z-invariant.

We consider now the possibilities for $C_G(a)$. Recall that the extended Dynkin diagram of $G = E_8$ is
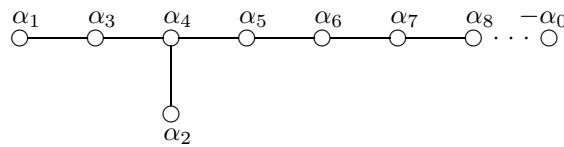
and the highest root $\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$.

Lemma 3.3. *Suppose that $o(a) \geq 6$. Then one of the following holds:*

(i) $C_G(a) = T_k D$, *where T_k is a central torus of rank $k \geq 1$, and D is a connected semisimple group;*

(ii) $C_G(a) = A_1 A_2 A_5$ *and $o(a) = 6$.*

Proof. This follows easily from 1.14: if (i) does not hold, then every power of a of prime order r has centralizer obtained by deleting a node from the extended Dynkin diagram of G corresponding to a coefficient r in the above expression for α_0. Repeating this process, we find that the only possibility with $o(a) \geq 6$ is that in conclusion (ii). □

Lemma 3.4. *Suppose that $o(a) \geq 6$ and $C_G(a) = T_1 D$, with T_1 a rank 1 torus and D semisimple. Then $C_G(T_1) = L_i$, a Levi subgroup corresponding to the Dynkin diagram of G with the node α_i deleted. Moreover $D \leq L_i$, and one of the possibilities in Table 1 holds.*

TABLE 1

L_i	D	$o(a)$	possible q
D_7	$A_1 A_1 D_5$ or $A_3 D_4$	$o(a^2) \leq 4$	$17, 19, 23, 25$
A_7	A_7	$o(a) \leq 6$	$17, 19$
$A_1 A_6$	$A_1 A_6$	$o(a) \leq 8$	
$A_1 A_2 A_4$	$A_1 A_2 A_4$	$o(a) \leq 12$	
$A_3 A_4$	$A_3 A_4$	$o(a) \leq 10$	
$A_2 D_5$	$A_2 D_5$	$o(a) \leq 8$	$17, 19, 23, 25$
	$A_2 A_1 A_1 A_3$	$o(a^2) \leq 8$	
$A_1 E_6$	$A_1 E_6$	$o(a) \leq 6$	$17, 19$
	$A_1 A_1 A_5$	$o(a^2) \leq 6$	
	$A_1 A_2 A_2 A_2$	$o(a^3) \leq 6$	
E_7	$A_1 D_6$ or A_7	$o(a^2) \leq 4$	$17, 19, 23, 25$
	$A_1 A_3 A_3$	$o(a^4) \leq 4$	
	$A_2 A_5$	$o(a^3) \leq 4$	

Proof. Observe that $C_G(T_1)$ is a Levi subgroup of G containing D, a semisimple group of rank 7; hence $C_G(T_1) = L_i$ for some i.

First suppose that $i = 8$, so $L_i = E_7$. Since the α_8-coefficient in the highest root α_0 is 2, the torus T_1 has just 5 distinct weights $2, 1, 0, -1, -2$ on $L(G)$; indeed, this can be seen by writing a typical element $T_1(c)$ of T_1 as

$$h_{\alpha_1}(c^2) h_{\alpha_2}(c^3) h_{\alpha_3}(c^4) h_{\alpha_4}(c^6) h_{\alpha_5}(c^5) h_{\alpha_6}(c^4) h_{\alpha_7}(c^3) h_{\alpha_8}(c^2).$$

If $a \in T_1$, then since $o(a) \geq 6$, every a-invariant subspace of $L(G)$ is also T_1-invariant; hence the subgroups $\langle a \rangle$ of X and T_1 of G satisfy (i) and (ii) of 3.2, contrary to our assumption (†). Therefore $a \notin T_1$, and so $a = a_1 a_2$ with $a_1 \in T_1$ and $1 \neq a_2 \in L_8 = E_7$ and $a_2 \notin Z(E_7)$. We have $C_{E_7}(a_2) = D$, a proper semisimple subgroup of E_7 of rank 7. Using 1.14 again, we see that the only possibilities for D are $A_1 D_6$, A_7, $A_1 A_3 A_3$, $A_2 A_5$, and $a_2^r \in Z(E_7)$ with $r = 2, 2, 4, 3$, respectively. Then $a^r \in T_1$; replacing a by a^r and using the previous argument, we conclude that $o(a^r) < 5$. This completes the proof for $i = 8$, and the other values of i are handled in the same way. $\square$

Lemma 3.5. *Assuming $o(a) \geq 6$, we have $SL_2(q) \cong S \leq C_X(a)' < C_G(a)' = D$, and one of the following holds:*

(1) $D = A_1 A_2 A_5$, $o(a) = 6$ and $q = 17$ or 19;
(2) D is as in row $1, 2, 6, 8$ or 11 of Table 1;
(3) $D = D_6 = L_{18}$;
(4) $D = E_6 = L_{78}$ and $p = 2$;

(5) *D is contained in one of the following subgroups of G:*

$$
\begin{array}{ll}
A_1A_3A_3,\ A_2A_5 & (\subseteq L_8 = E_7) \\
A_1A_1A_5,\ A_1A_2A_2A_2 & (\subseteq L_7 = A_1E_6) \\
A_2A_1A_1A_3 & (\subseteq L_6 = A_2D_5) \\
A_1A_1D_4 & (\subseteq L_{18} = D_6) \\
A_3A_4,\ A_1A_2A_4,\ A_1A_6 & (= L_5, L_4, L_3\ resp.) \\
A_2D_4,\ A_1D_5 & (= L_{16}, L_{17}\ resp.).
\end{array}
$$

Proof. By 3.3, either (1) holds or $C_G(a) = T_kD$ with $k \geq 1$, so assume the latter holds. When $k = 1$, the possibilities are as listed in Table 1, and visibly either (2) or (5) holds.

Now assume that $k = 2$, so $D \leq C_G(T_2) = L_{ij}$ for some i, j. If i or j is in $\{3, 4, 5\}$, or if $\{i, j\}$ is $\{1, 6\}$ or $\{1, 7\}$, then (5) holds. Also $L_{12}, L_{26}, L_{27}, L_{28}$, L_{67} and L_{68} lie in conjugates of subgroups listed in (5) of $L_3, L_4, L_7, L_3, L_{17}$ and L_{17}, respectively. And if $\{i, j\} = \{1, 8\}$ or $\{7, 8\}$ then $L_{ij} = D_6$ or E_6, and either $D = L_{ij}$ or D lies in $C_{L_{ij}}(a)$, a proper semisimple subgroup of L_{ij} of rank 6; the only possibilities for the latter are $A_1A_1D_4, A_3A_3$ (if $L_{ij} = D_6$), or $A_1A_5, A_2A_2A_2$ (if $L_{ij} = E_6$), all of which occur under (5). If $D = L_{18}$ then (3) holds. Finally, if $D = L_{78} = E_6$, then $p = 2$, as in (4): for if $p \neq 2$ then $Z(S) = \langle t \rangle \cong Z_2$, and $D \leq C_{L_{78}}(t)$, a contradiction.

To complete the proof, observe that if $k \geq 3$, then D lies in a Levi subgroup L_{ijk} of co-rank 3. We claim that all such Levi subgroups lie in conjugates of subgroups under (5). To see this, observe that as L_3, L_4, L_5, L_{16} and L_{17} are under (5), we may assume that $\{i, j, k\}$ does not contain $3, 4, 5, \{1, 6\}$ or $\{1, 7\}$. Hence $ijk = 128, 267, 268, 278$ or 678. One checks that in each case L_{ijk} lies in a subgroup under (5), giving the claim. This completes the proof. $\square$

Lemma 3.6. *Suppose that $a \in A_2 = C_G(E_6)$. Then p is not 2 or 3; in particular, case (4) of Lemma 3.5 does not occur.*

Proof. Assume $p = 2$ or 3. Then $q \geq 16$ if $p = 2$ and $q \geq 27$ if $p = 3$. Hence either $o(a) \geq 26$, or $o(a) = 15$ or 17 (with $q = 16$), or $o(a) = 11$ (with $q = 32$), or $o(a) = 21$ (with $q = 64$). Choose a torus $T_2 < A_2$ containing a. We may take T_2 to consist of diagonal 3×3 matrices, and write $a = \mathrm{diag}\,(\alpha, \beta, \alpha^{-1}\beta^{-1})$. The nontrivial composition factors of A_2 on $L(G)$ are $V_{A_2}(10), V_{A_2}(01)$ and $V_{A_2}(11)$. Hence the eigenvalues of a on $L(G)$ are

$$
1, \alpha, \beta, \alpha\beta, \alpha\beta^{-1}, \alpha^2\beta, \alpha\beta^2, \alpha^{-1}, \beta^{-1}, \alpha^{-1}\beta^{-1}, \alpha^{-1}\beta, \alpha^{-2}\beta^{-1}, \alpha^{-1}\beta^{-2}.
$$

If these are all distinct, then T_2 fixes the same subspaces of $L(G)$ as a, contrary to assumption (†). Therefore two of the above eigenvalues are equal, and so one of the following holds:

(i) $\alpha^k = 1$, $\beta^k = 1$ or $\alpha^k = \beta^{-k}$, for some $k \in \{1, 2, 3\}$;

(ii) $\alpha^l = \beta^l$, $\alpha^{2l} = \beta^{-l}$ or $\alpha^{-l} = \beta^{2l}$, for some $l \in \{1, 2\}$;

(iii) $\alpha^2 = \beta$, $\alpha = \beta^2$, $\alpha^3 = \beta^{-1}$, $\alpha^{-1} = \beta^3$, $\alpha^3 = \beta^{-2}$ or $\alpha^2 = \beta^{-3}$.

Consider (i). Here $a^k \in T_1$, where T_1 is a $W(A_2)$-conjugate of $\{\mathrm{diag}(1, c, c^{-1}) : c \in K^*\}$. The weights of T_1 on $L(G)$ are $2, 1, 0, -1, -2$, so T_1 and a^k fix the same subspaces of $L(G)$ provided $o(a^k) \geq 5$, which is true; this contradicts (†) (note that $a^k \notin Z(A)$, so $a^k \notin Z(X)$).

In case (ii), a^l lies in T_1, a $W(A_2)$-conjugate of $\{\mathrm{diag}(c, c, c^{-2}) : c \in K^*\}$. Then T_1 has weights $3, 2, 1, 0, -1, -2, -3$. Hence T_1 and a^l fix the same subspaces provided $o(a^l) \geq 7$, which is true; this again contradicts (†).

Finally in case (iii), a lies in T_1, a $W(A_2)$-conjugate of $\{\mathrm{diag}(c, c^2, c^{-3}) : c \in K^*\}$, which has weights $5, 4, 3, \ldots, -5$. Then T_1 and a fix the same subspaces since $o(a) \geq 11$. $\qquad\square$

Lemma 3.7. *Suppose $a \in A_1 A_1 = C_G(D_6)$. Then either q is prime or $q = 25$; in particular, this holds in case (3) of Lemma 3.5. Further, if $q = 25$ then $D \neq A_1 D_6$.*

Proof. The argument is similar to that of the previous lemma. Assume that q is not prime and that $q \neq 25$. Then either $q \geq 49$ or $q \in \{16, 27, 32\}$; hence either $o(a) \geq 15$ or $o(a) = 11$. We have $a \in T_2 < A_1 A_1 = C_G(D_6)$. Write elements of T_2 as (c, d) $(c, d \in K^*)$, where this represents the element $(\mathrm{diag}(c, c^{-1}), \mathrm{diag}(d, d^{-1}))$ of $A_1 A_1 = SL_2 \times SL_2$. Take $a = (\alpha, \beta)$. Since the nontrivial composition factors of $A_1 A_1$ on $L(G)$ are $1 \otimes 0, 0 \otimes 1, 2 \otimes 0, 0 \otimes 2$ and $1 \otimes 1$, the eigenvalues of a on $L(G)$ are

$$1, \alpha, \beta, \alpha^2, \beta^2, \alpha\beta, \alpha\beta^{-1}, \alpha^{-1}, \beta^{-1}, \alpha^{-2}, \beta^{-2}, \alpha^{-1}\beta^{-1}, \alpha^{-1}\beta.$$

As in the previous lemma, two of these must be equal. Hence one of the following holds:

(i) $\alpha^k = 1$ or $\beta^k = 1$ for some $k \in \{1, 2, 3, 4\}$;
(ii) $\alpha = \beta^{\pm 1}$ or $\alpha^2 = \beta^{\pm 2}$;
(iii) $\alpha = \beta^{\pm 2}$ or $\alpha^{\pm 2} = \beta$;
(iv) $\alpha = \beta^{\pm 3}$ or $\alpha^{\pm 3} = \beta$.

In (ii), a^2 lies in $T_1 = \{(c, c) : c \in K^*\}$ or $\{(c, c^{-1}) : c \in K^*\}$, both of which have weights $2, 1, 0, -1, -2$; hence T_1 and a^2 fix the same subspaces of $L(G)$ provided $o(a^2) \geq 5$, which is true.

In (iii) or (iv), $a \in T_1 = \{(c, c^m) : c \in K^*\}$ or $\{(c^m, c) : c \in K^*\}$, where $m \in \{\pm 2, \pm 3\}$. If $m = \pm 2$ the weights of T_1 are $4, 3, 2, \ldots, -4$, so T_1 and a fix the same spaces provided $o(a) \geq 9$; this is true by our assumption on q. And if $m = \pm 3$, the weights of T_1 are $\pm 6, \pm 4, \pm 3, \pm 2, \pm 1$; hence T_1 and a fix the same spaces if $o(a) \geq 13$ or $o(a) = 11$, which is again true.

Finally, consider (i). Here a^k lies in $T_1 = \{(1, c) : c \in K^*\}$ or $\{(c, 1) : c \in K^*\}$, which has weights $2, 1, 0, -1, -2$; hence T_1 and a^k fix the same subspaces provided $o(a^k) \geq 5$. By our assumption on q, this is true unless $k = 4$ and $q = 49$ (in which case $o(a)$ could be 16). So suppose $k = 4$ and $q = 49$, and without loss assume that $\alpha^4 = 1$. Recall, that $a = (\alpha, \beta) \in A_1 A_1$. The composition factors of $A_1 A_1$ on $L(G)$ are $0 \otimes 0, 1 \otimes 0, 0 \otimes 1, 2 \otimes 0, 0 \otimes 2$ and $1 \otimes 1$. Hence we check that $T_1 = \{(1, c) : c \in K^*\}$ fixes the same spaces as a itself. Finally, if $q = 25$ and $D = A_1 D_6$, then $o(a) = 8$ by 3.4, and we may take $a = (\alpha, \pm 1)$; as above, a and $T_1 = \{(1, c) : c \in K^*\}$ fix the same spaces. This completes the proof. $\qquad\square$

4. Proof of Theorem 1 for $p \neq 2$ and $q > 13$

We continue with the proof of Theorem 1. In this section we handle the case where $p \neq 2$ and $q > 13$. Recall, from the discussion following Lemma 3.2, that our basic strategy is to try to find a subgroup $\bar{S} \cong SL_2(K)$ of $D = C_G(a)'$ such that $\bar{S}$ contains S and fixes the same subspaces of $L(G)$ as S.

The next lemma provides some useful information for this strategy; it will be used in the following sections also, so we make no assumptions on p and only assume $q > 9$ in the statement.

To ease notation, we make the following definition. Suppose $D = E_1 \ldots E_k$, a product of simple factors E_i, and write $Z = Z(D)$. Denote by π_i the ith projection map $D/Z \to E_i Z/Z$. By the *projection* S_i of S in E_i, we mean a minimal preimage in E_i of the group $(SZ/Z)\pi_i$. If E_i is classical, with usual module V_i, we abuse notation by writing $V_i \downarrow S$ instead of $V_i \downarrow \tilde{S}_i$, where $\tilde{S}_i$ is the derived group of the preimage of S_i in a simply connected cover of E_i. And if each of the E_i is classical, we usually describe the embedding of S in D by giving the composition factors of $V_1 \downarrow S, \ldots, V_k \downarrow S$. We can always adjust these factors by applying (globally) any automorphism of S (this amounts to relabelling the elements of S).

Lemma 4.1. *Let E be a simple factor of D of classical type, where D is as in* $(1), (2), (3)$ *or* (5) *of Lemma 3.5, and let V be the usual module for E. Suppose that $V \downarrow S$ is not completely reducible. Then one of the following holds:*

(i) $p = 2$ and $V \downarrow S$ has an indecomposable section $1^{(2^i)}/0$ or $1^{(2^i)} \otimes 1^{(2^j)}/1^{(2^j)}$ (with $i \neq j$);

(ii) $p = 3$ and $V \downarrow S$ has an indecomposable section $1^{(3^i)}/1^{(3^{i+1})}$, $1^{(3^i)} \otimes 1^{(3^{i+1})}/0$ or $1^{(3^i)} \otimes 1^{(3^{i+1})}/2^{(3^{i+1})}$;

(iii) $p = 5$ and $V \downarrow S$ has an indecomposable section $3^{(5^i)}/1^{(5^{i+1})}$, $3^{(5^i)} \otimes 1^{(5^{i+1})}/0$, $2^{(5^i)}/1^{(5^i)} \otimes 1^{(5^{i+1})}$ or $2^{(5^i)} \otimes 1^{(5^{i+1})}/1^{(5^i)}$;

(iv) $q = p = 11$ and $V \downarrow S$ has an indecomposable section $4/4$, $5/5$ or $0/8/0$;

(v) $q = p = 13$ and $V \downarrow S$ has an indecomposable section $5/5$.

Proof. We have $\dim V \leq 12$; moreover, if $\dim V \geq 8$ then $E = D_4, D_5, D_6$ or A_7 (with q prime or $q = 25$ in the latter two cases). We use Proposition 1.2, which determines the 2-step indecomposables for $S = SL_2(q)$.

First assume that $q = p$. By 1.2(i), if there is an indecomposable W for S with composition factors a/b, then $a + b = p - 3$ or $p - 1$, whence $\dim W \geq a + 1 + b + 1 \geq p - 1$. As $q > 9$, we have $p - 1 = q - 1 \geq 10$. Therefore $E = D_5$ or D_6. Thus $V \downarrow S$ is self-dual, and it follows that either (iv) or (v) holds, or $E = D_6, p = 11$ and $V \downarrow S$ has an indecomposable section $a/b/a$. The dimension of this section is $2a + b + 3 \leq 12$, so $(a, b) = (0, 8)$ or $(1, 7)$. The first case is in conclusion (iv). And in the second case, the middle factor (of high weight 7) is symplectic, which is impossible.

Now assume $q = p^e$ with $e \geq 2$. By 1.2(ii), the following is a list of all 2-step indecomposables for S of dimension at most 10 (of dimension at most 12 if $q = 25$):

$p = 2 : 1^{(2^i)}/0, 1^{(2^i)} \otimes 1^{(2^j)}/1^{(2^j)}$

$p = 3 : 1^{(3^i)}/1^{(3^{i+1})}, 1^{(3^i)} \otimes 1^{(3^{i+1})}/0, 1^{(3^i)} \otimes 1^{(3^{i+1})}/2^{(3^{i+1})}, 1^{(3^i)} \otimes 2^{(3^{i+1})}/1^{(3^{i+1})}$

$\qquad 1^{(3^i)} \otimes 1^{(3^j)}/1^{(3^{i+1})} \otimes 1^{(3^j)}, 1^{(3^i)} \otimes 1^{(3^{i+1})} \otimes 1^{(3^j)}/1^{(3^j)}$

$p = 5 : 3^{(5^i)}/1^{(5^{i+1})}, 3^{(5^i)} \otimes 1^{(5^{i+1})}/0, 3 \otimes 1^{(5)}/2^{(5)} (q = 25),$

$\qquad 2^{(5^i)}/1^{(5^i)} \otimes 1^{(5^{i+1})}, 2^{(5^i)} \otimes 1^{(5^{i+1})}/1^{(5^i)}, 2 \otimes 1^{(5)}/1 \otimes 2^{(5)} (q = 25)$

$p = 7 : 5^{(7^i)}/1^{(7^{i+1})}, 4^{(7^i)}/1^{(7^i)} \otimes 1^{(7^{i+1})}, 3^{(7^i)}/2^{(7^i)} \otimes 1^{(7^{i+1})}.$

If $E = A_r$ then $\dim V \leq 8$ (and $q = 25$ if $\dim V = 8$), and the only possible indecomposables appearing in $V \downarrow S$ are among those in conclusions (i)-(iii). Otherwise, $E = D_r$ with $4 \leq r \leq 6$ (and $q = 25$ if $E = D_6$); as V is orthogonal, we see using

1.1 that the last three indecomposables listed for $p = 3$, the third, fifth and sixth listed for $p = 5$, and all the indecomposables listed for $p = 7$ cannot occur in $V \downarrow S$. Hence again one of (i)-(iii) holds. $\qquad\square$

From now on in this section we shall assume that

$$p \neq 2 \quad \text{and} \quad q > 13.$$

Lemma 4.2. *Suppose that D is as in $(1), (2), (3)$ or (5) of Lemma 3.5, but $D \neq A_1 E_6$ (and that $p \neq 2, q > 13$). Thus $D = E_1 \ldots E_k$, a product of classical simple factors E_i with usual module V_i; let S_i be the projection of S in E_i. Then one of the following holds:*

(i) for each i, $V_i \downarrow S_i$ is completely reducible;

(ii) for some j, $V_j \downarrow S_j$ is not completely reducible; moreover, there is a subgroup $\bar{S}_j \cong A_1(K)$ of E_j containing S_j, and the composition factors of $V_j \downarrow \bar{S}_j$ are as in Table 2 (up to a twist);

(iii) $q = 27$, $S < D = A_5$ or A_6 with embedding $1/1^{(3)}/1^{(9)}/0^m$ (where $m = 0$ or 1);

(iv) $q = 27$, $S < D = A_1 D_5$ with embedding $1^{(9)}$, $(1 \otimes 1^{(3)})^2/0^2$;

(v) $q = 27$, $S < D = A_1 D_5$ with embedding $1^{(9)}$, $(2^{(3)})^2/1 \otimes 1^{(3)}$.

<div align="center">

TABLE 2

p	E_j	$V_j \downarrow \bar{S}_j$
5	$A_{5+\delta}\,(\delta = 0, 1)$	$3/1^{(5)}/0^\delta$
	$A_{6+\delta}\,(\delta = 0, 1)$	$2/1 \otimes 1^{(5)}/0^\delta$
	$A_7\,(q = 25)$	$2 \otimes 1^{(5)}/1$
	$D_{5+\delta}\,(\delta = 0, 1)$	$3 \otimes 1^{(5)}/0^{2+2\delta}$
	$D_{5+\delta}\,(\delta = 0, 1)$	$2^2/1 \otimes 1^{(5)}/0^{2\delta}$
3	$A_{3+\delta}\,(\delta = 0, 1, 2, 3)$	$1/1^{(3)}/0^\delta$
	A_4	$1 \otimes 1^{(3)}/0$
	A_6	$1 \otimes 1^{(3)}/1^{(3^i)}/0$
	$D_{4+\delta}\,(\delta = 0, 1)$	$1^2/(1^{(3)})^2/0^{2\delta}$
	$D_{4+\delta}\,(\delta = 0, 1)$	$1 \otimes 1^{(3)}/0^{4+2\delta}$
	D_5	$1 \otimes 1^{(3)}/1^{(3^i)}/1^{(3^i)}/0^2$
	D_5	$1 \otimes 1^{(3)}/1^{(3^i)} \otimes 1^{(3^l)}/0^2$

</div>

Proof. Suppose (i) does not hold, so that some $V_j \downarrow S$ is not completely reducible. Write $E = E_j, V = V_j$. By 4.1, $p = 3$ or 5 and $V \downarrow S$ has a section as in 4.1(ii) or 4.1(iii). Let T be a Cartan subgroup of S.

Assume first that $p = 5$. Relabelling S, we may take it that $V \downarrow S$ has an indecomposable section $3/1^{(5)}$, $2 \otimes 1^{(5)}/1$, $3 \otimes 1^{(5)}/0$ or $2/1 \otimes 1^{(5)}$. In the first case, using 1.1 we see that $E = A_5, A_6$ or D_6. Suppose $E = A_5$ or A_6; then $V \downarrow S = 3/1^{(5)}$ or $(3/1^{(5)}) \oplus 0$. By 1.3(i), the restriction map $\text{Ext}^1_{A_1(K)}(3, 1^{(5)}) \to \text{Ext}^1_{A_1(q)}(3, 1^{(5)})$ is an isomorphism. It follows that there is a closed subgroup $A_1(K)$ of E containing S_j, as required for (ii). Now suppose $E = D_6$; then $D = E$, $q = 25$ by 3.7, and $V \downarrow S = 1^{(5)}/3/3/1^{(5)}$. Pick an element $t \in S$ of order 26; then t lies in a Levi subgroup A_5 of E, and we may take it that t lies in a 1-dimensional torus $T_1 = \{\text{diag}(c^5, c^{-5}, c^3, c^{-3}, c, c^{-1}) : c \in K^*\} < A_5$. As the nontrivial composition

factors of the restriction of $L(G)/L(A_5)$ to A_5 are just $V_{A_5}(\lambda_i)\,(1 \leq i \leq 5)$ (see [LS2, §2]), it follows that t and T_1 fix the same subspaces of $L(G)$, contrary to (†).

Next, if $V \downarrow S$ has an indecomposable section $2 \otimes 1^{(5)}/1$, then $E = A_7$ (use 1.1 again), and we obtain (ii) using 1.3(i), as above.

Now assume $V \downarrow S$ has an indecomposable section $3 \otimes 1^{(5)}/0$. Then $E = D_5$ or D_6. In the first case, $V \downarrow S$ is an indecomposable module $0/3 \otimes 1^{(5)}/0$, and S_j (the projection of S in E) fixes a singular 1-space of V. The stabilizer in E of this 1-space is a parabolic subgroup QL with unipotent radical Q and Levi subgroup $L = D_4T_1$, and Q is a KL-module of high weight λ_1. If we choose a subgroup $\tilde{S} \cong A_1(K)$ of L such that $Q \downarrow \tilde{S} = 3 \otimes 1^{(5)}$, then $S_j < Q\tilde{S}$. Moreover, by 1.3, $H^1(S,Q) \cong H^1(\tilde{S},Q)$, and clearly $C_Q(T) = 0$. Therefore by 1.6, S_j lies in a closed complement to Q in $Q\tilde{S}$, giving (ii). When $E = D_6$ we have $V \downarrow S = 0^2/3 \otimes 1^{(5)}/0^2$. Since $H^1(S, 3 \otimes 1^{(5)})$ has dimension 1 by 1.2(ii), in fact $V \downarrow S = V_2 \perp (0/3 \otimes 1^{(5)}/0)$, where V_2 is a non-degenerate 2-space, so S_j lies in a subgroup D_5 of E, and now the argument given for $E = D_5$ applies.

To complete the case $p = 5$, suppose $V \downarrow S$ has an indecomposable section $2/1 \otimes 1^{(5)}$. Then $E = A_6, A_7, D_5$ or D_6. If $E = A_6$ or A_7, we obtain (ii) using 1.3(i). Now suppose $E = D_5$. Then $V \downarrow S$ is an indecomposable module $2/1 \otimes 1^{(5)}/2$. The group $A_1(K)$ has a 10-dimensional orthogonal module $\bigwedge^2 V(4)$, and $\bigwedge^2 V(4) \downarrow A_1(K) = 2/1 \otimes 1^{(5)}/2$; moreover this module is indecomposable, since a 1-dimensional unipotent subgroup centralizes only a 1-space of weight 2 vectors in $\bigwedge^2 V(4)$. Hence E contains a subgroup $\tilde{S} \cong A_1(K)$ with this representation on the usual module. We may take it that both S_j and $\tilde{S}$ lie in the same parabolic subgroup QL stabilizing a totally singular 3-space. The unipotent radical Q has an $\tilde{S}$-series $1 < Q_1 < Q$ with $Q_1, Q/Q_1$ abelian, such that $Q_1 \downarrow \tilde{S} = V(2)$ and $(Q/Q_1) \downarrow \tilde{S} = 2 \otimes 1 \otimes 1^{(5)} = (3 \otimes 1^{(5)}) \oplus (1 \otimes 1^{(5)})$. By 1.2, $H^1(\tilde{S}, Q/Q_1)$ is 1-dimensional. Hence, under the action of $(Q/Q_1)Z(L)$ there are just two classes of closed complements to Q/Q_1 in $(Q/Q_1)\tilde{S}$, with representatives $\tilde{S}$ and $\bar{S}$, where $\bar{S} = L \cap Q\tilde{S}$. As $H^1(\tilde{S}, Q/Q_1) \cong H^1(S, Q/Q_1)$ by 1.3, and $C_{Q/Q_1}(T) = 0$, we deduce from 1.6 that S_j therefore lies in a conjugate of either $Q_1\tilde{S}$ or $Q_1\bar{S}$. Since also by 1.3, $H^1(\tilde{S}, Q_1) \cong H^1(S, Q_1) = 0$, S lies in a conjugate of $\tilde{S}$ or $\bar{S}$, giving the conclusion.

Finally, suppose $E = D_6$. If S_j lies in a subgroup D_5 of E, then the conclusion follows as before. Otherwise, $V \downarrow S = (1 \otimes 1^{(5)}/2/1 \otimes 1^{(5)}) \oplus 0$; also $q = 25$ and $D = E$, by 3.7. An element t of S of order 26 lies in a 1-dimensional torus

$$T_1 = \{\mathrm{diag}(c^6, c^4, c^{-4}, c^{-6}, c^2, 1, c^{-2}, c^6, c^4, c^{-4}, c^{-6}, 1) : c \in K^*\} < D_6$$

(matrices relative to a suitable basis). The nontrivial composition factors of $L(G) \downarrow E$ are $V_{D_6}(\lambda_i)$ $(i = 1, 2, 5, 6)$. Moreover, T_1 lies in a Levi subgroup A_4 of D_6, which by [LS2, 2.6] has all its nontrivial composition factors on $V_{D_6}(\lambda_i)$ $(i = 5, 6)$ among the modules $V_{A_4}(\lambda_1), \ldots, V_{A_4}(\lambda_4)$. Hence the eigenvalues on $L(G)$ of a typical element $T_1(c)$ of T_1 displayed above are among $c^{\pm k}$ with $0 \leq k \leq 12$. It follows that T_1 and t fix the same subspaces of $L(G)$, contrary to (†). This completes the proof for $p = 5$.

Now suppose that $p = 3$. By 4.1, $V \downarrow S$ has an indecomposable section $1 \otimes 1^{(3)}/0$, $1/1^{(3)}$ or $1 \otimes 1^{(3)}/2^{(3)}$. Consider the first case. Here $E = A_r, D_4$ or D_5 (note that $E \neq D_6$ by 3.7). Suppose $E = A_r$. Then $r \neq 7$ by 3.4, and $r = 4$ yields to the

usual argument using 1.3(i). If $r = 5$ then the projection S_j of S in E is $PSL_2(q)$, so $D = A_1 A_5$; but then $a \in C_G(A_1 A_5) = A_2 = C_G(E_6)$, contrary to 3.6. The same argument works if $r = 6$, unless $V \downarrow S = (1 \otimes 1^{(3)}/0) \oplus 1^{3^{(i)}}$; but then $S < A_1 A_4 < A_6$ and we can apply the 1.3(i) argument again.

When $E = D_4$ or D_5, one of the following holds:

(a) $V \downarrow S = V_m \perp (0/1 \otimes 1^{(3)}/0)$, with V_m ($m = 2$ or 4) a non-degenerate m-space on which S acts completely reducibly;

(b) $E = D_5$ and $V \downarrow S = 0/1 \otimes 1^{(3)}/1^{(3^i)} \otimes 1^{(3^{i+1})}/0$ (with S fixing a singular 1-space);

(c) $E = D_5$ and $V \downarrow S = (1 \otimes 1^{(3)}/0/1 \otimes 1^{(3)}) \perp V_1$ (with S fixing a singular 4-space).

In case (a), S_j lies in a subgroup of E of type $SO_m \times SO_6$, and we can use the usual H^1 argument for the SO_6 to obtain conclusion (ii) of the lemma. In case (b), observe that S_j lies in a subgroup $Q\tilde{S}$ of E fixing a singular 1-space, with $\tilde{S} \cong A_1(K)$ and $Q \downarrow \tilde{S} = (1 \otimes 1^{(3)}) \oplus (1^{(3^i)} \otimes 1^{(3^{i+1})})$. Now by 1.2, $H^1(S, Q)$ and $H^1(\tilde{S}, Q)$ both have dimension 2, and $C_Q(T) = 0$, so conclusion (ii) follows from 1.6. In case (c), the projection S_j of S in E is $PSL_2(q)$, so $D = A_1 D_5$ (note that D has rank at most 6 by 3.4). Let the projection of S in the factor A_1 be $1^{(3^k)}$. Pick $t \in S$ of order $q + 1$; then t lies in a rank 1 torus T_1 in $D = A_1 D_5$, with

$$T_1 = \{\text{diag}(c^{3^k}, c^{-3^k}), \text{diag}(c^4, c^4, c^2, c^2, c^{-4}, c^{-4}, c^{-2}, c^{-2}, 1, 1) : c \in K^*\}$$

(matrices in D_5 relative to a suitable basis). The nontrivial composition factors of $L(G) \downarrow A_1 D_5$ are $0 \otimes \lambda_i, 1 \otimes \lambda_i$ ($i = 1, 4, 5$) and $1 \otimes 0$, together with $L(A_1)$ and $L(D_5)$. Hence the eigenvalues of a typical element $T_1(c)$ of T_1 shown above on $L(G)$ are

$$c^{\pm 2.3^k}, c^{\pm 3^k \pm 6, 4, 2, 0}, c^{\pm 8, 6, 4, 2, 0}.$$

It follows that t and T_1 fix the same subspaces of $L(G)$, unless $q = 27$ and $3^k = 9$; this is the exceptional embedding in conclusion (iv) of the lemma.

Now suppose $V \downarrow S$ has an indecomposable section $1 \otimes 1^{(3)}/2^{(3)}$. Here $E = A_6, A_7, D_5$ or D_6. In the A_6, A_7 cases, the projection of S in E is $PSL_2(q)$, so D has a further factor; but then D has rank at least 7, which contradicts 3.4; and $E \neq D_6$ by 3.7. Therefore $E = D_5$; hence $D = D_5$ or $A_1 D_5$. We have $V \downarrow S = 2^{(3)}/1 \otimes 1^{(3)}/2^{(3)}$, indecomposable with S fixing a singular 3-space. Pick $t \in S$ of order $q + 1 \geq 28$. If $S < E$ then t lies in a 1-dimensional torus $T_1 = \{\text{diag}(c^6, c^{-6}, c^6, c^{-6}, 1, 1, c^4, c^2, c^{-2}, c^{-4}) : c \in K^*\}$ in a subgroup of type $SO_6 \times SO_4$ in E. The nontrivial composition factors of $L(G) \downarrow D_5$ have high weights $\lambda_1, \lambda_2, \lambda_4$ and λ_5 (see [LS2, §2]). Hence the highest weight of T_1 on $L(G)$ is 12. It follows that t and T_1 fix the same subspaces of $L(G)$, contrary to (†). Hence $S \not< E$, so $D = A_1 D_5$, with embedding $1^{(3^i)}, 2^{(3)}/1 \otimes 1^{(3)}/2^{(3)}$. Now an obvious adjustment of the previous T_1 argument goes through, unless $q = 27$ and $3^i = 9$, as in conclusion (v).

To complete the proof, suppose $V \downarrow S$ has an indecomposable section $1/1^{(3)}$. Here $E = A_r, D_4$ or D_5. In the D_4, D_5 cases, we have $V \downarrow S = (1/1^{(3)}/1^{(3)}/1) \perp V_k$, where $k = 0$ or 2 and V_k is a nondegenerate k-space. Thus $S_j < D_4 \leq E$. As $\text{Ext}^1_S(1, 1^{(3)})$ has dimension 1 by 1.2, $V \downarrow S$ has a submodule $1 \oplus 1^{(3)}$ which is totally singular. Hence S_j lies in a subgroup $Q\tilde{S}$ of E stabilizing this singular 4-space, where $\tilde{S} \cong A_1(K)$ and $Q \downarrow \tilde{S} = \bigwedge^2(1 \oplus 1^{(3)}) \cong (1 \otimes 1^{(3)}) \oplus 0^2$. By 1.2 we have

$H^1(\tilde{S}, Q) \cong H^1(S, Q)$; since also $C_Q(T) = C_Q(\tilde{T})$ (where $\tilde{T}$ is a Cartan subgroup of $\tilde{S}$), the conclusion now follows from 1.6.

Finally, suppose $E = A_r$. If S has only two nontrivial composition factors on V, then $S_j < A_3 A_{r-4} < E$, and we can use the usual 1.3(i) argument for the A_3 factor to obtain the conclusion. So assume that S has more than two nontrivial composition factors on V. In particular, $r \geq 5$. If $r = 7$, or if D has a further factor, 3.4 or 3.6 gives a contradiction. Thus $D = A_5$ or A_6, and $V \downarrow S = 1/1^{(3)}/1^{(3^k)}/0^m$, where $m = r - 5$. Let $t \in S$ have order $q + 1$; then t lies in a rank 1 torus $T_1 = \{\mathrm{diag}(c, c^3, c^{3^k}, c^{-1}, c^{-3}, c^{-3^k}, 1^m) : c \in K^*\}$ of D, and we check that t and T_1 fix the same subspaces of $L(G)$ unless $q = 27$ and $3^k = 9$, as in conclusion (iii) of the lemma. □

We now explain how we propose to use Lemma 4.2. The special configurations in 4.2(iii,iv,v) will be postponed until later (see 4.8). Thus suppose we are in the situation of conclusion (i) or (ii) of 4.2. Consider a module V_i such that $V_i \downarrow S_i$ is completely reducible. Write $V_i \downarrow S_i = \bigoplus_{r=1}^t W_{i_r}$, where the W_{i_r} are irreducible S_i-submodules if E_i is of type A_n, and are minimal non-degenerate S_i-submodules if E_i is of type D_n. Using 1.1, we see that the action of S_i on each W_{i_r} extends to an action of $A_1(K)$, preserving the quadratic form when $E_i = D_n$. Therefore S_i lies in a subgroup $R_i = \bar{S}_{i_1} \ldots \bar{S}_{i_t}$ of E_i, where each $\bar{S}_{i_r} \cong A_1(K)$.

Thus if 4.2(i) or (ii) holds, then S lies in a connected subgroup of D which is a commuting product of A_1's and contains each of the R_i. Abusing notation slightly, write $\bar{S}_1 \ldots \bar{S}_l$ for this commuting product, where each $\bar{S}_i \cong A_1(K)$. So we have

$$S < \bar{S}_1 \ldots \bar{S}_l \leq D.$$

Obviously $\bar{S}_1 \ldots \bar{S}_l$ is an image of $(SL_2(K)^l$. In the ensuing discussion we shall consider subgroups $SL_2(K)$ of $\bar{S}_1 \ldots \bar{S}_l$ containing S. When $l > 1$, there are many such groups, as described below.

Fix matrix groups $SL_2(q) < SL_2(K)$. For powers $p^{i_1}, \ldots, p^{i_l} < q$, the embedding $\phi : x \to (x^{(p^{i_1})}, \ldots, x^{(p^{i_l})})$ of $SL_2(q)$ in $SL_2(K)^l$ extends to an embedding $\bar{\phi} : \bar{x} \to (\bar{x}^{(p^{i_1})}, \ldots, \bar{x}^{(p^{i_l})})$ of $SL_2(K)$ in $SL_2(K)^l$. Every subgroup of $SL_2(K)^l$ which is isomorphic to $SL_2(q)$ and projects nontrivially to each factor is conjugate in $SL_2(K)^l$ to $\mathrm{Im}\, \phi$ for some such ϕ. Notice that it is possible to have such embeddings ϕ, ψ such that $\mathrm{Im}\, \phi = \mathrm{Im}\, \psi$ but $\mathrm{Im}\, \bar{\phi} \neq \mathrm{Im}\, \bar{\psi}$: for example, this is the case if $q = p^2$ and

$$\phi : x \to (x, x^{(p)}), \quad \psi : x \to (x^{(p)}, x).$$

In general, $\mathrm{Im}\, \phi = \mathrm{Im}\, \psi$ means that ψ is obtained from ϕ by applying a fixed field twist to each coordinate (and equating the q-power twist to 1).

Now let ρ be a surjective homomorphism $SL_2(K)^l \to \bar{S}_1 \ldots \bar{S}_l$. By the previous paragraph, we may assume that $S = (SL_2(q))\phi\rho$ for some ϕ as above. Then S lies in $\bar{S}_\phi = SL_2(K)\bar{\phi}\rho$, a closed subgroup of $\bar{S}_1 \ldots \bar{S}_l$ isomorphic to $SL_2(K)$. Consequently the first aim of our strategy, to find a subgroup $SL_2(K)$ of D containing S, is achieved in the situation of 4.2(i,ii).

Observe that if ψ is another embedding of $SL_2(q)$ such that $\mathrm{Im}\, \phi = \mathrm{Im}\, \psi$, then S also lies in $\bar{S}_\psi$. Thus if $l > 1$, then S lies in many subgroups $SL_2(K)$ of $\bar{S}_1 \ldots \bar{S}_l$, obtained from field twists of ϕ. If we can show that one of these subgroups $\bar{S}_\psi$ has

all its composition factors on $L(G)$ having high weight less than q, it will follow from 1.5 that S and $\bar{S}_\psi$ fix the same subspaces of $L(G)$, contradicting our assumption (†), as desired for Theorem 1. In the next few lemmas we show that this can usually be done.

Lemma 4.3. *Suppose that case (5) of 3.5 holds (with $p \neq 2$ and $q > 13$), and suppose also that 4.2(i) holds. Then either we can choose $\bar{S}$ with $S < \bar{S} \leq D$, such that every composition factor of $L(G) \downarrow \bar{S}$ has high weight less than q, or one of the following holds:*

(i) $q = 25$, $S < D \leq A_1A_3A_3$ *with embedding* $0^2, 3, 3^{(5)}$ *or* $1, 3, 3^{(5)}$;

(ii) $q = 25$, $S < D \leq A_3A_4$ *with embedding* $3^{(5)}, 4$ *or* $3^{(5)}, 3/0$.

Proof. We have $S < D \leq Y_1 \ldots Y_r$, a product of classical groups Y_i as in (5) of 3.5. In view of the natural embedding of D in $Y_1 \ldots Y_r$, we may replace D by $Y_1 \ldots Y_r$ in the discussion preceding the lemma. We know from that discussion that there is a closed subgroup $\bar{S} \cong SL_2(K)$ of D containing S. Let V_i be the usual module for Y_i. As usual, we describe the embedding of $\bar{S}$ in $Y_1 \ldots Y_r$ by giving the composition factors of $V_1 \downarrow \bar{S}, \ldots, V_r \downarrow \bar{S}$. We can certainly choose $\bar{S}$ so that all these composition factors have high weight less than q.

Suppose first that all the factors Y_i are of type A_m with $m \leq 4$. Projections of $\bar{S}$ in such factors are given by the following representations (up to field twists):

$$
\begin{aligned}
\bar{S} \to A_1 : &\quad 1 \text{ or } 0^2 \\
\bar{S} \to A_2 : &\quad 2, 1/0 \text{ or } 0^3 \\
\bar{S} \to A_3 : &\quad 3, 2/0, 1/1^{(p^i)}, 1 \otimes 1^{(p^i)}, 1/0^2 \text{ or } 0^4 \\
\bar{S} \to A_4 : &\quad 4, 3/0, 2/1^{(p^i)}, 2/0^2, 1 \otimes 1^{(p^i)}/0, 1/1^{(p^i)}/0, 1/0^3 \text{ or } 0^5.
\end{aligned}
$$

Each composition factor of $L(G) \downarrow Y_1 \ldots Y_r$ occurs either in $L(Y_i)$ or in a tensor product of modules among the V_i and $\bigwedge^2 V_i$ and their duals (see [LS2, §2]). The highest weight of a composition factor of $\bigwedge^2 V_{\bar{S}}(a)$ is $2a - 2$, and of $\bigwedge^2 (1^{(p^i)} \otimes 1^{(p^j)})$ (with $i > j$) is $2p^i$.

Now $Y_1 \ldots Y_r$ is one of

$$A_1A_3A_3, \ A_1A_2A_2A_2, \ A_1A_1A_2A_3, \ A_3A_4, \ A_1A_2A_4.$$

Suppose first that $V_i \downarrow \bar{S}$ has a composition factor $3^{(p^j)}$ or $4^{(p^j)}$ for some i, j. Then $p \geq 5$. Applying a field twist, as discussed in the preamble to the lemma, we can take the factor to be 3 or 4. Arguing only from the above observations, we check that the only possibilities where a composition factor of high weight at least q can occur in $L(G) \downarrow \bar{S}$ are:

(a) $\bar{S} < A_1A_3A_3$, with embedding $0^2, 3, 3^{(p^i)}$ or $1^{(p^j)}, 3, 3^{(p^i)}$: here $L(G) \downarrow \bar{S}$ has possible highest weight $6p^i$ or $p^j + 4 + 4p^i$;

(b) $\bar{S} < A_3A_4$, with embedding $3^{(p^i)}, 4$ or $3^{(p^i)}, 3/0$: here $L(G) \downarrow \bar{S}$ has possible highest weight $6p^i$ or $4p^i + 6$.

In both cases, the possible highest weight is at least q only if $p = 5$; and if $q = 5^e > 25$, we can apply a field twist to take $i \leq e - 2$ in all cases, giving highest weight less than q. Therefore a high weight at least q can only occur if $q = 25$ and conclusion (i) or (ii) of the lemma holds.

We may now assume that no composition factor of $V_i \downarrow \bar{S}$ is 3 or 4 (or a twist thereof). From [LS2, 2.1 and 2.3] we see that the number of factors V_i or $\bigwedge^2 V_i$ in a tensor product occurring in $L(G) \downarrow Y_1 \ldots Y_r$ is at most three; hence, referring to the list of possibilities for $Y_1 \ldots Y_r$ above, we see that the highest weight occurring in $L(G) \downarrow \bar{S}$ is at most $6p^{e-1}$, which is less than q if $p > 5$. Thus we may assume that $p = 3$ or 5.

For $p = 5$, a high weight q or more can be achieved only if three factors occur in a tensor product, each factor contributing $2p^{e-1}$ (since otherwise the highest weight is at most $4p^{e-1} + 2p^{e-2} < q$). Thus the embedding of $\bar{S}$ in $Y_1 \ldots Y_r$ has three representations of the form $2^{(p^{e-1})}$ or $1^{(p^{e-1})} \otimes 1^{(p^j)}$. This forces $\bar{S} < A_1 A_2 A_2 A_2$, with embedding $(0^2 \text{ or } 1^{(p^i)}), 2^{(p^{e-1})}, 2^{(p^{e-1})}, 2^{(p^{e-1})}$. Applying a field twist, this becomes $(0^2 \text{ or } 1^{(p^{i+1})}), 2, 2, 2$, and now $L(G) \downarrow \bar{S}$ has highest weight less than q.

Now consider $p = 3$. Here $o(a) \geq q - 1 \geq 26$, so we see from 3.2 and 3.4 that the semisimple group $D = C_G(a)'$ cannot have rank 7 or 8. Recall that the product $Y_1 \ldots Y_r$ is as in 3.5(5), with factors A_m ($m \leq 4$). Consequently D lies in one of the following products:

$$A_1 A_2 A_3, \ A_1 A_1 A_2 A_2, \ A_2 A_2 A_2, \ A_3 A_3, \ A_2 A_4, \ A_1 A_1 A_1 A_3, \ A_1 A_1 A_4$$

Arguing as above, and applying a field twist if necessary, we see that the only case where $L(G) \downarrow \bar{S}$ can have highest weight at least q is $\bar{S} < D \leq A_2 A_2 A_2$, with embedding $2, 2^{(3)}, 2^{(9)}$ and $q = 27$. But in this case $a \in C_G(A_2 A_2 A_2) = A_2 = C_G(E_6)$, which is not possible by 3.6. This completes the proof when all the Y_i are of type A_m with $m \leq 4$.

Next, suppose that $Y_1 \ldots Y_r$ has a factor A_m with $m \geq 5$, hence is $A_2 A_5$, $A_1 A_1 A_5$ or $A_1 A_6$. If the usual module for A_m has all $\bar{S}$-composition factors of dimension at most 5, then the arguments for the previous case (all factors A_n, $n \leq 4$) apply; hence the projection of $\bar{S}$ in A_m is one of the following (up to twists):

$$\begin{aligned} \bar{S} \to A_5 : &\quad 5 \text{ or } 2 \otimes 1^{(p^i)} \\ \bar{S} \to A_6 : &\quad 6, \ 5/0 \text{ or } 2 \otimes 1^{(p^i)}/0. \end{aligned}$$

By [LS2, 2.1 and 2.3], each composition factor of $L(G) \downarrow Y_1 \ldots Y_r$ occurs in $L(Y_i)$, in $\bigwedge^3 V_i$, or in a tensor product of modules among the V_i and $\bigwedge^2 V_i$ (and duals). Moreover, by [LS2, 2.13], the highest weights in $\bigwedge^3 V_{\bar{S}}(r)$, $\bigwedge^2(2 \otimes 1^{(p^i)})$, $\bigwedge^2(1 \otimes 2^{(p^i)})$, $\bigwedge^3(2 \otimes 1^{(p^i)})$, $\bigwedge^3(1 \otimes 2^{(p^i)})$ are $3r - 6$, $2p^i + 2$, $4p^i$, $3p^i$, $4p^i + 1$, respectively. Hence, as usual adjusting by a field twist if necessary, we see that $L(G) \downarrow \bar{S}$ has highest weight less than q when $p > 3$.

Now let $p = 3$. Then $o(a) \geq q - 1 \geq 26$, so by 3.2 and 3.4, $D = C_G(a)'$ has rank at most 6. Consequently $\bar{S}$ must lie in a product $A_1 A_5$, with embedding $0^2, 2 \otimes 1^{(3^i)}$ or $1^{(3^j)}, 2 \otimes 1^{(3^i)}$. Then $L(G) \downarrow \bar{S}$ has highest weight less than q, unless $i = e - 1$; but in this case we apply a field twist to adjust the embedding to 0^2 or $1^{(3^k)}, 2^{(3)} \otimes 1$, and now $L(G) \downarrow \bar{S}$ has highest weight less than q.

We have now handled all cases where the Y_i are all of type A_m. It remains to consider $Y_1 \ldots Y_r = A_1 A_1 D_4, A_2 D_4$ or $A_1 D_5$. The possible projections of $\bar{S}$ in the factor D_m ($m = 4$ or 5) are as follows (up to twists):

$\bar{S} \to D_4$: $\quad 6/0, 4/2^{(p^i)}, 4/0^3, 3/3, 2/2^{(p^i)}/0^2, 2/1^{(p^i)}/1^{(p^i)}/0, 2/0^5,$
$\qquad 1/1/1^{(p^i)}/1^{(p^i)}, 1/1/0^4, 1 \otimes 1^{(p^i)}/1^{(p^j)} \otimes 1^{(p^k)},$
$\qquad 1 \otimes 1^{(p^i)}/1^{(p^j)}/1^{(p^j)}, 3 \otimes 1^{(p^i)}, 1 \otimes 1^{(p^i)}/0^4, 1 \otimes 1^{(p^i)}/2^{(p^j)}/0$

$\bar{S} \to D_5$: $\quad 8/0, 6/2^{(p^i)}, 6/0^3, 4/4^{(p^i)}, 4/2^{(p^i)}/0^2, 4/0^5, 3/3/0^2, 2/2^{(p^i)}/2^{(p^j)}/0,$
$\qquad 2/2^{(p^i)}/1^{(p^j)}/1^{(p^j)}, 2/2^{(p^i)}/0^4, 2/1^{(p^i)}/1^{(p^i)}/0^3, 2/0^7,$
$\qquad 1/1/1^{(p^i)}/1^{(p^i)}/0^2, 1/1/0^6, 2 \otimes 2^{(p^i)}/0, 3 \otimes 1^{(p^i)}/0^2,$
$\qquad 1 \otimes 1^{(p^i)}/1^{(p^j)} \otimes 1^{(p^k)}/0^2, 1 \otimes 1^{(p^i)}/1^{(p^j)}/1^{(p^j)}/0^2, 1 \otimes 1^{(p^i)}/0^6,$
$\qquad 1 \otimes 1^{(p^i)}/4^{(p^j)}/0, 2/2^{(p^i)}/1^{(p^j)} \otimes 1^{(p^k)}, 2/1^{(p^i)} \otimes 1^{(p^j)}/0^3.$

By [LS2, 2.1 and 2.3],

$L(G) \downarrow A_1 A_1 D_4$ has nontrivial composition factors among $2 \otimes 0 \otimes 0, 0 \otimes 2 \otimes 0,$ $0 \otimes 0 \otimes \lambda_2, 0 \otimes 0 \otimes \lambda_1, 1 \otimes 1 \otimes 0, 1 \otimes 1 \otimes \lambda_i, 1 \otimes 0 \otimes \lambda_i, 0 \otimes 1 \otimes \lambda_i, 0 \otimes 0 \otimes \lambda_i$ $(i = 1, 3, 4),$

$L(G) \downarrow A_2 D_4$ has nontrivial composition factors $11 \otimes 0, 0 \otimes \lambda_2, 0 \otimes \lambda_j \ \lambda_i \otimes \lambda_j$ $(i = 1, 2, \ j = 1, 3, 4),$ and

$L(G) \downarrow A_1 D_5$ has nontrivial composition factors $2 \otimes 0, 0 \otimes \lambda_2, 0 \otimes \lambda_i \ 1 \otimes \lambda_i$ $(i = 4, 5).$

As before, for each embedding of $\bar{S}$, the highest weight of $L(G) \downarrow \bar{S}$ can be worked out using [LS2, 2.13]. We find that in every case but one, applying a field twist if necessary, $L(G) \downarrow \bar{S}$ has highest weight less than q; the exceptional case is

$$q = 25, \ S < A_1 D_5, \ \text{embedding } (0^2 \text{ or } 1), 4/4^{(5)}.$$

But in this case, S also lies in $A_1 A_3 A_3$, as in conclusion (i) of the lemma. $\qquad \square$

Lemma 4.4. *Suppose that case (5) of 3.5 holds, and also that 4.2(ii) holds. Then we can choose $\bar{S}$ with $S < \bar{S} \le D$, such that every composition factor of $L(G) \downarrow \bar{S}$ has high weight less than q.*

Proof. We have $S < \bar{S} \le D$, and by applying a suitable field twist we may take it that D has a factor E with usual module V such that $V \downarrow \bar{S}$ is as in Table 2.

Suppose first that $p = 5$. Then $E = A_5, A_6, D_5$ or D_6; as we are assuming 3.5(5) holds, D lies in $A_2 A_5, A_1 A_1 A_5, A_1 A_6$ or $A_1 D_5$. We may take the embedding of $\bar{S}$ in these groups to be one of the following:

$\bar{S} < A_2 A_5$: projection $(0^3, 1^{(5^i)}/0 \text{ or } 2^{(5^i)}), 3/1^{(5)}$ (with $5^i < q$)

$\bar{S} < A_1 A_1 A_5$: embedding $(0^2 \text{ or } 1^{(5^i)}), (0^2 \text{ or } 1^{(5^j)}), 3/1^{(5)}$ (with $5^i, 5^j < q$)

$\bar{S} < A_1 A_6$: embedding $(0^2 \text{ or } 1^{(5^i)}), (3/1^{(5)}/0 \text{ or } 2/1 \otimes 1^{(5)})$ (with $5^i < q$)

$\bar{S} < A_1 D_5$: embedding $(0^2 \text{ or } 1^{(5^i)}), (3 \otimes 1^{(5)}/0^2 \text{ or } 2^2/1 \otimes 1^{(5)})$ (with $5^i < q$)

In all cases we find that the highest weight of $\bar{S}$ on $L(G)$ is less than q, as required.

Now suppose $p = 3$. Then $E = A_r \ (3 \le r \le 6), D_4$ or D_5. As in the proof of 4.3, and using 3.4 to rule out all rank 7 possibilities, we see that D lies in one of the following subgroups $Y_1 \ldots Y_r$:

$$A_1 A_2 A_3, A_3 A_3, A_2 A_4, A_1 A_1 A_1 A_3, A_1 A_1 A_4, A_1 A_5, A_6, A_1 A_1 D_4, A_2 D_4, A_1 D_5.$$

If $D \le A_1 A_2 A_3$, the conclusion is clear unless the embedding of $\bar{S}$ is $(0^2 \text{ or } 1^{(3^i)}),$ $2^{(3^{e-1})}, 1/1^{(3)}$ (recall that $q = 3^e$). Applying a field twist, we can change $\bar{S}$ to take the embedding to be $(0^2 \text{ or } 1^{(3^{i+1})}), 2, 1^{(3)}/1^{(9)}$; this gives the conclusion unless

$i + 1 = e$, in which case we further twist to replace the $1^{(3^{i+1})}$ by just 1. Now the weights on $L(G)$ are less than q, as required.

This argument applies to all cases where the factors Y_i are all of type A_k, except for A_3A_3, in which case $\bar{S}$ could be indecomposable on both natural modules for the factors. Here the embedding is $1/1^{(3)}, (1/1^{(3)})^{(3^i)}$, and we can assume that $3^i < q$. The weights are all less than q unless $3^{i+1} = q$. We can now change $\bar{S}$ by applying a 3-power twist to take the embedding to be $1^{(3)}/1^{(9)}, 1/1^{(3)}$, for which the weights are less than q.

It remains to handle the cases where some Y_i is D_4 or D_5. For these the weights of $\bar{S}$ on $L(G)$ are less than q unless possibly $\bar{S} < A_1D_5$ with embedding $1 \otimes 1^{(3)}/1^{(3^i)}/1^{(3^i)}/0^2$ or $1 \otimes 1^{(3)}/1^{(3^i)} \otimes 1^{(3^l)}/0^2$ ($i < l$) in the D_5 factor. In the first case, $\bar{S} < A_1A_1D_3$, a case handled above. In the second case, if the factor $1^{(3^i)} \otimes 1^{(3^l)}$ splits off then $\bar{S} < A_1A_1A_1D_3$, a case already considered. Otherwise, by 1.2, we have $l = i + 1$. The weights of $\bar{S}$ on $L(G)$ are less than q unless $i + 1 = e$, in which case we twist to embedding $1^{(3)} \otimes 1^{(9)}/1 \otimes 1^{(3)}/0^2$, for which the weights are all less than q. $\square$

Continue to assume that $p \neq 2, q > 13$. By 3.5–3.7 and the previous two lemmas, we may now assume that either we are in one of the exceptional cases of 4.2(iii,iv,v), 4.3(i,ii), or D and q are as in Table 3.

TABLE 3

D	q
$A_1A_2A_5, A_1E_6, A_1D_6$	$17, 19$
$A_1A_1D_5, A_7, A_2D_5, A_3D_4$	$17, 19, 23, 25$
D_6	q prime or $q = 25$

Lemma 4.5. *Suppose that q is prime and D is as in Table 3, with $D \neq A_1E_6$. Then we can choose $\bar{S} \cong A_1(K)$ such that $S < \bar{S} \leq D$ and the highest weight of $L(G) \downarrow \bar{S}$ is less than q, except in the following case:*

$q = 17$, $S < D = A_1D_6$ *or* D_6, *with embedding* $10/0$ *in the D_6 factor.*

Proof. Since $q = p \geq 17$, it follows from 1.2 that S acts completely reducibly on the usual module for each factor of D. Hence there is a subgroup $\bar{S} \cong SL_2(K)$ of D containing S (see the preamble to 4.3). We see as in the proof of the previous lemma that either we can take the highest weight of $\bar{S}$ to be less than q, or D has a factor D_6 with S projecting via $10/0$, as in the conclusion. $\square$

Lemma 4.6. *If $q = 25$ and D is as in Table 3, then either we can choose $\bar{S} \cong A_1(K)$ such that $S < \bar{S} \leq D$ and the highest weight of $L(G) \downarrow \bar{S}$ is less than q, or one of the following holds:*

(i) *D has a factor D_5 or D_6, and the embedding of S in this factor is $4/4^{(5)}/0^m$ ($m = 0$ or 2);*

(ii) *$S < D = A_3D_4$, with embedding $3^{(5)}, 1^{(5)} \otimes 3$ or $3^{(5)}, 3/3$ or $3^{(5)}, 4/2$ or $3^{(5)}, 4/2^{(5)}$ or $3^{(5)}, 4/0^3$;*

(iii) *$S < D = A_7$, with embedding $3/3^{(5)}$.*

Proof. If S is not completely reducible on the usual module for one of the factors of D, then one of the possibilities given in Table 2 of 4.2 holds, from which we easily

see that $\bar{S}$ exists, with all weights on $L(G)$ less than q. Otherwise, S is completely reducible in all factors of D. The only way the desired $\bar{S}$ can fail to exist is for some $V_i \downarrow S$ to have a composition factor $3^{(5)}$ or $4^{(5)}$ which cannot be excluded by applying a field twist; all such possibilities are listed under conclusions (i)–(iii). □

The next lemma deals with the possibility $D = A_1 E_6$ excluded in 4.5.

Lemma 4.7. *Theorem 1 holds if $D = A_1 E_6$.*

Proof. Here $q = 17$ or 19, and by 3.4, $o(a) \leq 6$. Consequently the group A must be $U_3(17)$ or $L_3(19)$. Observe that the projection of S to the factor E_6 is a group $S_0 \cong L_2(p)$. Pick $u \in S$ of order p, and write $u = u_1 u_0$ with $u_1 \in A_1$, $u_0 \in S_0 < E_6$.

Assume first that u_0 is a regular or semiregular element of E_6. Now the projection of $C_A(a)$ to E_6 in fact contains $PGL_2(p)$. Under these circumstances, [ST2, Theorem 1] implies that there is a subgroup $\bar{S} \cong A_1(K)$ of E_6 containing S_0. Moreover, by [LT, Table 3], $\bar{S}$ is determined up to E_6-conjugacy; the composition factors of $L(E_6) \downarrow \bar{S}$ are as in [Se2, p.65]. It follows that $L(E_6) \downarrow \bar{S}$ has a unique composition factor with high weight 16. Hence the same is true of $L(E_8) \downarrow S$. Consequently $L(E_8) \downarrow A$ has a composition factor $V_A(aa)$ with $2a \geq 16$. By 1.5, $V_A(aa)$ has dimension either $(a+1)^3$ or $(a+1)^3 - (p-a-1)^3$. Since this is at most 248, we must have $p = 17$ and $a = 8$, in which case $V_A(aa)$ has dimension 217. Now $V_A(88) \downarrow S$ has two composition factors of high weight 15; moreover, $S < A_1 \bar{S} < A_1 E_6$, and

$$L(G) \downarrow A_1 E_6 = L(A_1)/L(E_6)/1 \otimes \lambda_1/1 \otimes \lambda_6/0 \otimes \lambda_1/0 \otimes \lambda_6/(1 \otimes 0)^2/0$$

(see [LS2, §2]). It follows that the restriction of the $A_1 E_6$-module $1 \otimes \lambda_1$ to S must have a composition factor of high weight 15. By [LS2, 2.5], we have $\lambda_1 \downarrow S_0 = 16/8/0$ or $12/8/4$; hence, by the previous sentence, in fact $\lambda_1 \downarrow S_0 = 16/8/0$. Then $1 \otimes \lambda_1 \downarrow S$ has two composition factors 15, whence $L(G) \downarrow S$ has four composition factors 15. This implies that $L(G) \downarrow A$ has a further factor $V_A(bc)$ with $b + c \geq 15$; but then $\dim V_A(88) + \dim V_A(bc) > 248$, a contradiction.

Now suppose that u_0 is not regular or semiregular in E_6. Then u_0 lies in the centralizer in E_6 of a semisimple element, hence lies in a subsystem subgroup $Y = D_5$, $A_1 A_5$ or $A_2 A_2 A_2$. Since $p \geq 17$, it follows using 1.7 that u_0 lies in a subgroup $\bar{S} \cong A_1(K)$ of Y such that $\bar{S}$ has restricted composition factors on the usual modules for all factors of Y. If we write M for the factor A_1 of $D = A_1 E_6$, then u lies in a diagonal subgroup $\tilde{S} \cong A_1(K)$ of $M\bar{S}$ (where neither projection involves a twist). The restrictions $L(E_8) \downarrow MY$ are given by [LS2, §2], and we deduce from this that all composition factors of $L(E_8) \downarrow \tilde{S}$ are restricted. It now follows from [LST, 1.14] that if U is a connected unipotent subgroup of $\tilde{S}$ containing u, then u and U fix the same subspaces of $L(E_8)$, contrary to (†). □

To complete the proof of Theorem 1 for $q > 13, p \neq 2$, all that remains is to handle the exceptional cases in the conclusions of 4.2, 4.3, 4.5 and 4.6.

Lemma 4.8. *None of the exceptional cases in 4.2(iii, iv, v) can occur.*

Proof. Consider first case 4.2(iii); here $q = 27$ and $S < A_5 < D$ with embedding $1/1^{(3)}/1^{(9)}$. From [LS2, Table 8.1] we have

$$L(G) \downarrow A_5 = L(A_5)/\lambda_1^6/\lambda_2^3/\lambda_3^2/\lambda_4^3/\lambda_5^6/0^{11}.$$

Hence

$$L(G) \downarrow S = (1 \otimes 1^{(3)} \otimes 1^{(9)})^2/(1 \otimes 1^{(3)})^8/(1 \otimes 1^{(9)})^8/(1^{(3)} \otimes 1^{(9)})^8/$$
$$1^{16}/(1^{(3)})^{16}/(1^{(9)})^{16}/2/2^{(3)}/2^{(9)}/0^{31}.$$

Recall that $S < A = A_2^\epsilon(27)$ or $B_2(27)$. If $A = B_2(27)$ then S contains a conjugate b of a (of order 28). However from the above restriction of $L(G)$ to S, we see that $\dim C_{L(G)}(b) = 34$, whereas $C_G(a)$ contains A_5, of dimension 35, a contradiction.

Therefore $A = A_2^\epsilon(27)$. Since $L(G) \downarrow S$ has two factors $1 \otimes 1^{(3)} \otimes 1^{(9)}$, it follows that $L(G) \downarrow A$ has a factor $\alpha \otimes \beta^{(3)} \otimes \gamma^{(9)}$ and its dual, where $\alpha, \beta, \gamma \in \{10, 01\}$. Moreover,

$$(\alpha \otimes \beta^{(3)} \otimes \gamma^{(9)}) \downarrow S = 1 \otimes 1^{(3)} \otimes 1^{(9)}/1 \otimes 1^{(3)}/1 \otimes 1^{(9)}/1^{(3)} \otimes 1^{(9)}/1/1^{(3)}/1^{(9)}/0.$$

Now $V_A(22) \downarrow S$ has three composition factors of high weight 2, hence is not present in $L(G) \downarrow A$. In order to accomodate the further six composition factors $1 \otimes 1^{(3)}$ in $L(G) \downarrow S$, it follows that $L(G) \downarrow A$ must have three factors of the form $\delta \otimes \epsilon^{(3)}$ with $\delta, \epsilon \in \{10, 01\}$, together with their duals; similarly for the other 2-fold twists.

The composition factors found so far for $L(G) \downarrow A$ account for all composition factors of $L(G) \downarrow S$ except $1^2/(1^{(3)})^2/(1^{(9)})^2/2/2^{(3)}/2^{(9)}/0^{11}$; in order to accomodate these, $L(G) \downarrow A$ must have further factors $11/11^{(3)}/11^{(9)}/00^8$.

Now a is an element of order 26 or 28 in A of the form $\mathrm{diag}(\omega, \omega, \omega^{-2})$ for some $\omega \in K^*$. We calculate that $\dim C_{V_A(\lambda)}(a)$ is at most 1 if $\lambda = \alpha \otimes \beta^{(3)} \otimes \gamma^{(9)}$, is 0 if $\lambda = \delta \otimes \epsilon^{(3)}$, and is 4 if $\lambda = 11$. Hence from the above we see that $\dim C_{L(G)}(a) \le 22$. This is a contradiction, as $C_G(a)$ contains A_5. This completes the argument for the case 4.2(iii).

The case 4.2(iv) is handled similarly. Here $S < D = A_1 D_5$, and from [LS2, §2],

$$L(G) \downarrow A_1 D_5 = L(A_1)/L(D_5)/(1 \otimes \lambda_1)^2/1 \otimes \lambda_4/1 \otimes \lambda_5/(0 \otimes \lambda_4)^2/$$
$$(0 \otimes \lambda_5)^2/(0 \otimes \lambda_1)^2/(1 \otimes 0)^4/0^4.$$

From the proof of 4.2, we see that the projection of S in the D_5 factor is as in case (c) for $p = 3$, namely $(1 \otimes 1^{(3)}/0/1 \otimes 1^{(3)}) \perp V_1$, the space $1 \otimes 1^{(3)}$ being a singular 4-space. We now calculate that

$$L(G) \downarrow S = (1 \otimes 1^{(3)} \otimes 1^{(9)})^8/(2 \otimes 1^{(9)})^2/(2^{(3)} \otimes 1^{(9)})^2/2 \otimes 2^{(3)}/$$
$$(1 \otimes 1^{(3)})^{16}/2^7/(2^{(3)})^7/2^{(9)}/(1^{(9)})^{12}/0^{18}.$$

If $A = B_2(27)$ then, as above, S contains a conjugate b of a, and from this restriction we see that $C_{L(G)}(b)$ has dimension 34, which is a contradiction as $C_G(a)' = A_1 D_5$. Thus $A = A_2^\epsilon(27)$. As in the previous case, $L(G) \downarrow A$ must have a composition factor $\alpha \otimes \beta^{(3)} \otimes \gamma^{(9)}$ with $\alpha, \beta, \gamma \in \{10, 01\}$. But the restriction of this to S has a composition factor $1 \otimes 1^{(9)}$, which is not present in $L(G) \downarrow S$.

Finally, consider 4.2(v). Here we calculate that

$$L(G) \downarrow S = (1 \otimes 2^{(3)} \otimes 1^{(9)})^2/(1 \otimes 1^{(3)} \otimes 1^{(9)})^2/(1 \otimes 2^{(3)})^4/$$
$$(2^{(3)} \otimes 1^{(9)})^4/(1 \otimes 1^{(3)})^6/(1 \otimes 1^{(9)})^4/(1^{(3)} \otimes 1^{(9)})^7/$$
$$2/(2^{(3)})^8/(2^{(9)})^3/1^4/(1^{(3)})^{12}/(1^{(9)})^8/0^8.$$

We obtain the usual contradiction if $A = B_2(27)$, so assume $A = A_2^\epsilon(27)$. In order to produce $(1 \otimes 2^{(3)} \otimes 1^{(9)})^2$ for S, $L(G) \downarrow A$ must have a factor $\lambda = \alpha \otimes \beta^{(3)} \otimes \gamma^{(9)}$ and its dual, where $\alpha, \gamma \in \{10, 01\}$, $\beta \in \{20, 02\}$ (note that $L(G) \downarrow A$ cannot have factors $\alpha \otimes 22^{(3)}$ or $\alpha \otimes 11^{(3)} \otimes \gamma^{(9)}$, as these are incompatible with $L(G) \downarrow S$). Removing the composition factors of $(\lambda + \lambda^*) \downarrow S$ from $L(G) \downarrow S$ still leaves two

factors $2^{(3)} \otimes 1^{(9)}$. To accommodate these, $L(G) \downarrow A$ must have $\delta^{(3)} \otimes \mu^{(9)}$ and its dual, where $\delta \in \{20, 02, 11\}$, $\mu \in \{10, 01\}$. Removing the restrictions of these to S leaves a further one or three factors $1^{(3)} \otimes 1^{(9)}$. At least one of these must occur in the restriction of a self-dual factor of $L(G) \downarrow A$, which must be $22^{(3)}$. Removing all S-composition factors of this from what is left of $L(G) \downarrow S$ leaves $(1 \otimes 2^{(3)})^2/(2^{(3)})^1/\dots$. The factors $1 \otimes 2^{(3)}$ must occur in the restriction of $\phi \otimes \psi^{(3)}$ and its dual, where $\phi \in \{10, 01\}$, $\psi \in \{20, 02, 11\}$; but this restriction has two composition factors $2^{(3)}$, which is a contradiction. This completes the proof. $\square$

Lemma 4.9. *None of the exceptional cases in 4.3 or 4.6 can occur.*

Proof. Suppose false, so $q = 25$. By considering the restriction of $L(G)$ to the subsystem subgroup containing D given in 4.3 or 4.6, we see that $L(G) \downarrow S$ has a composition factor $4 \otimes k^{(5)}$ or $4^{(5)} \otimes k$, where $k \in \{3, 4\}$. Applying a twist, we may assume the factor is $4 \otimes k^{(5)}$. Recall that X has a subgroup $A \cong A_2^\epsilon(q)$ or $B_2(q)$ which contains S. Let W be a composition factor of $L(G) \downarrow A$ such that $W \downarrow S$ has $4 \otimes k^{(5)}$ as a composition factor.

Observe first that W is not restricted, since the highest weight of $V_A(ab) \downarrow S$ $(a, b \leq 4)$ is $a + b$. Hence $W = ab \otimes cd^{(5)}$, with $a, b, c, d \leq 4$.

If $a + b < 4$ then $cd^{(5)} \downarrow S$ must have a composition factor $l \otimes k^{(5)} = l^{(25)} \otimes k^{(5)}$, with $1 \leq l \leq 4$. Then $5(c + d) \geq 25l + 5k \geq 40$, forcing $c = d = 4$. However, 44 (the Steinberg module) has dimension 5^3 (for $A = A_2^\epsilon(25)$) or 5^4 (for $A = B_2(25)$), so $ab \otimes 44^{(5)}$ has dimension greater than 248, a contradiction.

Therefore $a + b \geq 4$. Similarly we see that $c + d \geq k$. For $A = A_2^\epsilon(q)$ it is now easy to check that $\dim W > 248$ if W is self-dual (i.e. if $a = b$ and $c = d$), and that $\dim(W + W^*) > 248$ otherwise. And for $A = B_2(q)$ we check that $\dim W > 248$, by counting conjugates of subdominant weights and using [Pr]. $\square$

Lemma 4.10. *The exceptional case in 4.5 cannot occur.*

Proof. Suppose false, so that $q = 17$ and $S < D = D_6$ or $A_1 D_6$ with embedding $(0^2), 10/0$ or $1, 10/0$. By [LS2, §2], we have

$$L(G) \downarrow A_1 D_6 = L(A_1 D_6)/(1 \otimes \lambda_1)^2/1 \otimes \lambda_5/\lambda_6^2/0^3.$$

Moreover, by [LS2, 2.13], $\lambda_5 \downarrow S \cong \lambda_6 \downarrow S = 15/9/5$. Therefore $L(G) \downarrow S$ has a composition factor with high weight 15 or 16. If $A = B_2(17)$ then we can take a to have order 18, and the conclusion follows from the proof of 3.7. Therefore $A = A_2^\epsilon(17)$. Now $L(G) \downarrow A$ must have a composition factor $V_A(ab)$ with $a + b = 15$ or 16. By 1.8, this forces $a = b = 8$, in which case $\dim V_A(ab) = 217$. Now $V_A(88) = W_A(88)/W_A(77)$, from which we calculate that

$$V_A(88) \downarrow S = 16^1/15^2/14^2/\dots$$

To obtain the factor 16, the embedding of S in $A_1 D_6$ must be $1, 10/0$, whence $L(G) \downarrow S = 16^1/15^2/14^3/\dots$. The factor 14 not occurring in $V_A(88) \downarrow S$ must occur in the restriction of a self-dual A-composition factor of $L(G)$, which must be $V_A(77)$. However, $\dim V_A(88) + \dim V_A(77) > 248$, which is a contradiction. $\square$

This completes the proof of Theorem 1 for $p \neq 2$, $q > 13$.

5. COMPLETION OF THE PROOF OF THEOREM 1 FOR $p \neq 2$

In this section we complete the proof of Theorem 1 for $p \neq 2$. Since $q > 9$ by hypothesis, by the previous section it remains only to handle the cases $q = 11$ and $q = 13$. Thus let $G = E_8$ and suppose that $X = X(q) < G$ with $q = 11$ or 13. By the hypothesis of Theorem 1, $X \neq L_2(q)$, so X contains a subgroup A isomorphic to one of the following groups:

$$A_2^\epsilon(11), A_2^\epsilon(13), B_2(11), B_2(13).$$

As in the previous section, by 3.2 we may assume

(†)   there are no subgroups $Y \leq X$, $Z = Z^0 \leq G$, such that every
       Y-invariant subspace of $L(G)$ is Z-invariant and $X \cap Z \nleq Z(X)$.

A in a series of lemmas. The most complicated cases occur when $A = L_3(13)$ or $U_3(11)$, and we leave these until the final lemma.

We begin with a useful fact about self-dual indecomposables for $SL_2(p)$.

Lemma 5.1. *Let $S = SL_2(p)$ $(p = 11$ or $13)$, and let V be a self-dual indecomposable KS-module of the form $4/4$ $(p = 11)$ or $5/5$ $(p = 11$ or $13)$. Then $S < SO(V)$ only if $V = 5/5$ with $p = 11$; otherwise $S < Sp(V)$.*

Proof. Suppose $S < SO(V) = SO_{2r}$ $(r = 4$ or 5$)$. Then $S < P_r = QL$, the stabilizer of a maximal totally singular subspace, with unipotent radical Q; moreover, Q is abelian, and $Q \downarrow S \cong \bigwedge^2 V(r)$, which is $6/2$ if $r = 4$ and $8/4/0$ if $r = 5$. Since V is indecomposable, S does not lie in a conjugate of L, so $H^1(Q, S) \neq 0$. It follows from 1.2(i) that $r = 5$ and $p = 11$, as required. $\square$

Lemma 5.2. *Theorem 1 holds if $A = SL_3(13)$ or $SU_3(11)$.*

Proof. Write $Z(A) = \langle t \rangle$, a group of order 3. We claim that $C_G(t) = A_2 E_6$ or A_8. Now $A < C_G(t) = T_k E$, where E is a semisimple subsystem subgroup of G. Then $A < E = C_G(t)'$, whence $t \in Z(E)$. Therefore E has a simple factor of type A_2, A_5, E_6 or A_8. In the last case the claim holds. If there is an E_6 factor or a unique A_2 factor, then $C_G(t)$ contains the centralizer of this factor, whence $C_G(t) = A_2 E_6$, as in the claim. If there is an A_5 factor, this has centralizer $A_2 A_1$ in G, whence $C_G(t)$ has an A_2 factor. It remains to show that $C_G(t)$ cannot have more than one A_2 factor. If there is more than one, then the normalizer of the product of all the A_2 factors is A_2^4, whence $C_G(t) = A_2^4$, contradicting 1.14. Thus $C_G(t) = A_2 E_6$ or A_8, as claimed.

Suppose first that $C_G(t) = A_8$; in fact, then $C_G(t) = SL_9/Z_3$. But this group cannot contain A, since if it did, the preimage of $Z(A)$ in SL_9 would be cyclic of order 9, which is clearly impossible.

Now assume $C_G(t) = A_2 E_6$. If A lies in the A_2 factor, then these groups fix the same subspaces of $L(G)$; hence A projects nontrivially to the E_6 factor. Choose $a \in A$ of order 4 such that $C_A(a)^{(\infty)} = S \cong SL_2(p)$. The projection of S in E_6 centralizes an involution, hence lies in a subgroup $A_1 A_5$ or $T_1 D_5$; hence S lies in $T_1 A_1 A_1 A_5$ or $T_1 A_1 T_1 D_5$. The restriction of $L(G)$ to these subgroups is given by [LS2, §2], and for all but one possible embedding of S we conclude that $S < \bar{S} \cong SL_2(K)$, where all high weights of $L(G) \downarrow \bar{S}$ are less than q, giving the usual contradiction to (†). The one exception is $S < A_1 D_5$ with embedding $1, 8/0$ or $0^2, 8/0$ (note that projection into D_5 is not the indecomposable $4/4$, by 5.1). In this case, let S_0 be the projection of S in E_6, and consider the action of S_0 on

dimension 0, 14, 8, 0 according as $i = 1, 2, 5, 6$, so $\dim C_{L(G)}(b) \leq 36$, again a contradiction.

Similar contradictions are reached when $S < A_6$ or $A_1 D_5$: for the A_6 case we obtain a group $\bar{S}$ contradicting (†) (note that the irreducible embedding 6 is not possible as its image in A_6 is $PSL_2(q)$); and for the $A_1 D_5$ case we obtain $\bar{S}$ unless the projection of S into the D_5 factor is $8/0$, in which case we obtain a contradiction by considering $C_{L(G)}(b)$ as above. □

Lemma 5.5. *Theorem 1 holds if $A = L_3(13)$ or $U_3(11)$.*

Proof. Pick $a \in A$ of order 4 such that $C_A(a)^{(\infty)} = S \cong SL_2(p)$. Then $C_G(a)$ is one of the following groups (cf. [CG, Table 4]):

$$T_1 E_7, \, T_1 A_1 E_6, \, T_1 D_7, \, T_1 A_7, \, T_1 A_1 D_6, \, A_1 A_7, \, A_3 D_5.$$

We now argue in a series of steps.

Step 1. If $C_G(a) = T_1 E_7$, then the composition factors of $L(G) \downarrow A$ are among the modules $V_A(ab)$ with $ab \in \{00, 11, 30, 03\}$.

To see this, observe first that the weights of T_1 on $L(G)$ are $2, 1, 0, -1, -2$, and $2, -2$ occur with multiplicity 1; these are the only weights which agree on a, and the sum of the two corresponding weight spaces is the -1-eigenspace for a. It follows that any a-invariant subspace of $L(G)$ containing this eigenspace is also T_1-invariant, and so is any a-invariant space which intersects the eigenspace in 0 only. In view of (†), it follows that some composition factor of $L(G) \downarrow A$ has -1-eigenspace of dimension 1.

Suppose $W = V_A(cd) = V_A(\lambda)$ is a composition factor on which a has no eigenvalues -1. We may assume the composition factor is restricted. Note that $Z(A) = 1$, so $c \equiv d \mod 3$. Let α, β be fundamental roots of an A_2 root system. We may assume that $c \geq d$, $\alpha(a) = i$ and $\beta(a) = 1$. We know that $\lambda(a) = 1, i$ or $-i$. If $\lambda(a) = 1$ then $\lambda - 2\alpha - b\beta$ cannot be a weight of W for any $b \geq 0$, so $c \leq 1$ and $d \leq 1$. This forces $c = d = 0$. If $\lambda(a) = i$ then $\lambda - 3\alpha - b\beta$ cannot be a weight, so $c \leq 2$, $d \leq 2$, forcing $c = d = 1$. Finally, if $\lambda(a) = -i$ then $\lambda - \alpha - b\beta$ cannot be a weight, forcing $c = d = 0$, a contradiction.

Similarly, we find that the only possible composition factors on which a has exactly one eigenvalue -1 are $V_A(30)$ and $V_A(03)$. This proves Step 1.

Step 2. $C_G(a) \neq T_1 E_7$.

For suppose $C_G(a) = T_1 E_7$. Then by Step 1, the highest weight of S on $L(G)$ is 3. This is less than $(p-1)/3$, and one now checks that the argument of [ST1, Lemma 2] (second paragraph of proof) provides a rank 1 torus of G which fixes the same subspaces of $L(G)$ as some element of S of order $p-1$, contrary to (†).

Step 3. $C_G(a) \neq T_1 D_7$.

Suppose $C_G(a) = T_1 D_7$, so that $S < D_7$. Observe first that by [LS2, Table 8.1],

$$L(G) \downarrow D_7 = \lambda_2 / \lambda_1^2 / \lambda_6 / \lambda_7 / 0.$$

Arguing as in 4.2 and 4.3, we see that either $S < \bar{S} < D_7$ with $\bar{S} \cong SL_2(K)$ and all composition factors of $L(G) \downarrow \bar{S}$ of high weight less than p (in which case (†) is violated), or the embedding of S in D_7 is one of

$$12/0, \, 10/0^3, \, 8/4, \, 8/2/0^2, \, 8/1^2/0, \, 5/5/0^2 \, (p = 11)$$

(where as usual we specify the embedding by giving the composition factors of S on the natural 14-dimensional module for D_7). Note that the embeddings $6/6$ and $8/0^5$ are not possible, as these correspond to a subgroup $PSL_2(q)$ of D_7 rather than $SL_2(q)$.

In the ensuing arguments, observe that since $A = L_3(13)$ or $U_3(11)$, every composition factor of $L(G) \downarrow A$ is of the form $V_A(ab)$ with $a \equiv b$ mod 3.

If the embedding is $12/0$ then $p = 13$ and, using the restriction $L(G) \downarrow D_7$ given above, together with [LS2, 2.13], we see that $L(G) \downarrow S$ has three composition factors with high weight 12. Then by Proposition 1.9, $L(G) \downarrow A$ must have three composition factors among $V_A(12, 0)$, $V_A(93)$, $V_A(66)$. This forces $\dim L(G) > 248$, a contradiction.

If the embedding is $10/0^3$, then $L(G) \downarrow S$ has at least five composition factors with high weight 10, and again we obtain a contradiction using Proposition 1.9. If the embedding is $8/4$ then for $p = 13$ we find that $L(G) \downarrow S = 12^1/11^4/...$, and for $p = 11$, $L(G) \downarrow S = 10^2/9^4/8^4/7^6/...$, both of which conflict with 1.9. If the embedding is $8/2/0^2$ or $8/1^2/0$, we find that $L(G) \downarrow S = 11^4/...$ or $11^2/10^6/...$ (respectively) if $p = 13$, and is $10^2/9^8/...$ or $10^5/9^6/8^3/...$ if $p = 11$; all these possibilities conflict with 1.9.

Finally, the embedding $5/5/0^2$ is excluded exactly as in the proof of 5.3. This completes Step 3.

Step 4. $C_G(a) \neq T_1 A_7$, $T_1 A_1 D_6$, $A_1 A_7$ or $A_3 D_5$.

The proof of this is entirely similar to that of Step 3: supposing $C_G(a)$ is one of the above, we find that either $S < \bar{S} < C_G(a)'$ with $\bar{S}$ having all weights less than p, or the embedding of S in $C_G(a)'$ is one of the following:

$C_G(a)'$	embedding $S < C_G(a)'$
A_7	7
$A_1 A_7$	$1, 7$ or $1, 6/0 (p = 11)$
$A_1 D_6$	$1, 10/0$ or $1, 8/2$ or $1, 8/0^3$ or $1, 5/5 (p = 11)$
$A_3 D_5$	$3, 8/0$

(note that in the last two rows we have assumed that $S \not\leq D_7$, in view of the argument for Step 3). For each of the above embeddings, it is straightforward to work out the composition factors of $L(G) \downarrow S$, and in all cases we find as in Step 3 that these conflict with 1.9.

Step 5. $C_G(a) \neq T_1 A_1 E_6$.

The argument for this step is similar to the proof of 4.7. Let S_0 be the projection of S to the factor E_6, pick $u \in S$ of order p, and write $u = u_1 u_0$ with $u_1 \in A_1$, $u_0 \in S_0 < E_6$. Note that $S_0 \cong L_2(p)$.

Assume first that u_0 is a regular or semiregular element of E_6. As in the proof of 4.7, [ST2] then implies that $S_0 < \bar{S} < E_6$, where $\bar{S} \cong A_1(K)$; and by [Se2, p.65] and [LS2, 2.5(ii)] we have, writing $V_{27} = V_{E_6}(\lambda_1)$, either

(i) $p = 13$, $L(E_6) \downarrow \bar{S} = W(22)/W(16)/W(14)/W(10)/W(8)/W(2)$ and $V_{27} \downarrow \bar{S} = W(16)/W(8)/0$ (recall $W(r)$ denotes the Weyl module of high weight r), or

(ii) $L(E_6) \downarrow \bar{S} = W(16)/W(14)/W(10)^2/W(8)/W(6)/W(4)/W(2)$ and $V_{27} \downarrow \bar{S} = W(12)/W(8)/W(4)$.

Moreover, by [LS2, §2],

$$L(G) \downarrow A_1 E_6 = L(A_1 E_6)/1 \otimes \lambda_1/1 \otimes \lambda_6/0 \otimes \lambda_1/0 \otimes \lambda_6/(1 \otimes 0)^2/0.$$

Note that S projects nontrivially into the A_1 factor (since the image of S in the E_6 factor is $PSL_2(q)$). In case (i), we deduce that $L(G) \downarrow S$ has at least 3 composition factors with high weight 10, and none with high weight 12. Using 1.9 we find that this leads to a contradiction. In case (ii) with $p = 13$, we have $L(G) \downarrow S = 12^2/11^4/10^3/9^2/....$; and in case (ii) with $p = 11$, we have $L(G) \downarrow S = 10^2/9^4/8^5/.....$ Both of these restrictions conflict with the possibilities given by 1.9.

Therefore u_0 is not regular or semiregular in E_6, so, as in the proof of 4.7, it lies in a subsystem subgroup $Y = D_5, A_1 A_5$ or $A_2 A_2 A_2$. Using 1.7 for the D_5 case, we see that there is a subgroup $\bar{S} \cong A_1(K)$ of Y containing u_0 and having restricted composition factors on the usual modules for all factors of Y. The restrictions $V_{27} \downarrow Y$ are given by [LS2, Table 8.7], and using [LS2, 2.13] we deduce that all composition factors of $V_{27} \downarrow \bar{S}$ are restricted. Therefore, if U is a connected unipotent subgroup of $\bar{S}$ containing u_0, then by [LST, 1.14], u_0 and U fix the same subspaces of V_{27}. Now $S_0 = A_1(p)$ is not irreducible on V_{27}; hence neither is $\langle S_0, U \rangle$. Thus $\langle S_0, U \rangle^0$ is a proper nontrivial connected subgroup of E_6, hence lies in a maximal connected subgroup, M say. The possibilities for M are given by [Se2, Theorem 1].

Suppose M contains a maximal torus of E_6. If M is reductive, then M' is a subsystem subgroup, hence lies in a subgroup $D_5, A_1 A_5$ or $A_2 A_2 A_2$. We find in the usual way that either S lies in a subgroup $A_1(K)$ having all composition factors on $L(G)$ of high weight less than p, or $S_0 < D_5$ with embedding $8/0$. But in this case $S < A_1 D_5 < D_7$, which was excluded in Step 3.

Now suppose M is parabolic. Choose a minimal parabolic $P = QL$ of E_6 containing S_0, with unipotent radical Q and Levi subgroup L. The L-composition factors of Q have the structure of KL-modules, with high weights given by [LS2, 3.1]. Using 1.2, we find that either $H^1(S_0, V) = 0$ for all such modules V, or $L = D_5$ and the embedding of S_0 in L modulo Q is $8/0$. In the latter case, $S < QA_1 D_5$, so S has the same composition factors on $L(G)$ as a suitable subgroup $A_1(p)$ of $A_1 D_5 < D_7$, and the proof of Step 3 gives a contradiction. And in the former case, S_0 lies in a conjugate of L, a case dealt with in the previous paragraph.

Thus M does not contain a maximal torus of E_6. The argument of the previous two paragraphs shows that S_0 lies in no subgroup of maximal rank in E_6. Then by [Se2], $M = A_2, G_2, F_4$ or $A_2 G_2$; the restrictions of $L(E_6)$ and V_{27} to each of these subgroups are given in [LS2, §2]. The fact that S_0 lies in no maximal rank subgroup of E_6 implies that S_0 also lies in no maximal rank subgroup of M (since otherwise S_0 would centralize a nontrivial semisimple element t in the centre of a maximal rank subgroup of M, hence lie in the maximal rank subgroup $C_{E_6}(t)$).

Write $R = \langle S_0, U \rangle$. Observe first that R does not act irreducibly in dimension 27 or 26, so R is not equal to one of the maximal subgroups G_2, A_2 or F_4 of E_6. Also, if R lies in a maximal G_2, then by [Se2, pp.193 and 65], $R = A_1$ and $S_0 < R$ has composition factors as in (ii) above, giving a contradiction as before. And if R lies in a maximal A_2, then $R = A_1$ has all composition factors on $L(G)$ of high weight less than p.

Now suppose R lies in a maximal subgroup $A_2 G_2$ of E_6. Then $S_0 < A_2 G_2$. Let S_1 be the projection of S_0 in the factor G_2. Then $S_1 = L_2(p) < G_2 < B_3$. There is a subgroup $\bar{S}_1 \cong A_1(K)$ of B_3 containing S_1, with all weights on $L(B_3)$ less than p.

Hence by 1.5, $\bar{S}_1$ fixes all S_1-invariant subspaces of $L(B_3)$, and in particular fixes $L(G_2)$. Therefore $S_1 < \bar{S}_1 < G_2$; as S_1 is not contained in a proper maximal rank subgroup of G_2, we have $V_{G_2}(\lambda_1) \downarrow \bar{S}_1 = 6$ and $L(G_2) \downarrow \bar{S}_1 = 10/2$. It follows that $S < \bar{S} < A_1 A_2 G_2$ with $\bar{S} \cong A_1(K)$ having all weights on $L(G)$ less than p.

Finally, suppose R lies in a maximal subgroup F_4 of E_6. By [Se2] again, R lies in a maximal subgroup A_1 or $A_1 G_2$ of this F_4. The subgroup $A_1 G_2$ lies in a maximal $A_2 G_2$ of E_6, a case handled in the previous paragraph. Hence $R = A_1$; now R has composition factors as in (i) above, giving a contradiction as before.

This completes the proof of Step 5, and hence of the lemma. $\qquad\square$

Lemmas 5.2–5.5, together with the remarks at the beginning of this section, complete the proof of Theorem 1 for $p \neq 2$.

6. PROOF OF THEOREM 1 FOR $p = 2$

In this section we complete the proof of Theorem 1 by handling the case where $p = 2$. Thus we assume that $X = X(q) < G$ with $q = 2^e \geq 16$, and $G = E_8$ (as we may assume by 3.1). As before, by Proposition 2 we can take it that $X \neq L_2(q)$ or $^2B_2(q)$. Moreover, $X \neq A_2^\epsilon(16)$ by the hypothesis of Theorem 1.

Observe that X contains a subgroup A, where $A = A_2^\epsilon(q)$ or $B_2(q)$ if $q \geq 32$, and $A = B_2(16)$ or $G_2(16)$ if $q = 16$. Pick $a \in A$ of order divisible by $(q - \epsilon)/(3, q - \epsilon)$ if $A = A_2^\epsilon(q)$, and of order $q + 1$ otherwise, such that $C_A(a)^{(\infty)} = S \cong SL_2(q)$, generated by long root subgroups of A. Note that either $o(a) \geq 17$ or $o(a) = 11$ (which occurs only if $A = U_3(32)$).

In view of 3.2, we continue to adopt the assumption (†) made immediately after 3.2.

We begin with several preliminary results which will be required in the proof.

Lemma 6.1. *The restrictions to S of $V_A(10), V_A(01)$ and $V_A(11)$ are as in Table 4.*

TABLE 4

A	$V_A(10) \downarrow S$	$V_A(01) \downarrow S$	$V_A(11) \downarrow S$	$\dim V_A(10), V_A(01), V_A(11)$
$A_2^\epsilon(q)$	$1/0$	$1/0$	$1^2/1^{(2)}/0^2$	$3, 3, 8$
$B_2(q)$	1^2	$1/0^2$	$1^4/(1^{(2)})^2/0^4$	$4, 4, 16$
$G_2(q)$	$1^2/0^2$	$1^4/1^{(2)}/0^4$	$(1 \otimes 1^{(2)})^2/1^{10}/$	$6, 14, 64$
			$(1^{(2)})^9/0^{18}$	

Proof. This is elementary for the high weights 10 and 01, and follows for 11 from the following facts: for A_2, the module 11 is the quotient of $10 \otimes 01$ by a trivial submodule 00; for B_2, $11 = 10 \otimes 01$; and for G_2, 11 is the quotient of $10 \otimes 01$ by a submodule $10 \oplus 01$. $\qquad\square$

Lemma 6.2. (i) *If $L(G) \downarrow S$ has a composition factor $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_3})} \otimes 1^{(2^{i_4})}$, then $A \neq B_2(q)$ or $G_2(q)$; moreover, if this composition factor occurs with multiplicity 1, then $A \neq A_2^\epsilon(q)$ also.*

(ii) *If $L(G) \downarrow S$ has a composition factor $1^{(2^{i_1})} \otimes \ldots \otimes 1^{(2^{i_r})}$ with $r = 3$ or 4, and has no composition factor $1^{(2^{i_1})} \otimes \ldots \otimes 1^{(2^{i_{r-1}})}$, then $A \neq A_2^\epsilon(q)$ or $G_2(q)$.*

(iii) *If $L(G) \downarrow S$ has a composition factor $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_3})}$ with multiplicity 1, then it has $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}$ as a composition factor also.*

(iv) *If $L(G) \downarrow S$ has a composition factor $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_3})}$ with multiplicity less than 8, or has composition factors $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_3})}$, $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}$, the latter with multiplicity less than 8, or has two different composition factors $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_3})}$ and $1^{(2^{j_1})} \otimes 1^{(2^{j_2})} \otimes 1^{(2^{j_3})}$, then $A \neq G_2(q)$.*

(v) *If $L(G) \downarrow S$ has composition factors $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_3})}$ with multiplicity 2, $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}$ with multiplicity less than 4, and no composition factor $1^{(2^{i_2})} \otimes 1^{(2^{i_3})}$, then $A \neq B_2(q)$.*

(vi) *If $L(G) \downarrow S$ has composition factors $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_3})}$ with multiplicity a, $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{j})}$ ($j \neq i_3$) with multiplicity b, and $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}$ with multiplicity less than $a + b$, then $A \neq A_2^\epsilon(q)$.*

Proof. (i) Note first that $L(G) \downarrow S$ has no r-fold tensor composition factors with $r \geq 5$, since $\dim L(G) < 3^5$. By 6.1, $L(G) \downarrow A$ must also have a 4-fold tensor composition factor. If $A = B_2(q)$ or $G_2(q)$, this has dimension at least 4^4 or 6^4, which is greater than 248, a contradiction; and if $A = A_2^\epsilon(q)$ and the 4-fold factor in $L(G) \downarrow S$ occurs with multiplicity 1, then $L(G) \downarrow A$ has a self-dual 4-fold factor, of dimension at least 8^4, again a contradiction.

(ii) Suppose $A = A_2^\epsilon(q)$ or $G_2(q)$. By 6.1 and dimensions, $L(G) \downarrow A$ must have a composition factor $10^{(2^{i_1})} \otimes \ldots \otimes 10^{(2^{i_r})}$ (for type A_2, each 10 could also be 01); however the restriction of this to S has a composition factor $1^{(2^{i_1})} \otimes \ldots \otimes 1^{(2^{i_{r-1}})}$, contrary to the hypothesis of (ii).

(iii) This follows from (ii) unless $A = B_2(q)$. For this case, by 6.1, the multiplicity 1 assumption forces $L(G) \downarrow A$ to have a composition factor $01^{(2^{i_1})} \otimes 01^{(2^{i_2})} \otimes 01^{(2^{i_3})}$; however, the restriction of this to S has a composition factor $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}$.

(iv) Suppose $A = G_2(q)$. The hypothesis implies that $L(G) \downarrow A$ has a 3-fold composition factor, which by dimensions must be $10^{(2^{i_1})} \otimes 10^{(2^{i_2})} \otimes 10^{(2^{i_3})}$. The restriction of this to S conflicts with the assumptions in (iv).

(v) Suppose $A = B_2(q)$. Again $L(G) \downarrow A$ must have a 3-fold composition factor $\mu_1^{(2^{i_1})} \otimes \mu_2^{(2^{i_2})} \otimes \mu_3^{(2^{i_3})}$, where each $\mu_i \in \{10, 01\}$. Each possibility for the μ_i conflicts with the assumptions on multiplicities.

(vi) This follows from an easy extension of the argument for (ii). $\qquad\square$

Lemma 6.3. *Let $e \geq 5$, and take $\mathbb{Z}_e = \{0, 1, \ldots, e-1\}$ to act additively on itself (i.e. $a \in \mathbb{Z}_e$ sends $x \to x + a$). Let $k \in \{3, 4, 5\}$, and let $i_1 \ldots i_k$ be an unordered k-tuple of elements of $\mathbb{Z}_e$. Then $i_1 \ldots i_k$ is in the $\mathbb{Z}_e$-orbit of a k-tuple $j_1 \ldots j_k$ satisfying one of the following conditions:*

(i) *$j_r \leq e - 3$ for all r;*

(ii) *taking $j_1 \leq \ldots \leq j_k$, we have $j_k = e - 2$, $j_{k-1} \leq e - 3$, and, if $j_{k-1} = e - 3$, then $j_{k-2} \leq e - 4$;*

(iii) *$e = k = 5$ and $j_1 \ldots j_k = 01234, 01223, 00223, 01133$ or 01233.*

Proof. Assume that (i) and (ii) fail. If $e = k = 5$ and all the i_j are distinct, then clearly $i_1 \ldots i_k = 01234$, as in (iii). So suppose this is not the case. Then the number of distinct i_j is less than e. Take $i_1 \leq \ldots \leq i_k$. Applying an element of $\mathbb{Z}_e$, we can assume that $i_k < e - 1$. Therefore as (i) fails, we have $i_k = e - 2$; we also have $i_1 = 0$, since otherwise we can apply the element -1 of $\mathbb{Z}_e$ to obtain conclusion (i).

Thus we have $i_1 = 0, i_k = e - 2$. Applying the element 2 of $\mathbb{Z}_e$, we obtain (i) again, unless some i_j is $e - 3$ or $e - 4$. As (ii) fails, we deduce that either

(a) $i_1 \ldots i_k$ contains $0, e - 3, e - 3, e - 2$, or

(b) $i_1 \ldots i_k$ contains $0, e - 2, e - 2$.

Consider case (a). If $k \leq 4$, application of the element 3 of $\mathbb{Z}_e$ gives the tuple 3001, which satisfies (ii). Now suppose $k = 5$, so the tuple is $0, i, e - 3, e - 3, e - 2$. Applying $3 \in \mathbb{Z}_e$ gives the tuple $3, i + 3, 0, 0, 1$; this satisfies (i) or (ii) unless either $i = e - 4$ or $e = 5, i = 0$. In the latter case the tuple is 33001, in the same orbit as 00223, as in (iii). And if $i = e - 4$ the tuple is in the orbit of 40112. As (ii) fails, $e = 5$; then this is in the orbit of 01223, under (iii).

Now consider case (b). If $k \leq 4$ the tuple is $0, i, e - 2, e - 2$, which is in the same $\mathbb{Z}_e$-orbit as $2, i + 2, 0, 0$; this satisfies (ii) unless $i = e - 3$, in which case the tuple is in the orbit of 3011, which satisfies (ii).

Now suppose $k = 5$, so the tuple is $0, i, j, e - 2, e - 2 \ (i \leq j)$, in the same orbit as $2, i + 2, j + 2, 0, 0$. As (ii) fails, either $j = e - 3$, or $i = j = e - 4$, or $e = 5, i = 0, j = 1$. In the last case the tuple is 00223, under (iii). If $i = j = e - 4$ the tuple is in the orbit of 40022; as (i),(ii) fail, we must have $e = 5$ here, in which case the tuple is in the orbit of 01133, under (iii). Finally, suppose $j = e - 3$. Then the tuple is in the orbit of $3, i + 3, 0, 1, 1$; as (ii) fails, either $i = e - 4$ or $e = 5, i = 0$. In the latter case the tuple is 01133, under (iii). In the former, the tuple is in the orbit of 40122; as (ii) fails, $e = 5$ and this is in the orbit of 01233, under (iii). □

Corollary 6.4. *Let $e \geq 5$, $k \in \{3, 4\}$, and let $i_1 \ldots i_k$ be an unordered k-tuple of elements of $\mathbb{Z}_e$; if $k = 4$, assume that $i_1, \ldots, i_k$ are not all distinct. Then $i_1 \ldots i_k$ is in the $\mathbb{Z}_e$-orbit of a k-tuple $j_1 \ldots j_k$ satisfying one of the following conditions:*

(i) *$j_r \leq e - 3$ for all r;*

(ii) *$e = 5$ and $j_1 \ldots j_k = 0013, 0023, 0113$ or 013;*

(iii) *$e = 6$ and $j_1 \ldots j_k = 0024$ or 024.*

Proof. Consider first the case where $k = 4$, and take $i_1 \leq \ldots \leq i_4$. As in the proof of the previous result, we may take $i_4 = e - 2, i_1 = 0$. By 6.3 we may assume that 6.3(ii) holds. Thus $i_1 \ldots i_4$ is either $0, i, e - 3, e - 2$ (with $i \leq e - 4$) or $0, i, j, e - 2$ (with $i \leq j \leq e - 4$). In the first case the tuple is in the orbit of $3, i + 3, 0, 1$. This satisfies (i) unless $i = e - 4$ or $e = 5, i = 0$; in the latter the tuple is in the orbit of 0023, under (iii), and in the former, the tuple has 4 distinct elements, contrary to hypothesis.

Now suppose the tuple is $0, i, j, e - 2$ with $i \leq j \leq e - 4$. This is in the orbit of $2, i + 2, j + 2, 0$, which satisfies (i) unless $j = e - 4$. Since $i_1, \ldots i_4$ are not all distinct, i must be 0 or $e - 4$. Hence the tuple is in the orbit of 0244 or 0024. These satisfy (i) unless $e = 5$ or 6, in which case they lie in the orbits of tuples under (iii). This completes the case $k = 4$, and the case $k = 3$ is entirely similar. □

As in previous sections, in the course of our proof of Theorem 1 we shall study the embedding of our subgroup $S \cong SL_2(q)$ in $C_G(a)$. As before, define $D = C_G(a)'$, a connected semisimple group. By Lemmas 3.4–3.7 we know that D is as in 3.5(5). In particular, either D has a factor D_4 or D_5, or D is a product of factors A_r. For the D_4, D_5 possibilities we need to study in some detail the embeddings of subgroups $SL_2(q)$ in these groups.

For this purpose we now introduce some notation for certain connected semisimple subgroups of D_4, D_5 (in characteristic 2). It is convenient to treat these groups

in the forms SO_8, SO_{10}. First, we denote by A_1 a connected fundamental subgroup SL_2 in SO_8 or SO_{10}. Thus A_1^4 denotes a subgroup $SO_4 SO_4$ in SO_8, naturally embedded. Taking a subgroup SO_4 of SO_8 fixing pointwise a non-degenerate 4-space, we write B_1 for a subgroup SO_3 of this SO_4 fixing a nonsingular vector. Then SO_6 contains a subgroup B_1^2, SO_8 a subgroup B_1^3, and SO_{10} a subgroup B_1^4; in each case, the subgroup B_1^k fixes a unique nonsingular 1-space. Note that if v is a nonsingular vector then $(SO_{10})_v = SO_9$, and a surjective morphism $SO_9 \to Sp_8$ sends $B_1^4 \to C_1^4 = (Sp_2)^4$, naturally embedded in Sp_8. Finally, SO_8 contains a subgroup $Sp_2 \otimes Sp_2 \otimes Sp_2$, which we denote by C_1^3; and C_1^2 denotes a subgroup $Sp_2 \otimes Sp_2$ of this.

Let $S_0 = A_1(2^e)$, and fix an injection $\iota : S_0 \to SL_2(K)$. Whenever $S_0 = A_1(2^e)$ maps homomorphically into a commuting product $A_1^k B_1^l$, up to conjugacy the projections of S_0 to the factors are either trivial, or are field twists of the injection ι. In the following result, when we say S_0 has "distinct twists on the B_1 factors", we mean that the twists involved in all the nontrivial projections to B_1 factors are all distinct.

Finally, if Y is a connected reductive group, we shall denote by QY a connected group with unipotent radical Q.

Lemma 6.5. *Let* $S_0 = SL_2(2^e)$ $(e \geq 4)$, *and suppose* $S_0 < D_r$ *with* $r = 4$ *or* 5.
  (i) *If* $r = 4$ *then* S_0 *lies in one of the following connected subgroups of* D_4:

$$A_1^4, \; QA_1^3, \; B_1^3, \; QB_1^2, \; QC_1^2, \; C_1^3.$$

In the last four cases S_0 *has distinct twists on the* B_1 *and* C_1 *factors.*
  (ii) *If* $r = 5$ *then* S_0 *lies in one of the following connected subgroups of* D_5:

$$QA_1^4, \; B_1^4, \; A_1^2 B_1^2, \; QA_1 B_1^2, \; QB_1^3, \; QC_1^2, \; C_1^3.$$

In the third case $A_1^2 B_1^2 < SO_4 SO_6$. *Also,* S_0 *has distinct twists on the* B_1 *and* C_1 *factors in all cases with such factors.*

Proof. In the course of the proof we shall require information on the possible embeddings of S_0 in the symplectic groups Sp_4, Sp_6, Sp_8, as well as in SO_6, SO_8, SO_{10}. It is convenient to start with the smallest dimensions and work our way upwards. For a classical group $G = Sp_{2n}$ or SO_{2n}, we shall denote the natural G-module by V_{2n}. If $S_0 < G$, let W be an irreducible S_0-submodule of V_{2n}; then W has dimension 1,2,4 or 8, and W is either non-degenerate or totally singular, or, when G is orthogonal, a nonsingular 1-space.

Suppose that $S_0 < Sp_4$, with natural module V_4. As above, let W be an irreducible S_0-submodule of V_4. If $W = V_4$ then $V_4 \downarrow S_0 = 1^{(2^{i_1})} \otimes 1^{(2^{i_2})}$, and S_0 lies in a connected subgroup $Sp_2 \otimes Sp_2 = C_1 \otimes C_1$ of Sp_4. If W is a non-degenerate 2-space then $S_0 < Sp_2 \perp Sp_2 = C_1 C_1$. If W is a 1-space, then $V_4 \downarrow S_0 = 0/1^{(2^i)}/0$; we can take this to be indecomposable, as otherwise we are in the previous case. Here S_0 lies in a parabolic subgroup $QC_1 T_1$ of Sp_4, and the unipotent radical Q has a 1-dimensional subgroup Z such that Q/Z is an S_0-module of high weight 1; moreover, S_0 acts trivially on Z. Choose $\tilde{S}_0 < C_1$ such that $S_0 \cong \tilde{S}_0$ and $QS_0 = Q\tilde{S}_0$. Then $H^1(\tilde{S}_0, Q/Z)$ is 1-dimensional by 1.2, so under the action of $(Q/Z)T_1$ there are just 2 classes of complements to Q/Z in $(Q/Z)\tilde{S}_0$, both of which give complements to Q in $Q\tilde{S}_0$. One of these, $\tilde{S}_0$, lies in C_1; the other, S_0, lies in an indecomposable B_1 contained in $SO_4 < Sp_4$. Consequently $S_0 < B_1 < C_1 \otimes C_1 = SO_4$ in this case. The last possibility is that W is a singular 2-space. A graph automorphism τ of

Sp_4 maps the stabilizer of W to the stabilizer of a 1-space. Hence by the previous case, S_0^τ lies in $C_1 \otimes C_1$, whence S_0 lies in $C_1 C_1$. Summarising, we have shown that if $S_0 < Sp_4$ then S_0 lies in either $C_1 \otimes C_1$ or $C_1 C_1$.

Next suppose $S_0 < SO_6$, with natural module V_6. If W is non-degenerate of dimension 2 or 4, then $S_0 < SO_4 = A_1^2$. If W is a singular 1-space or 2-space, then $S_0 < QA_1^2$ or QA_1. The only other possibility is that W is a nonsingular 1-space, in which case $V_6 \downarrow S_0 = 0/V_4/0$, where V_4 is symplectic 4-space. Applying the previous paragraph, S_0 maps into a subgroup $C_1 C_1$ or $C_1 \otimes C_1$ of $Sp(V_4)$. Thus S_0 lies in a subgroup B_1^2 or A_1^2 of SO_6. Moreover, if $S_0 < B_1^2$ with equal twists on the factors, then $V_6 \downarrow S_0 = 0/1^{(2^i)}/1^{(2^i)}/0$; as $\text{Ext}_{S_0}^1(1^{(2^i)}, 0)$ is 1-dimensional by 1.2, it follows that S_0 fixes a singular 2-space, yielding $S_0 < QA_1$. Summarising, if $S_0 < SO_6$ then either $S_0 < QA_1^2$ or $S_0 < B_1^2$ with distinct twists on the B_1 factors.

Next consider $S_0 < Sp_6$, with natural module V_6. If W is non-degenerate then $S_0 < C_1 C_2$, so by the Sp_4 case handled above, S_0 lies in $C_1(C_1 \otimes C_1)$ or C_1^3. If W is a singular 2-space then $S_0 < QA_1 C_1$. Moreover, either $V_6 \downarrow S_0 = 1^{(2^i)}/1^{(2^j)}/1^{(2^i)}$, completely reducible by 1.2, or $V_6 \downarrow S_0 = 1^{(2^i)}/0^2/1^{(2^i)}$; hence either $S_0 < A_1 C_1$ or $S_0 < QA_1$. Finally, if W is a 1-space then $S_0 < QC_2$, hence S_0 lies in $Q(C_1 \otimes C_1)$ or $QC_1 C_1$.

Similarly, if $S_0 < Sp_8$ then S_0 lies one of the subgroups $C_1 \otimes C_1 \otimes C_1$ (with S_0 irreducible on V_8), $C_2 C_2$, $C_1 C_3$, $QA_1 C_2$, QC_3 or QA_3 (in the last case W is a singular 4-space), and we can apply the previous conclusions for subgroups of C_2, C_3 to refine this list.

Now we are ready to consider $S_0 < SO_8$, with natural module V_8. If $W = V_8$ then $S_0 < C_1^3$, with distinct twists on the factors, and if W is a non-degenerate 4-space then $S_0 < A_1^4$. If W is a singular 4-space, then $W \downarrow S_0 = 1^{(2^i)} \otimes 1^{(2^j)}$ and $S_0 < QC_1^2 < QA_3$. If W is a singular 2-space then $S_0 < QA_1^3$. And if W is a singular 1-space, then $V_8 \downarrow S_0 = 0/V_6/0$ with V_6 an SO_6-space, so by the SO_6 case above, S_0 lies in QA_1^2 or QB_1^2 (with distinct twists in the latter case).

To complete the analysis of $S_0 < SO_8$, it remains to handle the case where W is a nonsingular 1-space. Here $V_8 \downarrow S_0 = 0/V_6/0$, where V_6 is symplectic 6-space, and S_0 maps into Sp_6. By the Sp_6 case above, S_0 maps into a subgroup $C_1(C_1 \otimes C_1), C_1^3, QA_1 C_1, Q(C_1 \otimes C_1)$ or $QC_1 C_1$ of Sp_6. In the first case $S_0 < B_1 A_1^2 < A_1^4$ (in SO_8). In the second, $S_0 < B_1^3$; and if S_0 has equal twists on two of the B_1 factors here, then $V_8 \downarrow S_0 = 0/1^{(2^i)}/1^{(2^i)}/...$, and an application of 1.2 shows that S_0 fixes a 2-space $1^{(2^i)}$, putting us in a previous case. In the third case $S_0 < QA_1 B_1$, which lies in QA_1^3 in SO_8. In the fourth case $S_0 < QA_1^2$. Finally, in the last case $S_0 < QB_1^2$, and if S_0 has equal twists in the B_1 factors then $S_0 < QA_1$. This completes the proof of part (i) of the lemma.

Now consider $S_0 < SO_{10}$, with natural module V_{10}. If W has dimension 8 then $S_0 < C_1^3$ (with distinct twists). If W is a non-degenerate 4-space then $S_0 < SO_4 SO_6$, so by the SO_6 case S_0 lies in QA_1^4 or $A_1^2 B_1^2$. If W is a singular 4-space then $V_{10} \downarrow S_0 = (1^{(2^i)} \otimes 1^{(2^j)})^2/0^2$, and $S_0 < QC_1^2$. And if W is a singular 1-space, then $S_0 < QSO_8$, and conclusion (ii) follows from the SO_8 case (i) (note that the possibility $S_0 < QC_1^3$ only arises in the SO_8 proof when S_0 is irreducible on V_8, in which case $V_{10} \downarrow S_0$ is completely reducible by 1.2, giving $S_0 < C_1^3$ with distinct twists).

Next suppose that W is a nonsingular 1-space, so $V_{10} \downarrow S_0 = 0/V_8/0$, where V_8 is Sp_8-space. By the Sp_8 case above, S_0 maps into a subgroup $C_1 \otimes C_1 \otimes C_1$,

C_2C_2, C_1C_3, QA_1C_2, QA_3 or QC_3 of Sp_8. In the first case we have $S_0 < C_1^3$ with distinct twists. Next, if S_0 maps into C_2C_2 then, by the Sp_4 case above, S_0 lies in A_1^4, $A_1^2B_1^2$ or B_1^4 in SO_{10}. Moreover we may take it that S_0 has distinct twists on the B_1 factors, since equal twists on a pair of B_1 factors allow us to replace the group B_1^2 by QA_1, putting S_0 inside QA_1^3 or $QA_1B_1^2$, as in the conclusion of (ii). If S_0 maps into C_1C_3 then, by the Sp_6 case, S_0 maps into $C_1^2(C_1 \otimes C_1)$, C_1^4, $C_1^2A_1$, QA_1C_1, $QC_1(C_1 \otimes C_1)$, or QC_1^3. Hence in SO_{10}, S_0 lies in $A_1^2B_1^2$, B_1^4, $A_1B_1^2$, QA_1B_1, $QA_1^2B_1$ or QB_1^3. The only subgroup here not in conclusion (ii) is $QA_1^2B_1$, which lies in QA_1^4. Next, if S_0 maps into QA_1C_2 in Sp_8, then $S_0 < QA_1^3$ or $QA_1B_1^2$ in SO_{10}. If $S_0 < QA_3$ then $S_0 < QC_1^2$. Finally in this case (W a nonsingular 1-space), if S_0 maps into QC_3 in Sp_8, then S_0 fixes a singular 1-space in V_{10}, putting us in a previous case.

It remains to handle the case where W is a singular 2-space. Here $S_0 < QA_1D_3$, so by the SO_6 case, S_0 lies in either QA_1^3 or $QA_1B_1^2$ (with distinct twists), as in (ii).

This completes the proof of the lemma. □

Corollary 6.6. *If $D = C_G(a)'$ has a factor D_4 or D_5, then one of the following holds:*

(i) $S < QA_1^k < D$ with $k \leq 5$ (the A_1^k being fundamental A_1's in G);

(ii) $S < QA_1B_1^3 < A_2D_4$ or A_1D_5;

(iii) $S < QA_1^2B_1^2 < A_1D_5$;

(iv) $S < A_1B_1^4 < A_1D_5$;

(v) $S < A_1^3B_1^2 < A_1D_5$.

In cases (ii)–(v), *S has distinct twists on the B_1 factors.*

Proof. Observe that $D \neq A_1A_1D_4$, since otherwise $a \in C_G(A_1A_1D_4) = A_1A_1 = C_G(D_6)$, contradicting 3.7. Hence, as D satisfies 3.5(5), we see that $D = A_rD_4$ or A_sD_5 with $r \leq 2, s \leq 1$. The projection S_0 of S in the factor D_4 or D_5 satisfies 6.5(i) or 6.5(ii). We see from this that either the result holds, or S_0 lies in a subgroup C_1^3 or QC_1^2 of the factor D_4 or D_5. If $S_0 < C_1^3$, then $S_0 < C_1^3 < D_4$, and $N_G(D_4)$ contains a triality automorphism of this D_4. This triality sends the subgroup C_1^3 to a subgroup B_1^3 of D_4, and hence a conjugate of S lies in a subgroup $A_rB_1^3$ or $A_sB_1^3$ of D, as in (ii) of the conclusion. A similar argument applies if $S_0 < QC_1^2$; here $S_0 < QC_1^2 < D_4$, and application of triality gives a conjugate of S lying in $A_r(QB_1^2)$ or $A_s(QB_1^2)$, yielding (ii) again. □

In the next lemma we consider the other possibilities for D.

Lemma 6.7. *Suppose that D is a product of factors A_m. Then conclusion* (i) *or* (v) *of Corollary 6.6 holds.*

Proof. Recall that D is as in 3.5(5). Note also that $D \neq A_3A_4$ or A_1A_6 by 3.4. Assume that 6.6(i) does not hold. Then D must have a factor $A_{3+\delta}$ ($\delta \geq 0$) with natural module V such that $V \downarrow S$ has a composition factor $1^{(2^i)} \otimes 1^{(2^j)}$; and $D \neq A_1A_5$, since otherwise $a \in C_G(A_1A_5) = A_2 = C_G(E_6)$, contrary to 3.6. Moreover, if D has a factor A_4, then the natural module for this factor restricts to S as $1^{(2^i)} \otimes 1^{(2^j)}/0$; this splits by 1.2, and hence S projects into a subgroup A_3 of this factor. We conclude that one of the following holds:

(a) $S < D = A_5$ or A_6, with embedding $1^{(2^i)} \otimes 1^{(2^j)}/1^{(2^k)}/0^m$ or $1^{(2^i)} \otimes 1^{(2^j)}/0^{2+m}$ ($m \leq 1$);

(b) $S < A_1^3 A_3$, with projection $1^{(2^i)} \otimes 1^{(2^j)}$ in the A_3 factor;

(c) $S < A_3 A_3$, with projection $1^{(2^i)} \otimes 1^{(2^j)}$ in one of the A_3 factors;

(d) $S < D = A_1 A_2 A_3$, with projection $1^{(2^i)} \otimes 1^{(2^j)}$ in the A_3 factor.

Consider first case (b), $S < A_1^3 A_3$. Now $A_1^3 A_3 < A_1^3 D_4$ in G, and $N_G(A_1^3 D_4)$ contains an element τ of order 3 inducing a triality automorphism on the D_4 factor and permuting the A_1 factors cyclically. Since the projection of S to the A_3 factor is $1^{(2^i)} \otimes 1^{(2^j)}$, we see that for some k, S^{τ^k} projects into a subgroup B_1^2 in the D_4 factor. Hence $S < A_1^3 B_1^2 < A_1^3 D_4$. Moreover this $A_1^3 B_1^2 < A_1^3 D_3 < A_1^3 D_4$, so $S < A_1^3 B_1^2 < A_1^3 D_3 < A_1 D_5$, as in 6.6(v). Note that the twists on the B_1 factors are distinct.

Now suppose case (a) holds, and let V be the natural module for $D = A_5$ or A_6. If $V \downarrow S$ is completely reducible then $S < A_1 A_3 < D$, as in case (b) handled above. So assume $V \downarrow S$ is not completely reducible. Then applying a twist to the labelling of S, we can take $V \downarrow S = 1 \otimes 1^{(2^i)}/1^{(2^j)}/0^m$ for some i, j. If $j \neq 0, i$ then by 1.2, $V \downarrow S = (1 \otimes 1^{(2^i)}) \oplus (1^{(2^j)}/0^m)$, so $S < A_3 A_2 < A_3 A_3$, as in case (c), which we handle below.

Thus we assume now that $j = 0$ or i. Applying a twist, we may take $j = 0$, so $V \downarrow S = 1 \otimes 1^{(2^i)}/1/0^m$. Recall that $q = 2^e$ with $e \geq 4$, and the nontrivial composition factors of $L(G) \downarrow D$ are of the form $V_D(\lambda_r)$, together with adjoint modules of the factors ([LS2, §2]). If $i \leq e-3$ then we can choose a rank 1 torus T_1 in D (namely $T_1 = \{\mathrm{diag}(c^{2^i+1}, c^{2^i-1}, c^{-2^i-1}, c^{-2^i+1}, c, c^{-1}, 1^m) : c \in K^*\}$) containing an element $t \in S$ of order $q+1$, such that t and T_1 fix the same subspaces of $L(G)$, contrary to (†). If $i = e-1$, we can twist to take $V \downarrow S = 1^{(2)} \otimes 1/1^{(2)}/0^m$ and apply the same T_1 argument. And if $i = e-2$ we twist to $V \downarrow S = 1^{(4)} \otimes 1/1^{(4)}/0^m$, and the T_1 argument applies unless $q = 16$. Thus to complete this case (a), assume now that

$$q = 16, \quad V \downarrow S = 1 \otimes 1^{(4)}/1/0^m.$$

By [LS2, Table 8.1], $L(G) \downarrow A_5 = L(A_5)/\lambda_1^6/\lambda_2^3/\lambda_3^2/\lambda_4^3/\lambda_5^6/0^{11}$. Using [LS2, 2.13] we deduce from this that

$$\begin{aligned}
L(G) \downarrow S = \ & (1 \otimes 1^{(2)})^2/(1 \otimes 1^{(4)})^{16}/(1 \otimes 1^{(8)})^2/(1^{(2)} \otimes 1^{(4)})^8/ \\
& (1^{(2)} \otimes 1^{(8)})^1/1^{16}/(1^{(2)})^9/(1^{(4)})^{16}/(1^{(8)})^8/0^{34}.
\end{aligned}$$

As $q = 16$, we have $A = B_2(16)$ or $G_2(16)$. If $A = B_2(16)$ then S contains a conjugate b of a (of order 17). But from $L(G) \downarrow S$ we see that $\dim C_G(b) = 34$, whereas $\dim C_G(a) \geq \dim A_5 T_3 = 38$, a contradiction. Now suppose $A = G_2(16)$. Since $L(G) \downarrow S$ has composition factor $1^{(2)} \otimes 1^{(8)}$ with multiplicity 1, we see from Lemma 6.1 that $L(G) \downarrow A$ must have a composition factor $01 \otimes 01^{(4)}$; but the restriction of this to S has composition factor $1 \otimes 1^{(8)}$ with multiplicity 4, which contradicts $L(G) \downarrow S$ given above. This completes the argument for case (a).

Next consider case (c), $S < A_3 A_3$. If $S < A_1^2 A_3$ then we are in case (b), so assume this is not so. Then the embedding of S in $A_3 A_3$ is either $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}, 1^{(2^{i_3})} \otimes 1^{(2^{i_4})}$ or $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}, 1^{(2^{i_3})}/0^2$, and in the latter case the second representation is indecomposable. By [LS2, p.60], the composition factors of $L(G) \downarrow A_3 A_3$ are among $L(A_3 A_3)$, $\lambda_i \otimes \lambda_j$, $\lambda_i \otimes 0$ and $0 \otimes \lambda_j$, and include $\lambda_1 \otimes \lambda_1$.

Suppose first that the embedding is $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}, 1^{(2^{i_3})} \otimes 1^{(2^{i_4})}$. If $i_1, \ldots, i_4$ are

all distinct, then $L(G) \downarrow S$ has a composition factor $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_3})} \otimes 1^{(2^{i_4})}$ (coming from $\lambda_1 \otimes \lambda_1$), and either no $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_3})}$ or no $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_4})}$; this contradicts 6.2(i,ii). Therefore $i_1, \ldots, i_4$ are not all distinct. If $e \geq 5$ we can twist to take $i_1 \ldots i_4$ to satisfy (i), (ii) or (iii) of 6.3. Now $S < \bar{S} < A_3 A_3$, where $\bar{S} \cong SL_2(K)$ has embedding $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}, 1^{(2^{i_3})} \otimes 1^{(2^{i_4})}$ in $A_3 A_3$, and it is easily checked that $\bar{S}$ has all weights on $L(G)$ less than q. Then by 1.5, S and $\bar{S}$ fix the same subspaces of $L(G)$, contradicting (†). Finally suppose $e = 4$. We can twist to take the embedding the embedding of S in $A_3 A_3$ to be

$$1 \otimes 1^{(2)}, 1^{(2)} \otimes 1^{(4)} \text{ or } 1 \otimes 1^{(4)}, 1 \otimes 1^{(2)} \text{ or } 1 \otimes 1^{(4)}, 1^{(2)} \otimes 1^{(4)} \text{ or }$$
$$1 \otimes 1^{(2)}, 1 \otimes 1^{(2)} \text{ or } 1 \otimes 1^{(4)}, 1 \otimes 1^{(4)}.$$

Choose $\bar{S} \cong SL_2(K)$ with $S < \bar{S} < A_3 A_3$ and $\bar{S}$ having one of these embeddings in $A_3 A_3$. Then in the first, second and fourth embeddings, $\bar{S}$ has all weights on $L(G)$ less than $q = 16$. In the third case, $\bar{S}$ could have a composition factor of high weight 16 coming from a composition factor $\lambda_2 \otimes \lambda_2$ for $L(G) \downarrow A_3 A_3$. By [LS2, p.60], the presence of a composition factor $\lambda_2 \otimes \lambda_2$ implies that there are no composition factors $\lambda_i \otimes \lambda_2$ with $i \in \{1, 3\}$; so we see that $\bar{S}$ has highest weight 16 on $L(G)$, and has no composition factor of high weight 1. Hence 1.4 implies that S and $\bar{S}$ fix the same subspaces of $L(G)$, contradicting (†). The same argument handles the final embedding $1 \otimes 1^{(4)}, 1 \otimes 1^{(4)}$.

Now suppose that the embedding of S in $A_3 A_3$ is $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}, 1^{(2^{i_3})}/0^2$. Assume first that $e \geq 5$. Then we can take $i_1 i_2 i_3$ (in some order) to satisfy the conclusion of 6.4. If $i_r \leq e - 3$ for all r, then the T_1 argument applies (i.e. there is a torus T_1 in $A_3 A_3$ containing an element $t \in S$ of order $q + 1$ and fixing the same subspaces of $L(G)$ as t, contrary to (†)). Otherwise, by 6.4, we have $e = 5$ or 6 and $\{i_1, i_2, i_3\} = \{0, 1, 3\}$ or $\{0, 2, 4\}$ respectively. In the first case, either the T_1 argument applies, or the embedding of S is $1^{(2)} \otimes 1^{(8)}, 1/0^2$. From the two possible restrictions $L(G) \downarrow A_3 A_3$ given in [LS2, p.60], we deduce that $L(G) \downarrow S$ has 8 composition factors $1 \otimes 1^{(2)} \otimes 1^{(8)}$, but no composition factors $1 \otimes 1^{(8)}$; hence $A \neq A_2^\epsilon(q)$ by 6.2(ii). Thus $A = B_2(q)$, whence S contains a conjugate b of a (of order $q + 1$). Again using [LS2, p.60], we see that $C_G(b)$ has dimension 34. Now $A_3 A_3$ lies in a subgroup $D_4 D_4$ of G, which in turn lies in a subgroup D_8. Consideration of the eigenvalues of b on the natural D_8-module shows that $C_{D_8}(b)'$ is either A_1^5 or $A_1 D_4$. Since $C_G(b)$ has dimension 34, this forces $C_G(b) = A_1 D_4 T_3$, contrary to the hypothesis of the lemma.

To complete case (c), assume that $e = 4$ (and the embedding of S in $A_3 A_3$ is $1^{(2^{i_1})} \otimes 1^{(2^{i_2})}, 1^{(2^{i_3})}/0^2$). If we can twist to take $(i_1, i_2, i_3) = (0, 1, 2), (0, 2, 1), (0, 1, 0)$ or $(0, 1, 1)$, then the T_1 argument applies. The only remaining possibilities are $(i_1, i_2, i_3) = (1, 2, 0)$ or $(0, 2, 0)$. For $A = B_2(16)$ these are handled as in the previous paragraph. For $A = G_2(16)$ in the $(0, 2, 0)$ case, we find that $L(G) \downarrow S$ has composition factor $1 \otimes 1^{(2)}$ with multiplicity 2, conflicting with 6.1. And for $A = G_2(16)$ in the $(1, 2, 0)$ case, we work out the composition factors of $L(G) \downarrow S$ (using [LS2, p.60]), and show that these conflict with 6.2. This completes case (c).

It remains to consider case (d), $S < D = A_1 A_2 A_3$. We can take it that the projections of S to each of the 3 factors are nontrivial (otherwise we are in a previous case). Say the embedding is $1^{(2^{i_1})}, 1^{(2^{i_2})}/0, 1^{(2^{i_3})} \otimes 1^{(2^{i_4})}$. Now D lies in a

maximal rank subgroup $A_1 A_2 A_5$ of G, and from [LS2, §2] we have

$$L(G) \downarrow A_1 A_2 A_5$$
$$= L(A_1 A_2 A_5)/1 \otimes \lambda_1 \otimes \lambda_1/1 \otimes \lambda_2 \otimes \lambda_5/1 \otimes 0 \otimes \lambda_3/0 \otimes \lambda_1 \otimes \lambda_2/0 \otimes \lambda_2 \otimes \lambda_4.$$

If $i_1, \ldots, i_4$ are all distinct then $L(G)$ has a composition factor $1^{(2^{i_1})} \otimes \ldots \otimes 1^{(2^{i_4})}$ (from $1 \otimes \lambda_1 \otimes \lambda_1$), but no composition factor $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_3})}$, contrary to 6.2(i,ii). Thus $i_1, \ldots, i_4$ are not all distinct.

Assume now that $e \geq 5$. Then we can take $i_1 \ldots i_4$ to satisfy the conclusion of 6.4. If $i_r \leq e-3$ for all r then the T_1 argument applies. Otherwise, $i_1 \ldots i_4$ (in some order) is one of $0013, 0023, 0113$ (all with $e = 5$) or 0024 (with $e = 6$). If $i_2 > 0$ then $S < \bar{S} < D$, where $\bar{S} \cong SL_2(K)$ with embedding $1^{(2^{i_1})}, 1^{(2^{i_2})}/0, 1^{(2^{i_3})} \otimes 1^{(2^{i_4})}$ (the point being that by 1.2, for $SL_2(K)$, the indecomposable $1^{(2^{i_2})}$ exists only if $i_2 > 0$, in which case $\mathrm{Ext}_S(1^{(2^{i_2})}, 0) \cong \mathrm{Ext}_{SL_2(K)}(1^{(2^{i_2})}, 0)$). But now $\bar{S}$ has all weights on $L(G)$ less than q. Hence $i_2 = 0$, and the embedding of S in D is one of

$1, 1/0, 1^{(2)} \otimes 1^{(8)}$ or $1^{(2)}, 1/0, 1 \otimes 1^{(8)}$ or $1^{(8)}, 1/0, 1 \otimes 1^{(2)}$ $(e = 5)$,
$1, 1/0, 1^{(4)} \otimes 1^{(8)}$ or $1^{(4)}, 1/0, 1 \otimes 1^{(8)}$ or $1^{(8)}, 1/0, 1 \otimes 1^{(4)}$ $(e = 5)$,
$1^{(2)}, 1/0, 1^{(2)} \otimes 1^{(8)}$ $(e = 5)$,
$1, 1/0, 1^{(4)} \otimes 1^{(16)}$ or $1^{(4)}, 1/0, 1 \otimes 1^{(16)}$ or $1^{(16)}, 1/0, 1 \otimes 1^{(4)}$ $(e = 6)$.

Twist the embeddings in the first and third rows by 2^2, those in the second row by 2^3, and those in the fourth row by 2^2; so for example, the embeddings in the first row now read

$1^{(4)}, 1^{(4)}/0, 1^{(4)} \otimes 1$ or $1^{(8)}, 1^{(4)}/0, 1^{(4)} \otimes 1$ or $1, 1^{(4)}/0, 1^{(4)} \otimes 1^{(8)}$ $(e = 5)$.

Now $S < \bar{S} < D$, with $\bar{S}$ embedded in D with the new labelling, and we see that the highest weight of $\bar{S}$ on $L(G)$ is less than q. This completes the proof for $e \geq 5$.

To complete the proof for case (d), assume finally that $e = 4$. Then we can take $i_1 \ldots i_4$ (in some order) to be $0012, 0112, 0122$ or 0022 (in all other cases the T_1 argument applies). As above, $i_2 = 0$.

For the tuple 0012, the embedding of S is $1, 1/0, 1^{(2)} \otimes 1^{(4)}$ or $1^{(2)}, 1/0, 1 \otimes 1^{(4)}$ or $1^{(4)}, 1/0, 1 \otimes 1^{(2)}$. In the last case, the T_1 argument applies. In the second case, $L(G) \downarrow S$ has composition factor $1 \otimes 1^{(2)} \otimes 1^{(4)}$ with multiplicity 3, so $A \neq G_2(16)$ by 6.2(iii); also if $b \in S$ of order 17, we calculate from $L(G) \downarrow D$ above that $C_{L(G)}(b)$ has dimension 24, so $A \neq B_2(16)$ (as otherwise b is conjugate to a, whereas $C_G(a) = A_1 A_2 A_3 T_2$, of dimension 28). In the first case, we twist to take the embedding to be $1^{(2)}, 1^{(2)}/0, 1^{(4)} \otimes 1^{(8)}$. As explained before, we have $S < \bar{S} < D$, where $\bar{S} \cong SL_2(K)$ has this embedding in D. The composition factors of $\bar{S}$ on $L(G)$ of high weight 16 or more are $1^{(16)}$ and $1^{(2)} \otimes 1^{(16)}$, both of which are irreducible for S; moreover, $\bar{S}$ has no composition factors 1 or $1^{(2)} \otimes 1$. Hence S and $\bar{S}$ fix the same subspaces of $L(G)$, by 1.4, contradicting (†).

The other tuples $0112, 0122$ and 0022 are handled in a similar fashion. This completes the proof of the lemma. $\quad\square$

In view of the lemma just proved, it remains to deal with S as in cases (i)–(v) of Corollary 6.6. To handle cases (ii)–(v), we require information about the restriction of $L(G)$ to the subgroups $A_1 B_1^4$ and $A_1^2 B_1^3$, which is provided by the next lemma. In the conclusion, by a "1-fold tensor" we mean a module for the product $A_1 B_1^4$ or $A_1^2 B_1^3$ in which only one of the simple factors acts nontrivially.

Lemma 6.8. (i) *For $A_1 B_1^4 < A_1 D_5 < G$, the nontrivial composition factors of $L(G) \downarrow A_1 B_1^4$ are as follows:*

> *1-fold tensors 1 or $1^{(2)}$,*
> $0 \otimes 1^{(2)} \otimes 1^{(2)} \otimes 0 \otimes 0, \dots, 0 \otimes 0 \otimes 0 \otimes 1^{(2)} \otimes 1^{(2)}$ *(one for each pair of B_1 factors, each with mult. 1),*
> $1 \otimes 1^{(2)} \otimes 0 \otimes 0 \otimes 0, \dots, 1 \otimes 0 \otimes 0 \otimes 0 \otimes 1^{(2)}$ *(one for each B_1 factor, each with mult. 2),*
> $1 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ *(mult. 2),* $0 \otimes 1 \otimes 1 \otimes 1 \otimes 1$ *(mult. 4).*

(ii) *For $A_1^3 B_1^2 = A_1(A_1^2 B_1^2) < A_1(D_2 D_3) < A_1 D_5 < G$, the nontrivial composition factors of $L(G) \downarrow A_1^3 B_1^2$ are:*

> *1-fold tensors 1 or $1^{(2)}$ (only $1^{(2)}$ for B_1 factors),*
> $0 \otimes 1 \otimes 1 \otimes 0 \otimes 0, \ 0 \otimes 1 \otimes 1 \otimes 1^{(2)} \otimes 0, \ 0 \otimes 1 \otimes 1 \otimes 0 \otimes 1^{(2)},$
> $0 \otimes 0 \otimes 0 \otimes 1^{(2)} \otimes 1^{(2)}, 1 \otimes 1 \otimes 1 \otimes 0 \otimes 0, \ 1 \otimes 0 \otimes 0 \otimes 1^{(2)} \otimes 0,$
> $1 \otimes 0 \otimes 0 \otimes 0 \otimes 1^{(2)}, 0 \otimes 1 \otimes 0 \otimes 1 \otimes 1, \ 0 \otimes 0 \otimes 1 \otimes 1 \otimes 1,$
> $1 \otimes 1 \otimes 0 \otimes 1 \otimes 1, \ 1 \otimes 0 \otimes 1 \otimes 1 \otimes 1$ *(both with mult. 2).*

Proof. By [LS2, §2], $L(G) \downarrow A_3 D_5 = L(A_3)/L(D_5)/\lambda_2 \otimes \lambda_1/\lambda_1 \otimes \lambda_4/\lambda_3 \otimes \lambda_5$, whence $L(G) \downarrow A_1 D_5$

$$= (L(A_3) \downarrow A_1)/L(D_5)/(1 \otimes \lambda_1)^2/(0 \otimes \lambda_1)^2/1 \otimes \lambda_4/(0 \otimes \lambda_4)^2/1 \otimes \lambda_5/(0 \otimes \lambda_5)^2.$$

Let $V = V_{D_5}(\lambda_1)$, the usual 10-dimensional module. Then from the definition of $B_1^4 < D_5$, we have $V \downarrow B_1^4 = 1^{(2)}/1^{(2)}/1^{(2)}/1^{(2)}/0^2$ (where $1^{(2)}$ stands for a 1-fold tensor, one for each factor B_1), and $V \downarrow A_1^2 B_1^2 = 1 \otimes 1/1^{(2)}/1^{(2)}/0^2$ (where $1 \otimes 1$ has the A_1^2 acting, and the $1^{(2)}$ have a single factor B_1 acting). It follows, using [LS2, 2.6 and 2.7], that for $m = 4, 5$, we have

$$V_{D_5}(\lambda_m) \downarrow B_1^4 = 1 \otimes 1 \otimes 1 \otimes 1, \ \ V_{D_5}(\lambda_m) \downarrow A_1^2 B_1^2 = 1 \otimes 0 \otimes 1 \otimes 1/0 \otimes 1 \otimes 1 \otimes 1.$$

Using this and the above description of $L(G) \downarrow A_1 D_5$, we obtain the conclusion. □

Lemma 6.9. *Case (iv) of 6.6 does not hold.*

Proof. Suppose 6.6(iv) holds, so $S < A_1 B_1^4$ with distinct twists of the B_1 factors. If S projects nontrivially to each B_1 factor, say with projections $1^{(2^{i_1})}, \dots, 1^{(2^{i_4})}$, then we see from 6.8(i) that $L(G) \downarrow S$ has a composition factor $1^{(2^{i_1})} \otimes \dots \otimes 1^{(2^{i_4})}$, but no composition factor $\bigotimes_{r \in A} 1^{(2^{i_r})}$ for some 3-set $A \subseteq \{1, 2, 3, 4\}$. This contradicts 6.2(i,ii).

Hence S projects trivially to some B_1 factor. Let $1^{(2^{i_1})}, \dots, 1^{(2^{i_k})}$ be the nontrivial projections of S into the factors of $A_1 B_1^4$; thus $k \le 4$. Then $S < \bar{S} < A_1 B_1^4$, where $\bar{S} \cong SL_2(K)$ also has these projections.

If $e \ge 5$ then we can twist to take $i_r \le e - 2$ for all r. Then from 6.8(i) we see that the weights of $\bar{S}$ on $L(G)$ are all less than q. Then by 1.5, S and $\bar{S}$ fix the same subspaces of $L(G)$, contradicting (†).

Now suppose $e = 4$. If S projects trivially into two or more of the B_1 factors, then we can twist to take $i_r \le 2$ for all r, giving a contradiction as above. Therefore S projects trivially into just one B_1 factor. We can take the projections into the other B_1 factors to be $1, 1^{(2)}, 1^{(4)}$; then the weights of $\bar{S}$ on $L(G)$ are all less than q, unless the projection into the A_1 factor is $1^{(8)}$. But then by 6.8(i), $L(G) \downarrow S$ has a composition factor $1 \otimes 1^{(2)} \otimes 1^{(4)} \otimes 1^{(8)}$, contrary to 6.2(i) (recall that $A = B_2(16)$ or $G_2(16)$ when $q = 16$). □

Lemma 6.10. *Case (v) of 6.6 does not hold.*

Proof. Assume 6.6(v) holds, so $S < A_1^3 B_1^2$ with distinct twists on the B_1 factors. Observe that $S < \bar{S} < A_1^3 B_1^2$, where $\bar{S} \cong SL_2(K)$.

Suppose first that $e \geq 5$. If S has distinct nontrivial twists on all five factors of $A_1^3 B_1^2$, we obtain a contradiction using 6.8(ii) together with 6.2(i,ii). Otherwise, we can twist to take all nontrivial projections of $\bar{S}$ to be of the form $1^{(2^i)}$ with $i \leq e - 2$, and then the highest weight of $\bar{S}$ on $L(G)$ is less than q, giving the usual contradiction.

Now suppose $e = 4$. Assume first that S projects nontrivially into all five factors of $A_1^3 B_1^2$, with projections $1^{(2^{i_1})}, \dots, 1^{(2^{i_5})}$. If all $i_r \leq 2$ then, twisting if necessary, $\bar{S}$ has all weights less than q, so some $i_r = 3$. Then we can twist to take $i_1 \dots i_5$ (in some order) to be 00123. Now $L(G) \downarrow S$ has no 4-fold tensor composition factor, by 6.2(i); it follows from 6.8(ii) that (with the factors ordered as in 6.8(ii)), the projections of S in $A_1 A_1 A_1 B_1 B_1$ are $1, 1^{(2)}, 1^{(4)}, 1, 1^{(8)}$ or $1, 1^{(2)}, 1^{(8)}, 1, 1^{(4)}$ or $1, 1^{(4)}, 1^{(8)}, 1, 1^{(2)}$. In the first case, by 6.8(ii), $L(G) \downarrow S$ has a composition factor $1 \otimes 1^{(4)} \otimes 1^{(8)}$, but no composition factor $1 \otimes 1^{(8)}$, contrary to 6.2(iii). We obtain similar contradictions in the other cases.

Therefore S projects trivially to at least one of the factors of $A_1^3 B_1^2$. The only case where $\bar{S}$ cannot be chosen to have all weights less than q is that in which S has four nontrivial projections $1, 1^{(2)}, 1^{(4)}, 1^{(8)}$. By 6.2(i), $L(G) \downarrow S$ has no 4-fold tensor composition factor. Hence by 6.8(ii), we see that the projections of S in $A_1 A_1 A_1 B_1 B_1$ can be taken to be one of the following (up to reordering the 3 A_1 factors and the 2 B_1 factors):

(a) $1^{(2^{i_1})}, 1^{(2^{i_2})}, 1^{(2^{i_3})}, 1, 0^2$ (where $\{i_1, i_2, i_3\} = \{1, 2, 3\}$),

(b) $0^2, 1^{(4)}, 1^{(8)}, 1, 1^{(2)}$,

(c) $0^2, 1^{(2)}, 1^{(8)}, 1, 1^{(4)}$.

In cases (a) and (b), the highest weight of $\bar{S}$ on $L(G)$ is 16, and there are no composition factors of high weight 1, so 1.4 implies that S and $\bar{S}$ fix the same subspaces of $L(G)$, a contradiction. And in case (c), the composition factors of $\bar{S}$ having high weight 16 or more are $1^{(16)}$ and $1^{(2)} \otimes 1^{(16)}$; using 6.8(ii) we check that $\bar{S}$ has no composition factors 1 or $1^{(2)} \otimes 1$, so again 1.4 gives a contradiction. □

The next lemma is useful in dealing with case (ii) of 6.6, in which $S < QA_1 B_1^3$, lying in either $A_2 D_4$ or $A_1 D_5$. In the conclusion there is a certain subgroup $B_1 B_1' B_1''$ of D_5; this is a commuting product of three subgroups SO_3 of SO_{10}, where the radicals of the associated 3-spaces may be different.

Lemma 6.11. *Let $S_0 = SL_2(2^e)$ ($e \geq 4$). Suppose that $S_0 < QB_1^3 < D_5$, and S_0 has three distinct nontrivial twists on the B_1 factors. Then there is a subgroup $B_1 B_1' B_1'' \cong B_1^3$ of D_5 such that $QB_1^3 = QB_1 B_1' B_1''$ and $S_0 < B_1 B_1' B_1''$.*

Proof. Let V_{10} be the usual 10-dimensional D_5-module. Now $QB_1^3 < QD_4 < D_5$, so S_0 fixes both a singular vector v and a nonsingular vector w in V_{10}. Write V_8 for the C_4-space $w^\perp / \langle w \rangle$. By hypothesis, $V_8 \downarrow S_0$ has three nontrivial composition factors $1^{(2^i)}, 1^{(2^j)}, 1^{(2^k)}$, where i, j, k are distinct. Pulling back to V_{10}, we see that S_0 fixes just three indecomposable 3-spaces in V_{10} of the form $0/1^{(2^i)}$, $0/1^{(2^j)}$ and $0/1^{(2^k)}$; call these W_i, W_j, W_k respectively, and let w_i, w_j, w_k be nonzero vectors fixed by S_0 in W_i, W_j, W_k.

Now $(D_5)_{W_i}$ is a subgroup $B_1 B_3$ of $B_4 = (D_5)_{w_i}$; similarly $(D_5)_{W_j} = B_1' B_3'$ and $(D_5)_{W_k} = B_1'' B_3''$. The group B_1 acts trivially on the image of W_j mod $\langle w_i \rangle$, and also on $\langle w_i \rangle$, hence acts trivially on W_j, and similarly on W_k. Hence B_1 lies in the kernel of the action of $(D_5)_{W_j}$ on W_j, which is B_3', and consequently $[B_1, B_1'] = 1$. Similarly $[B_1, B_1''] = [B_1', B_1''] = 1$. Finally, $(D_5)_{W_i, W_j, W_k} = B_1 B_1' B_1''$, so this group contains S_0. □

Lemma 6.12. *Case (ii) of 6.6 does not hold.*

Proof. Suppose 6.6(ii) holds, so $S < Q A_1 B_1^3$, lying in either $A_1 D_5$ or $A_2 D_4$, and S has distinct twists on the B_1 factors. Moreover, by 6.7, $D = C_G(a)'$ is not a product of factors A_m, and hence D has a factor D_4 or D_5.

Suppose first that the projections of S to the factors $A_1 B_1^3$ are all nontrivial; say they are $1^{2^{(i_1)}}, \ldots, 1^{2^{(i_4)}}$. If these are all distinct, then by 6.8(i), $L(G) \downarrow S$ has a composition factor $1^{2^{(i_1)}} \otimes \ldots \otimes 1^{2^{(i_4)}}$, but no $1^{2^{(i_1)}} \otimes 1^{(2^{i_3})} \otimes 1^{2^{(i_4)}}$, contrary to 6.2(i,ii). Hence $i_1, \ldots, i_4$ are not all distinct; since the B_1 twists are distinct, we may take $i_1 = i_2$ (where i_1 is the twist on the A_1 factor), and i_2, i_3, i_4 are distinct.

If $Q A_1 B_1^3 < A_1 D_5$, then by the previous lemma we have $S < A_1 B_1 B_1' B_1''$, and the conclusion follows from 6.9. Therefore $Q A_1 B_1^3 < A_2 D_4$, with $Q A_1 < A_2$ and $B_1^3 < D_4$. Observe that if $i_1 > 0$, then $S < \bar{S} < A_2 D_4$ with $\bar{S} \cong SL_2(K)$, by 1.2.

When $e \geq 5$, we can take $i_1 \ldots i_4$ to be as in (i), (ii), or (iii) of 6.4. In case (i), the usual T_1 argument applies: there is a rank 1 torus T_1 in $A_1 B_1^3$ fixing the same subspaces of $L(G)$ as some element of order $q+1$ in S. In cases (ii) and (iii) of 6.4, we have $(i_1, \ldots, i_4) = (0, 0, 1, 3), (0, 0, 2, 3), (1, 1, 0, 3)$ $(e = 5)$ or $(0, 0, 2, 4)$ $(e = 6)$. For the $(1, 1, 0, 3)$ case, we have $S < \bar{S} < A_2 D_4$ as observed above, and $\bar{S}$ has all weights on $L(G)$ less than q. In the $(0, 0, 1, 3), (0, 0, 2, 3), (0, 0, 2, 4)$ cases, $L(G) \downarrow S$ has composition factors $1 \otimes 1^{(2)} \otimes 1^{(8)}, 1 \otimes 1^{(4)} \otimes 1^{(8)}, 1 \otimes 1^{(4)} \otimes 1^{(16)}$ (respectively), but no $1 \otimes 1^{(8)}, 1 \otimes 1^{(4)}, 1 \otimes 1^{(4)}$ (respectively), so $A \neq A_2^\epsilon(q)$ by 6.2(ii). And for $A = B_2(q)$, picking a conjugate b of a lying in S, we find from $L(G) \downarrow A_2 D_4$ that $C_{L(G)}(b)$ has dimension 22, contradicting the fact that $C_G(a)'$ has a factor D_4 or D_5.

To complete the case of four nontrivial projections, consider $e = 4$. Here we can take $(i_1, \ldots, i_4) = (0, 0, 1, 2), (1, 0, 1, 2)$ or $(2, 0, 1, 2)$. Then $L(G) \downarrow S$ has composition factors $1 \otimes 1^{(2)} \otimes 1^{(4)}, 1 \otimes 1^{(4)}$, the latter with multiplicity less than 8, so $A \neq G_2(16)$ by 6.2(iv). And for $A = B_2(16)$ we find that $\dim C_{L(G)}(a) = 30$ or 26, which is impossible.

Now suppose S has just three nontrivial projections to the factors of $A_1 B_1^3$, say $1^{2^{(i_1)}}, 1^{(2^{i_2})}, 1^{2^{(i_3)}}$. If the projection to the A_1 factor is trivial, then using the previous lemma we see that either $S < B_1^3 < D_4$ or $S < B_1 B_1' B_1'' < D_5$, and the conclusion follows from 6.9. Hence the projection to the A_1 factor is nontrivial; say it is $1^{2^{(i_1)}}$.

When $i_r \leq e - 3$ for all r, the usual T_1 argument gives a contradiction. Hence by 6.4, we can take $i_1 i_2 i_3$ (in some order) to be 012, 013 or 024 with $e = 4, 5$ or 6, respectively.

For the $e = 6$ case, we can twist to take $(i_1, i_2, i_3) = (4, 2, 0)$, and now the usual T_1 argument works.

Now consider the $e = 5$ case. If $i_1 = 3$, the T_1 argument works; and if $i_1 = 1$, we twist to take $(i_1, i_2, i_3) = (3, 2, 0)$, and now the T_1 argument again works. In the remaining case, $(i_1, i_2, i_3) = (0, 1, 3)$. Here $L(G) \downarrow S$ has a composition factor

$1 \otimes 1^{(2)} \otimes 1^{(8)}$, but no $1 \otimes 1^{(2)}$, so $A \neq A_2^\epsilon(q)$ by 6.2(ii). Thus $A = B_2(32)$, and S contains a conjugate b of a of order 33. Now $S < QA_1B_1^2 < A_2D_4$ or A_1D_5, from which we calculate that $C_{L(G)}(b)$ has dimension 34, whence $C_G(a)^0 = A_1D_4T_3$. Thus $S < QA_1B_1^2 < A_1D_4$, where $QB_1^2 < D_4$. The proof of 6.11 shows that in fact $S < A_1B_1B_1' < A_1D_4$, where $QB_1^2 = \bar{Q}B_1B_1'$. Hence $S < \bar{S} < A_1D_4$, and $\bar{S}$ has all weights on $L(G)$ less than q.

To conclude the case of three nontrivial projections, consider the $e = 4$ case, in which $\{i_1, i_2, i_3\} = \{0, 1, 2\}$. We find that $L(G) \downarrow S$ has a composition factor $1 \otimes 1^{(2)} \otimes 1^{(4)}$, but either no $1 \otimes 1^{(4)}$ or no $1^{(2)} \otimes 1^{(4)}$, so $A \neq G_2(16)$. Thus $A = B_2(16)$, and working with a conjugate b of a lying in S we calculate that $C_{L(G)}(b)$ has dimension 34, 38 or 38, according as $i_1 = 0, 1$ or 2; consequently $C_G(a)'$ is A_1D_4, A_2D_4 or A_2D_4 in the respective cases. Now it follows as before that $S < \bar{S} < C_G(a)'$, where $\bar{S}$ has all weights on $L(G)$ less than q.

Finally, when S has fewer than three nontrivial projections, the usual T_1 argument works in all cases, after twisting suitably. This completes the proof. $\square$

We next handle case (i) of 6.6, in which $S < QA_1^k < D$ with $k \leq 5$ (and A_1^k is a commuting product of fundamental A_1's).

Lemma 6.13. *Case (i) of 6.6 does not hold if $q \geq 32$.*

Proof. We begin by discussing the composition factors of $L(G) \downarrow A_1^5$. Now A_1^5 lies in a maximal commuting product A_1^8 of fundamental A_1's in G, and $N_G(A_1^8)/A_1^8 \cong AGL_3(2)$, acting 3-transitively on the set of 8 factors (see [As, Theorem 2]). Further, A_1^8 lies in the subgroup D_4D_4 of G, with each D_4 containing A_1^4. As $AGL_3(2)$ has only one orbit on 5-sets, we may take our A_1^5 to lie in D_4D_4, with 4 of the A_1 factors lying in a D_4 factor; thus $A_1^5 < A_1D_4 < D_4D_4 < G$. Now by [LS2, 2.1], $L(G) \downarrow D_4D_4 = L(D_4D_4)/\lambda_1 \otimes \lambda_1/\lambda_3 \otimes \lambda_3/\lambda_4 \otimes \lambda_4$, whence

$$
\begin{aligned}
L(G) \downarrow A_1D_4 = \ & (L(D_4) \downarrow A_1)/L(D_4)/(1 \otimes \lambda_1)^2/(0 \otimes \lambda_1)^4/(1 \otimes \lambda_3)^2/ \\
& (0 \otimes \lambda_3)^4/(1 \otimes \lambda_4)^2/(0 \otimes \lambda_4)^4.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
L(D_4) \downarrow A_1^4 &= 1 \otimes 1 \otimes 1 \otimes 1/1^{(2)}/1^{(2)}/1^{(2)}/1^{(2)}/0^4, \\
V_{D_4}(\lambda_1) \downarrow A_1^4 &= 1 \otimes 1 \otimes 0 \otimes 0/0 \otimes 0 \otimes 1 \otimes 1, \\
V_{D_4}(\lambda_3) \downarrow A_1^4 &= 1 \otimes 0 \otimes 1 \otimes 0/0 \otimes 1 \otimes 0 \otimes 1, \\
V_{D_4}(\lambda_4) \downarrow A_1^4 &= 1 \otimes 0 \otimes 0 \otimes 1/0 \otimes 1 \otimes 1 \otimes 0.
\end{aligned}
$$

From this we see that $L(G) \downarrow A_1^5$ has no 5-fold tensor composition factors, and has a unique 4-fold tensor composition factor (coming from $L(D_4) \downarrow A_1^4$).

Suppose now that $q = 2^e \geq 32$, and $S < QA_1^k$ with nontrivial projections $1^{2^{(i_1)}}, \ldots, 1^{2^{(i_k)}}$ ($k \leq 5$) in to the A_1 factors. If $i_1 \ldots i_k$ is as in (i) or (ii) of 6.3, then the usual T_1 argument works: an element b of order $q + 1$ in S lies in a rank 1 torus

$$
T_1 = \{(c^{2^{i_1}}, \ldots, c^{2^{i_k}}) : c \in K^*\} < A_1^k
$$

(where each entry $c^{2^{i_r}}$ stands for the matrix $\mathrm{diag}(c^{2^{i_r}}, c^{-2^{i_r}})$ in the corresponding factor A_1); the weights of T_1 on $L(G)$ are all at most 2^{e-1}, so b and T_1 fix the same subspaces of $L(G)$, contrary to (†).

Therefore by 6.3, we have $e = k = 5$, and we may take $i_1 \ldots i_5$ to be one of the following (in some order):

$$01234, \ 01223, \ 00223, \ 01133, \ 01233.$$

Take $1^{(2^{i_1})}$ to be the projection of S into the factor A_1 of A_1^5 not lying in the D_4 chosen above (where $A_1^5 < A_1 D_4$).

If $i_1 \ldots i_5 = 01234$, then from the above description, $L(G) \downarrow S$ has a unique composition factor $1^{2^{(i_2)}} \otimes \ldots \otimes 1^{2^{(i_5)}}$, contrary to 6.2(i).

Now suppose $i_1 \ldots i_5 = 01223$. For this case, the above T_1 argument works unless $i_1 = 1$, in which case we calculate that

$$L(G) \downarrow S = (1 \otimes 1^{(2)} \otimes 1^{(4)})^4/(1 \otimes 1^{(2)} \otimes 1^{(8)})^2/(1^{(2)} \otimes 1^{(4)} \otimes 1^{(8)})^4/$$
$$(1^{(2)} \otimes 1^{(8)})^2/(1 \otimes 1^{(2)})^0/\ldots$$

If $A = A_2^\epsilon(q)$, the presence of ten 3-fold tensor composition factors in $L(G) \downarrow S$ implies the same for $L(G) \downarrow A$, whence $\dim L(G) \geq 10.27$, a contradiction. And if $A = B_2(q)$, the composition factors $(1 \otimes 1^{(2)} \otimes 1^{(8)})^2/(1^{(2)} \otimes 1^{(8)})^2/(1 \otimes 1^{(2)})^0$ conflict with 6.2(v).

Next consider $i_1 \ldots i_5 = 00223$. The usual T_1 argument works unless $i_1 = 0$, so suppose this is the case. Then

$$L(G) \downarrow S = (1 \otimes 1^{(4)} \otimes 1^{(8)})^4/(1 \otimes 1^{(16)})^1/(1^{(16)})^1/\ldots$$

and the $(1 \otimes 1^{(4)} \otimes 1^{(8)})^4$ are the only 3-fold tensor composition factors appearing. If $A = A_2^\epsilon(q)$, then the single factor $1 \otimes 1^{(16)}$ must occur in the restriction of a self-dual composition factor of $L(G) \downarrow A$, which must be $11^{(8)} \otimes 11^{(16)}$ (see 6.1); but the restriction of this module to S has composition factor $1^{(16)}$ with multiplicity 6, a contradiction. And if $A = B_2(q)$, then the single factor $1 \otimes 1^{(16)}$ must occur in the restriction of a factor $01 \otimes 01^{(16)}$ in $L(G) \downarrow A$ (see 6.1), but the restriction of this to S has $1^{(16)}$ with multiplicity 2.

Now suppose $i_1 \ldots i_5 = 01133$. If $i_1 = 1$ or 3, we find that

$$L(G) \downarrow S = (1 \otimes 1^{(2)} \otimes 1^{(16)})^1/(1 \otimes 1^{(16)})^0/\ldots \text{ or } (1 \otimes 1^{(4)} \otimes 1^{(8)})^1/(1 \otimes 1^{(4)})^0/\ldots$$

respectively, contrary to 6.2(iii). Thus $i_1 = 0$. Here

$$L(G) \downarrow S = (1 \otimes 1^{(2)} \otimes 1^{(8)})^8/(1 \otimes 1^{(2)})^0/\ldots,$$

so $A \neq A_2^\epsilon(q)$ by 6.2(ii). Hence $A = B_2(q)$, and S contains a conjugate b of a. We calculate that $C_G(b)$ has dimension 34. As $S < QA_1^5 < C_G(a)'$, it follows that $C_G(a)' = A_1 D_4$. Since a subgroup A_1^4 of D_4 cannot normalize a nontrivial unipotent group therein, it follows that $Q = 1$, and hence $S < \bar{S} < A_1 D_4$, where $\bar{S} \cong SL_2(K)$ has all weights on $L(G)$ less than q.

To conclude, let $i_1 \ldots i_5 = 01233$. If $i_1 = 3$ then $L(G) \downarrow S$ has a unique 4-fold tensor composition factor, contrary to 6.2(i). Otherwise, according as $i_1 = 0, 1$ or 2, we find that

$$L(G) \downarrow S = (1^{(2)} \otimes 1^{(4)} \otimes 1^{(16)})^1/(1^{(4)} \otimes 1^{(16)})^0/\ldots,$$
$$\text{or } (1 \otimes 1^{(4)} \otimes 1^{(16)})^1/(1^{(4)} \otimes 1^{(16)})^0/\ldots$$
$$\text{or } (1 \otimes 1^{(2)} \otimes 1^{(16)})^1/(1^{(2)} \otimes 1^{(16)})^0/\ldots,$$

respectively. All of these possibilities conflict with 6.2(iii). This completes the proof. $\qquad \square$

Lemma 6.14. *Case (i) of 6.6 does not hold if $q = 16$.*

Proof. As in the previous proof, let $S < QA_1^k$ ($k \leq 5$), with nontrivial projections $1^{2^{(i_1)}}, \ldots, 1^{2^{(i_k)}}$. If $k \leq 3$, we can take $i_1 \ldots i_k$ to be one of $1, 11, 14, 111, 112, 114, 122, 124$, and the usual T_1 argument works in all cases.

Now suppose $k = 4$. Then either the T_1 argument works, or we can twist to take $i_1 \ldots i_4$ to be one of the following:

$$0123, \ 0122, \ 0112, \ 0022.$$

Consider the first case. If $S < QA_1^4$ with the A_1^4 lying in a subgroup D_4 of G, then $L(G) \downarrow S$ has a 4-fold tensor factor, contrary to 6.2(i). Otherwise, we find that

$$L(G) \downarrow S = (1 \otimes 1^{(2)} \otimes 1^{(4)})^2 / (1 \otimes 1^{(2)} \otimes 1^{(8)})^2 / (1 \otimes 1^{(4)} \otimes 1^{(8)})^2 /$$
$$(1^{(2)} \otimes 1^{(4)} \otimes 1^{(8)})^2 / \ldots .$$

Hence we see using 6.1 that $\dim L(G) \geq 4.4^3$ if $A = B_2(16)$, and $\dim L(G) \geq 2.6^3$ if $A = G_2(16)$, both of which are false.

Now suppose $i_1 \ldots i_4 = 0122$. If $S < QA_1^4$ with the $A_1^4 < D_4$, then $L(G) \downarrow S = (1 \otimes 1^{(2)} \otimes 1^{(8)})^1 / (1 \otimes 1^{(8)})^0 / \ldots$, contrary to 6.2(iii). So assume the $A_1^4 \not< D_4$. Then $L(G) \downarrow S = (1 \otimes 1^{(2)} \otimes 1^{(4)})^4 / \ldots$, ruling out $A = G_2(16)$ by 6.2(iv). Thus $A = B_2(16)$, and S contains a conjugate b of a. We can choose a subgroup $D_4 D_4$ containing b, and lying in a subgroup D_8 of G, such that b has weights $4^2, -4^2, 0^4$ and $3, -3, 1, -1, 4^2, -4^2$ on natural modules for the D_4 factors, and weights $4^4, -4^4, 3, -3, 1, -1, 0^4$ on the natural D_8-module. Hence $C_{D_8}(b)^0 = A_1 A_1 A_3 T_3$. Moreover, from the above description of $L(G) \downarrow A_1^5$ we calculate that $\dim C_{L(G)}(b) = 24$, so it follows that $C_G(a)^0 = A_1 A_1 A_3 T_3$. Now $S < QA_1^4 < A_1 A_1 A_3$. Hence $Q = 1$ and $S < \bar{S} < A_1^4$, where $\bar{S} \cong SL_2(K)$ has all weights on $L(G)$ less than q.

Next consider $i_1 \ldots i_4 = 0112$. If $S < QA_1^4$ with the A_1^4 not in D_4, then the usual T_1 argument works. Otherwise, we find that $L(G) \downarrow S$ has no 3-fold tensor factors, and has $1 \otimes 1^{(8)} / 1^0$; this is not possible, by 6.1.

To complete the $k = 4$ case, let $i_1 \ldots i_4 = 0022$. The T_1 argument works if $S < QA_1^4$ with $A_1^4 < D_4$, so suppose this is not the case. If $A = B_2(16)$, then we work out $C_G(a)^0$ as above, and find that it is $A_3 A_3 T_2$. Then $S < \bar{S} < A_1^4 < A_3 A_3$, where $\bar{S}$ has all weights less than q. Now suppose $A = G_2(16)$. Here we find that

$$L(G) \downarrow S = (1 \otimes 1^{(8)})^4 / (1^{(2)} \otimes 1^{(4)})^4 / (1 \otimes 1^{(4)})^{16} / (1^{(4)})^{24} / (1^{(8)})^6 / (1^{(2)})^6 / 1^{24} / 0^{32}.$$

By 6.1, the factors $(1 \otimes 1^{(8)})^4$ must occur in the restriction of a composition factor $10 \otimes 10^{(8)}$ or $10 \otimes 01^{(4)}$ of $L(G) \downarrow A$.

Suppose there is a factor $10 \otimes 10^{(8)}$. This restricts to S as $(1 \otimes 1^{(8)})^4 / (1^{(8)})^4 / 1^4 / 0^4$. The remaining factors $(1 \otimes 1^{(4)})^{16}$ in $L(G) \downarrow S$ must come from $(10 \otimes 10^{(4)})^4$ in $L(G) \downarrow A$, and this restricts to S as $(1 \otimes 1^{(4)})^{16} / 1^{16} / (1^{(4)})^{16} / 0^{16}$. Now the remaining factors $(1^{(2)} \otimes 1^{(4)})^4$ must come from $10^{(2)} \otimes 10^{(4)}$, which restricts to S as $(1^{(2)} \otimes 1^{(4)})^4 / (1^{(2)})^4 / (1^{(4)})^4 / 0^4$. Of the factors in $L(G) \downarrow S$, there remain $1^4 / (1^{(2)})^2 / (1^{(4)})^4 / (1^{(8)})^2 / 0^8$, of total dimension 32. These factors must come from composition factors of $L(G) \downarrow A$ which are twists of 10 (say a of them), 01 (say b of them) and 00. Since $10 \downarrow S = 1^2 / 0^2$ and $01 \downarrow S = 1^4 / 1^{(2)} / 0^4$ (see 6.1), counting 2-dimensional composition factors for S gives $2a + 5b = 12$, while counting trivial composition factors yields $2a + 4b = 8$. Therefore $b = 4$, forcing a to be negative, a contradiction. This completes the argument when $L(G) \downarrow A$ has a composition

factor $10 \otimes 10^{(8)}$. The other case, in which there is a composition factor $10 \otimes 01^{(4)}$, is handled by an entirely similar argument.

The case where $k = 4$ is now settled.

Assume from now on that $k = 5$. We have $A_1^5 < A_1 D_4$; take $1^{(2^{i_1})}$ to be the projection of S into the A_1 not lying in the D_4. Then either the usual T_1 argument works, or we can twist to take $i_1 \ldots i_5$ to be one of the following (in some order):

$$00022, \ 00112, \ 00122, \ 01112, \ 01122, \ 01222, \ 00123.$$

Moreover, either the T_1 argument works, or we have $i_1 = 0$ in the second case, and $i_1 \neq 2$ in the fourth case. Let b be an element of order $q + 1$ in S. In the following table we present some information for each of the above possibilities, which in almost all cases is sufficient to obtain a contradiction:

$i_1, i_2 \ldots i_5$	some comp. factors of $L(G) \downarrow S$	$C_G(b)'$
$0, 0022$	$(1^{(2)} \otimes 1^{(8)})^1 / (1 \otimes 1^{(8)})^2$	$A_1 D_4$
$2, 0002$	$(1 \otimes 1^{(2)} \otimes 1^{(4)})^1$	$A_2 D_4$
$0, 0112$	$(1 \otimes 1^{(2)} \otimes 1^{(4)})^4$	$A_1 A_4$ or $A_1^2 A_2 A_3$
$0, 0122$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^1 / (1^{(2)} \otimes 1^{(8)})^0$	
$1, 0022$	$(1 \otimes 1^{(2)} \otimes 1^{(4)})^8 / (1 \otimes 1^{(2)})^0$	$A_2 D_4$
$2, 0012$	$(1 \otimes 1^{(2)} \otimes 1^{(4)})^4$	$A_1 A_1 A_3$
$0, 1112$	$(1 \otimes 1^{(2)} \otimes 1^{(4)})^6 / (1 \otimes 1^{(2)})^0$	$A_2 D_4$
$1, 0112$	$(1 \otimes 1^{(2)} \otimes 1^{(4)})^2$	$A_1 A_4$ or $A_1^2 A_2 A_3$
$0, 1122$	$(1 \otimes 1^{(2)} \otimes 1^{(4)})^8 / (1 \otimes 1^{(2)})^0$	$A_1 D_4$
$1, 0122$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^1 / (1 \otimes 1^{(8)})^0$	
$2, 0112$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^4$	$A_2 A_3$
$0, 1222$	$(1 \otimes 1^{(2)} \otimes 1^{(4)})^6$	$A_2 D_4$
$1, 0222$	$(1 \otimes 1^{(2)} \otimes 1^{(4)})^6$	$A_2 D_4$
$2, 0122$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^1$	$A_1 A_3$
$0, 0123$	$1 \otimes 1^{(2)} \otimes 1^{(4)} \otimes 1^{(8)}$	
$1, 0023$	$(1^{(2)} \otimes 1^{(4)} \otimes 1^{(8)})^2$	$A_2 A_3$
$2, 0013$	$(1^{(2)} \otimes 1^{(4)} \otimes 1^{(8)})^2$	$A_1 A_3$
$3, 0012$	$(1^{(2)} \otimes 1^{(4)} \otimes 1^{(8)})^2$	$A_1 A_1 A_3$

In each case, using 6.1 and 6.2, we see that the information in the composition factors column rules out $A = G_2(16)$, and also rules out $A = B_2(16)$ in rows 4, 10 and 15 of the table. Therefore $A = B_2(16)$ (and we are not in row 4, 10 or 15), and the element $b \in S$ is conjugate to a. We know that $S < QA_1^5 < C_G(a)'$. This implies that $C_G(a)'$ is $A_1 D_4$, $A_2 D_4$ or $A_1^2 A_2 A_3$; the last case is ruled out by 3.4, and in the first case, $S < \bar{S} < A_1 D_4$ with $\bar{S} \cong SL_2(K)$ having all weights on $L(G)$ less than q. Thus $C_G(a)' = A_2 D_4$. Then $S < (QA_1) A_1^4 < A_2 D_4$ with $QA_1 < A_2$ and $A_1^4 < D_4$. If $i_1 > 0$ then using 1.2 we have $S < \bar{S} < A_2 D_4$ with all weights of $\bar{S}$ less than q. The surviving cases are those in rows 7 and 12 of the table. For row 7, twist to take $i_1, i_2 \ldots i_5 = 3, 0001$; then $S < \bar{S} < A_2 D_4$, where $\bar{S}$ has highest weight 16 on $L(G)$, and has no composition factor of high weight 1. The conclusion now follows from 1.4. For row 12, we twist to $2, 3000$ and apply the same argument. □

To complete the proof of Theorem 1, it remains to handle case (iii) of 6.6.

Lemma 6.15. *Case (iii) of 6.6 does not hold if $q \geq 32$.*

Proof. Assume 6.6(iii) holds, so $S < QA_1^2B_1^2 < A_1(QA_1B_1^2) < A_1D_5$. Moreover, $C_G(a)' = A_1D_5$: for otherwise by 6.5 and 6.7, we are in a different case of 6.6, already ruled out. Thus $C_G(a)^0 = A_1D_5T_2$, of dimension 50.

If S projects trivially to one of the B_1 factors then $S < QA_1^4$ (as $B_1 < A_1^2$), contrary to 6.13 and 6.14. Hence S projects nontrivially to both the B_1 factors (and with distinct twists, by 6.6). Let the nontrivial projections of S to the factors $A_1(A_1B_1B_1) < A_1D_5$ be $1^{(2^{i_1})}, \dots, 1^{(2^{i_k})}$, written in the order of the factors given (so $1^{(2^{i_{k-1}})}, 1^{(2^{i_k})}$ are the projections to the B_1 factors). Take $i_{k-1} < i_k$.

If $k = 2$, then the usual T_1 argument goes through, so assume $k \geq 3$. Also, if $k = 4$ and $i_1, \dots, i_4$ are all distinct, then by 6.8(ii), $L(G) \downarrow S$ has a composition factor $1^{(2^{i_1})} \otimes \dots \otimes 1^{(2^{i_4})}$ but no composition factor $1^{(2^{i_1})} \otimes 1^{(2^{i_2})} \otimes 1^{(2^{i_4})}$, contrary to 6.2(i,ii).

Thus if $k = 4$ then $i_1, \dots, i_4$ are not all distinct. When $i_r \leq e - 3$ for all r, the usual T_1 argument goes through. As $e \geq 5$ by hypothesis, we may therefore apply 6.4 to take one of the following to hold:

(1) $e = 5$ and $i_1 \dots i_k = 0013, 0023, 0113$ or 013 (in some order);

(2) $e = 6$ and $i_1 \dots i_k = 0024$ or 024.

Consider first case (1). We have $S < A_1(QA_1B_1^2) < A_1D_5$. Given $i_1, \dots, i_k$, the composition factors of $L(G) \downarrow S$ can be worked out from the restrictions $L(G) \downarrow A_1D_5$ and $V_{D_5}(\lambda_m) \downarrow A_1B_1^2$ ($m = 1, 4, 5$) given in the proof of 6.8(ii). From 6.8(ii), we see that if neither i_{k-1} nor i_k is 3, then the usual T_1 argument goes through. Hence we may take it that $i_k = 3$. In the table below, for each possibility for $(i_1, \dots, i_k)$, we give various composition factors of $L(G) \downarrow S$ and their multiplicities; we also give the dimension of $C_G(b)$, where b is an element of order $q + 1$ in S. In the left hand column, a symbol "–" indicates a trivial projection.

$(i_1, \dots, i_k)$	$L(G) \downarrow S$	$\dim C_G(b)$
$(0, 0, 1, 3)$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^8/(1 \otimes 1^{(8)})^0/....$	24
$(0, 1, 0, 3)$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^4/(1^{(2)} \otimes 1^{(16)})^3/(1 \otimes 1^{(2)})^6/1^8/....$	20
$(1, 0, 0, 3)$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^4/(1^{(2)} \otimes 1^{(16)})^3/(1 \otimes 1^{(2)})^6/1^8/....$	20
$(0, 0, 2, 3)$	$(1 \otimes 1^{(4)} \otimes 1^{(8)})^8/(1 \otimes 1^{(4)})^0/....$	24
$(0, 2, 0, 3)$	$(1 \otimes 1^{(4)} \otimes 1^{(8)})^4/(1^{(2)} \otimes 1^{(4)} \otimes 1^{(8)})^2/$ $(1^{(4)} \otimes 1^{(8)})^4/....$	16
$(2, 0, 0, 3)$	$(1 \otimes 1^{(4)} \otimes 1^{(8)})^4/(1^{(2)} \otimes 1^{(4)} \otimes 1^{(8)})^2/$ $(1^{(4)} \otimes 1^{(8)})^4/....$	16
$(0, 1, 1, 3)$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^4/(1 \otimes 1^{(4)} \otimes 1^{(8)})^2/(1 \otimes 1^{(8)})^4/....$	16
$(1, 0, 1, 3)$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^4/(1 \otimes 1^{(4)} \otimes 1^{(8)})^2/(1 \otimes 1^{(8)})^4/....$	16
$(1, 1, 0, 3)$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^8/(1 \otimes 1^{(2)})^0/....$	32
$(0, -, 1, 3)$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^4/(1 \otimes 1^{(2)})^0/....$	34
$(-, 0, 1, 3)$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^8/(1 \otimes 1^{(2)})^0/....$	34
$(1, -, 0, 3)$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^8/(1 \otimes 1^{(2)})^0/....$	38
$(-, 1, 0, 3)$	$(1 \otimes 1^{(2)} \otimes 1^{(8)})^8/(1 \otimes 1^{(2)})^0/....$	38

If $A = B_2(q)$ then S contains a conjugate b of a; but $C_G(a)$ has dimension 50, which conflicts with the dimension of $C_G(b)$ given in the table.

Therefore $A = A_2^\epsilon(q)$. By 6.2(ii,vi), the information given in the second column of the table rules out all possibilities for $(i_1, \dots, i_k)$ except $(0, 1, 0, 3)$ and $(1, 0, 0, 3)$.

In these cases we have

$$L(G) \downarrow S = (1 \otimes 1^{(2)} \otimes 1^{(8)})^4 / (1^{(2)} \otimes 1^{(16)})^3 / (1 \otimes 1^{(2)})^6 / 1^8 / ....$$

The 3-fold tensor factors force $L(G) \downarrow A$ to have four composition factors of the form $\alpha \otimes \beta^{(2)} \otimes \gamma^{(8)}$ with $\alpha, \beta, \gamma \in \{10, 01\}$. As $1^{(2)} \otimes 1^{(16)}$ occurs with odd multiplicity, it must occur as a composition factor in a self-dual factor of $L(G) \downarrow A$, which must therefore by 6.1 be $11 \otimes 11^{(8)}$. Now $(11 \otimes 11^{(8)}) \downarrow S$ and the four factors $(\alpha \otimes \beta^{(2)} \otimes \gamma^{(8)}) \downarrow S$ account for all eight composition factors 1 in $L(G) \downarrow S$. However, two of the composition factors $1 \otimes 1^{(2)}$ are still to be accounted for in other composition factors of $L(G) \downarrow A$. By 6.1, there is no irreducible A-module V such that $V \downarrow S$ has $1 \otimes 1^{(2)}$ but not 1, so this is impossible.

Now consider case (2), in which $e = 6$ and $i_1 \ldots i_k = 0024$ or 024. The usual T_1 argument works unless $i_k = 4$. As above, we calculate the following information.

$(i_1, \ldots, i_k)$	$L(G) \downarrow S$	$\dim C_G(b)$
$(0,0,2,4)$	$(1 \otimes 1^{(4)} \otimes 1^{(16)})^8 / (1 \otimes 1^{(4)})^0 / ....$	20
$(0,2,0,4)$	$(1 \otimes 1^{(4)} \otimes 1^{(16)})^4 / (1^{(2)} \otimes 1^{(4)} \otimes 1^{(16)})^2 /$	
	$(1^{(4)} \otimes 1^{(16)})^4 / ....$	16
$(2,0,0,4)$	$(1 \otimes 1^{(4)} \otimes 1^{(16)})^4 / (1^{(2)} \otimes 1^{(4)} \otimes 1^{(16)})^2 /$	
	$(1^{(4)} \otimes 1^{(16)})^4 / ....$	16
$(0,-,2,4)$	$(1 \otimes 1^{(4)} \otimes 1^{(16)})^8 / (1 \otimes 1^{(4)})^0 / ....$	34
$(-,0,2,4)$	$(1 \otimes 1^{(4)} \otimes 1^{(16)})^8 / (1 \otimes 1^{(4)})^0 / ....$	34
$(2,-,0,4)$	$(1 \otimes 1^{(4)} \otimes 1^{(16)})^8 / (1 \otimes 1^{(4)})^0 / ....$	34
$(-,2,0,4)$	$(1 \otimes 1^{(4)} \otimes 1^{(16)})^8 / (1 \otimes 1^{(4)})^0 / ....$	34

Now we obtain a contradiction as before. This completes the proof of the lemma.
$\square$

Lemma 6.16. *Case (iii) of 6.6 does not hold if $q = 16$.*

Proof. Assume $q = 16$ and 6.6(iii) holds. Then $A = B_2(16)$ or $G_2(16)$. As in the previous lemma, $\dim C_G(a) = 50$.

Again let the nontrivial projections of S to the factors $A_1(A_1 B_1 B_1) < A_1 D_5$ be $1^{(2^{i_1})}, \ldots, 1^{(2^{i_k})}$. As in the previous lemma, the projections to the B_1 factors are nontrivial and distinct. Twisting if necessary, we can take $i_1 i_2 \in \{00, 01, 02, 11, 12, 22, 0-, 1-\}$ and $i_3 i_4 \in \{01, 02, 12\}$. Moreover, the T_1 argument goes through in all cases where $i_3 i_4 = 01$, except when $i_1 i_2 = 12$ or 22.

For all the possibilities for $i_1 \ldots i_k$, we calculate $L(G) \downarrow S$ and $\dim C_G(b)$ as in the previous lemma. In all cases we find that $\dim C_G(b) < 50$, ruling out $A = B_2(16)$ in the usual way. Therefore $A = G_2(16)$. If $i_3 i_4 \neq 02$ or $i_1 i_2 \notin \{02, 0-\}$, we find that $L(G) \downarrow S$ is either

$$(1 \otimes 1^{(2)} \otimes 1^{(4)})^r / .... \text{ with } r < 8,$$

or

$$(1 \otimes 1^{(2)} \otimes 1^{(4)})^8 / (1^{(2^i)} \otimes 1^{(2^j)})^0 / .... \text{ for some } i, j \in \{0, 1, 2\}.$$

Both possibilities contradict 6.2(ii,iv).

Finally, consider the cases where $i_3 i_4 = 02$ and $i_1 i_2 = 02$ or $0-$. Assume first that $(i_1, i_2, i_3, i_4) = (0, 2, 0, 2)$. Then we find that

$$L(G) \downarrow S = (1 \otimes 1^{(2)})^2 / (1 \otimes 1^{(4)})^{12} / (1 \otimes 1^{(8)})^6 / (1^{(2)} \otimes 1^{(8)})^3 / (1^{(4)} \otimes 1^{(8)})^2 / ....$$

The composition factor $1^{(2)} \otimes 1^{(8)}$ appears with odd multiplicity. By 6.1, the only way this can happen is if $L(G) \downarrow A$ has a composition factor $01 \otimes 01^{(4)}$ (of dimension 196). However, $(01 \otimes 01^{(4)}) \downarrow S$ has $1 \otimes 1^{(4)}$ with multiplicity 16, whereas it only occurs with multiplicity 12 in $L(G) \downarrow S$, a contradiction. Similar contradictions are reached for the other possibilities for $(i_1, \ldots, i_4)$. □

This completes the proof of Theorem 1.

7. Proof of Theorem 4

Let G be a simple algebraic group over the algebraically closed field K of characteristic p. Let $X = X(q) < G$ with $q = p^e$ and X not of the same type as G, and suppose that X is irreducible on each G-composition factor of $L(G)$. We aim to show that X and G are as in the conclusion of Theorem 4. We may as well assume that G is of adjoint type; as X is irreducible on each G-composition factor of $L(G)$, we then have $Z(X) = Z(G) = 1$. The G-composition factors of $L(G)$ are given by 1.10.

We now embark upon the proof of Theorem 4. We first dispose of classical groups.

Lemma 7.1. *Theorem 4 holds if G is classical.*

Proof. First observe that X does not lie in a parabolic subgroup of G: for if $X < P$, a parabolic, then X leaves invariant the series $0 < L(R_u(P)) < L(P) < L(G)$, hence cannot be irreducible on all nontrivial G-composition factors of $L(G)$, by 1.10. We now apply [ST1, Theorem 1]; this implies that X lies in a closed, connected, simple subgroup $\bar{X}$ of G of the same type as X. Then X fixes $L(\bar{X})$.

We now claim that $\dim \bar{X} < \dim G - 2$. For suppose $\dim \bar{X} \geq \dim G - 2$. If a maximal unipotent subgroup of $\bar{X}$ is also one of G, then $\bar{X} = G$ (see [Se4, 1.6] for instance), which is not the case. The dimension of a simple algebraic group is equal to the rank of the group plus twice the dimension of a maximal unipotent subgroup; it follows that $\bar{X}$ is a subgroup of maximal rank in G, and that a maximal unipotent subgroup of $\bar{X}$ has dimension just one less than a maximal unipotent subgroup of G. It is trivial to check that this cannot happen.

It follows from the previous paragraph and 1.10 that (G, p) must be $(B_n, 2)$ or $(C_n, 2)$. Moreover, $\bar{X}$ is irreducible on $V_G(\lambda_1)$ and $V_G(\lambda_2)$. At this point [Se1, Theorem 1] determines all possibilities for $\bar{X}$: either $\bar{X} = D_n$, or $n = 3$ and $\bar{X} = G_2$, as in the conclusion of Theorem 4. □

In view of 7.1, we assume from now on that G is of exceptional type.

Let V be a nontrivial G-composition factor of $L(G)$, so that X is irreducible on V. Write $V = V_G(\lambda)$ with λ as in 1.10 (up to twists).

If $P_X = Q_X L_X$ is a parabolic subgroup of X with unipotent radical Q_X and Levi subgroup L_X, then by [BT], there is a parabolic subgroup $P = QL$ of G containing P_X and such that $Q_X \leq Q$, the unipotent radical of P. The next result follows from the argument in [ST1, p.565] (third paragraph).

Lemma 7.2. *We have $C_V(Q_X) = C_V(Q)$, and P_X and P are both irreducible on this space. The high weight of $C_V(Q)$ as an L-module is the restriction of λ to L. Assuming this action is nontrivial, we have $\lambda = \lambda_i$ for some i, and if L_0 is the simple factor of L involving the root α_i, then the possibilities for $C_V(Q) \downarrow L_0$ are*

as in Table 5. Moreover, if $(G, L_0) = (E_7, E_6)$ or (E_6, D_5), then Q and Q_X are abelian.

TABLE 5

G	L_0	$C_V(Q) \downarrow L_0$ (up to auts. of L_0)
E_8	D_7 or A_r $(r \leq 7)$	$V_{L_0}(\lambda_1)$
E_7	E_6, D_5 or A_r $(r \leq 6)$	$V_{L_0}(\lambda_1)$
E_6	D_5, D_4	$V_{L_0}(\lambda_4)$
	A_r $(r \leq 4)$	$V_{L_0}(\lambda_1)$
F_4	$B_3, C_3(p=2)$, or A_r $(r \leq 2)$	$V_{L_0}(\lambda_1)$

Lemma 7.3. X is not $A_1(q)$.

Proof. Suppose $X = A_1(q)$, and let t be a generator for a Cartan subgroup of X. By the representation theory of X, $C_V(t)$ has dimension 1 unless V is the Steinberg module for X; the latter is impossible as $\dim V$ cannot be equal to q (see 1.10). However, t lies in a maximal torus of G, so $C_V(t)$ must certainly be bigger than 1, which is a contradiction. $\square$

Lemma 7.4. X is not $A_2^\epsilon(q)$.

Proof. Suppose that $X = A_2^\epsilon(q)$. Now V is a self-dual X-module, so by [ST1], V lifts to a self-dual $A_2(K)$-module. This is a tensor product of twists of modules of the form $V_{A_2}(aa)$, where $0 \leq a \leq p-1$.

Now by 1.8, either $W_X(aa) = V_X(aa)$, of dimension $(a+1)^3$, or $W_X(aa)$ has two composition factors, namely $V_X(aa)$ and $V_X(p-a-2, p-a-2)$; in the latter case, $2a+2 > p \geq a+2$ and $V_X(p-a-2, p-a-2) = W_X(p-a-2, p-a-2)$, whence $\dim V_X(aa) = (a+1)^3 - (p-a-1)^3$.

If $p = 2$ then V is a tensor product of twists of $V_X(11)$, which has dimension 8; but from 1.10 we see that $\dim V$ is not a power of 8. Hence $p > 2$.

Suppose first that $G = E_8$. Then $\dim V = 248 = 8.31$, so there is a restricted module $V_X(aa)$ of dimension $2^r.31$ $(r \leq 3)$. Therefore $2^r.31$ must be equal to $(a+1)^3 - (p-a-1)^3$. This has a factor $2a+2-p$, which, since p is odd, forces $2a+2-p = 1$. Then $2^r.31 = (a+1)^3 - a^3$, which is not possible.

Now let $G = E_7$. Then $\dim V = 133 = 7.19$ (recall $p \neq 2$). As 133 is not a difference of cubes, we must have $V \downarrow X \cong V_X(aa) \otimes V_X(bb)^{(p^i)}$ with $(a+1)^3 - (p-a-1)^3 = 7$ and $(b+1)^3 - (p-b-1)^3 = 19$. This is impossible.

Next consider $G = E_6$ or F_4. Here $\dim V = 52, 77$ or 78, none of which is the product of a difference of cubes.

Finally, if $G = G_2$, then as 14 is not a product of differences of cubes, we must have $p = 3$ and $\dim V = 7$. Subgroups $A_2^\epsilon(q)$ of G_2 are identified by [LST, Theorem 2]: they lie in maximal rank connected subgroups A_2. However these are not irreducible on both $V_G(\lambda_1)$ and $V_G(\lambda_2)$, so no examples arise here. $\square$

Lemma 7.5. *Theorem 4 holds if $G = G_2$.*

Proof. Suppose $G = G_2$. By 7.3, $\mathrm{rk}(X) \geq 2$ (recall that $\mathrm{rk}(X)$ denotes the rank of the simple algebraic group corresponding to X).

Since a parabolic of X lies in a parabolic of G (by [BT]), we deduce that X is of type $A_2^\epsilon(q), B_2(q), G_2(q), {}^3D_4(q)$ or ${}^2A_3(q)$. The first case is out by 7.4, and

$X \neq G_2(q)$ as X is assumed not to have the same type as G; also $X \neq {}^3D_4(q)$ since ${}^3D_4(q)$ has no nontrivial representation of dimension 7. By the argument in [ST1, p.565] (third paragraph), up to twists, $V \downarrow X$ is 10, 01 or 11 if $X = B_2(q)$, and is 101 or 010 if $X = {}^2A_3(q)$. As $\dim V = 14 - 7\delta_{p,3}$, the only possibility is that $X = {}^2A_3(q)$, $V \downarrow X = 101$ with $p = 2$. But here X has a parabolic subgroup $Q_X L_X$ with $L_X' \cong A_1(q^2)$ acting on $C_V(Q_X)$ as $1 \otimes 1^{(q)}$, which does not agree with the action of a parabolic of G containing $Q_X L_X$. $\square$

We assume from now on that $G \neq G_2$.

Lemma 7.6. X is not $B_2^\epsilon(q)$ or $G_2^\epsilon(q)$.

Proof. Suppose that $X = B_2^\epsilon(q)$ or $G_2^\epsilon(q)$, and let $V \downarrow X \cong V_X(ab) \cong V_X(a_0 b_0) \otimes \ldots$ with $a_0, b_0 \leq p - 1$. At several points in the proof we make use of the sum formula in [An], which can be used to calculate the dimensions of many modules $V_X(cd)$ with c, d reasonably small.

First assume $p = 2$ or 3. By [GS] for $X = G_2^\epsilon(q)$, and by the sum formula [An] for $X = B_2^\epsilon(q)$, the possibilities for $\dim V_X(a_0 b_0)$ are as follows:

X	p	$a_0 b_0$	$\dim V_X(a_0 b_0)$
$B_2^\epsilon(q)$	2	$10, 01, 11$	$4, 4, 16$ (resp.)
$G_2(q)$	2	$10, 01, 11$	$6, 14, 64$
$B_2(q)$	3	$10, 20, 01, 11, 21, 02, 12, 22$	$5, 14, 4, 16, 40, 10, 25, 81$
$G_2^\epsilon(q)$	3	$10, 20, 01, 11, 21, 02, 12, 22$	$7, 27, 7, 49, 189, 27, 189, 729$

As $\dim V$ is a product of dimensions in the table, we have a contradiction.

Thus $p > 3$. At this point we make use of 7.2. Take a parabolic subgroup $P_X = Q_X L_X$ of X, with $L_X' \cong A_1(q)$. Then P_X lies in a parabolic subgroup QL of G with $Q_X \leq Q$, and L_X is irreducible on the space $C_V(Q)$, which is given as L-module in Table 5. We can choose P_X so that $C_V(Q) \downarrow L_X \cong V_{L_X}(a)$ or $V_{L_X}(b)$.

We claim that one of the following holds:

$$\begin{aligned} G = E_8 : &\quad a_0, b_0 \leq 7, \\ G = E_7 : &\quad a_0, b_0 \leq 6, \\ G = E_6 : &\quad a_0, b_0 \leq 4, \\ G = F_4 : &\quad a_0, b_0 \leq 6. \end{aligned}$$

To see this we argue as follows. If the claim is false, then from Table 5 we see that $L_0 = D_7, E_6, D_5, D_5$ or D_4 and L_X' is irreducible on $V_{L_0}(\lambda)$, where $\lambda = \lambda_1, \lambda_1, \lambda_1, \lambda_4$ or λ_4, respectively. This is impossible for $V_{D_r}(\lambda_1)$ and $V_{D_4}(\lambda_4)$, by 1.1, and for $V_{D_5}(\lambda_4)$, by [LS2, 2.13]. Finally, if $L_0 = E_6$ then $X \neq G_2(q)$ since Q_X is abelian by 7.2, so $X = B_2(q)$. We shall see in the next paragraph that $V_X(ab)$ is restricted; hence a or b must be 26. But it is easy to see that this forces $\dim V > \dim L(G) = 133$, a contradiction. This establishes the claim.

As $p > 3$, $\dim V$ is 248, 133, 78 or 52. The irreducible KX-modules of dimension 8 or less, dividing $\dim V$, are as follows:

$$\begin{aligned} X = B_2(q) : &\quad V_X(01) \text{ of dimension } 4, \\ X = G_2(q) : &\quad V_X(10) \text{ of dimension } 7. \end{aligned}$$

Hence, if $V_X(ab)$ is not restricted, then it is a tensor product of one of the above modules with a twist of a restricted module $V_X(cd)$; the latter module has dimension 62 or 13 (for $X = B_2(q)$, $G = E_8$ or F_4), or dimension 19 (for $X = G_2(q)$, $G = E_7$). When $X = G_2(q)$, $G = E_7$, we have $C_V(Q) \downarrow L_X \cong V_{L_X}(1) \otimes V_{L_X}(c)^{(p^i)}$ or $V_{L_X}(d)^{(p^i)}$, whence $c \leq 2$ and $d \leq 6$ by 7.2. It is easy to see that no such module $V_X(cd)$ has dimension 19: use [GS] for the small weights, and count conjugates for the others. A similar argument applies when $X = B_2(q)$. Here $c \leq 7, d \leq 3$. Moreover, since $Z(X) = 1$ and $V \downarrow X \cong 01 \otimes cd^{(p^i)}$, the module $V(cd)$ must admit the action of the simply connected group of type B_2, and hence d must be odd; in particular, the central element -1 has determinant 1 on $V(cd)$, so $\dim V(cd)$ is even, hence equal to 62. Now [Pr] shows that all weights of the Weyl module $W_{B_2}(cd)$ appear as weights of $V_{B_2}(cd)$, and counting conjugates of such weights shows that the only possibilities with $\dim V_{B_2}(cd) \leq 62$ are among the following:

$$cd = 01, 03, 11, 13, 21, 23, 31, 41.$$

The dimensions of these modules can be calculated using the sum formula in [An]; we find that none of them is equal to 62.

We conclude that $V_X(ab)$ is restricted (that is, $a = a_0, b = b_0$). Now suppose $X = B_2(q)$. As $Z(X) = 1$, b is even. The dimensions of $V_X(ab)$ with $a, b \leq 7$ can be calculated using the sum formula in [An], and we find that the only cases where this dimension is less than or equal to 248 are as follows:

ab	$\dim V_{B_2}(ab)$
06	84
16	231
04	35
14	105
24	$220 - 105\delta_{p,5} - 71\delta_{p,7}$
02	10
12	35
22	$81 - 13\delta_{p,5} - 10\delta_{p,7}$
32	$154 - 68\delta_{p,5}$
42	$260 - 54\delta_{p,7} - 81\delta_{p,11}$
52	$199 \ (p = 7)$
10	5
20	$14 - \delta_{p,5}$
30	$30 - 5\delta_{p,7}$
40	$55 - \delta_{p,7}$
50	$91 - 30\delta_{p,11}$
60	$140 - 14\delta_{p,11} - 55\delta_{p,13}$
70	$204 - 5\delta_{p,11} - 30\delta_{p,13}$

None of these dimensions is equal to $\dim V$, a contradiction.

Now let $X = G_2(q)$, and recall that $p > 3$. Again, application of the sum formula shows that the only cases where $V_X(ab)$ (with $a, b \leq 7$) has dimension at most 248

are:

ab	$\dim V_{G_2}(ab)$
40	$182 - 27\delta_{p,11}$
30	77
21	$189 - 64\delta_{p,11}$
20	$27 - \delta_{p,7}$
13	$244\ (p = 13)$
12	$286 - 189\delta_{p,5} - 38\delta_{p,7}$
11	$64 - 26\delta_{p,7}$
10	7
03	$273 - 77\delta_{p,5} - 125\delta_{p,11}$
02	77
01	14

The only possibility where $\dim V_X(ab) = \dim V$ is

$$G = E_8,\ p = 7 \text{ and } V \downarrow X \cong V_X(12).$$

In this case, let α, β be fundamental roots for a G_2 root system, and let $t = h_\alpha(-1) \in X$. We calculate $\dim C_V(t)$, where $V = V_X(12)$. (We shall then obtain a contradiction by comparing this with the known possibilities for $\dim C_{L(G)}(t)$.) All the weights appearing in V and their multiplicities are listed in [GS]. The dominant weights which are subdominant to 12 are

$$12, 40, 21, 02, 30, 11, 20, 01, 10, 00,$$

and these appear in $V_X(12)$ with multiplicities 1,1,2,2,3,4,6,6,8,8, respectively. The orbits of these weights under the Weyl group of X are as follows:

$$
\begin{array}{ll}
\text{orbit of } 12: & 12, 1(-3), 8(-3), 8(-5), 7(-5), 7(-2) \text{ and negatives} \\
\text{orbit of } 40: & 40, 4(-4), 8(-4) \text{ and negatives} \\
\text{orbit of } 21: & 21, 2(-3), 7(-3), 7(-4), 5(-4), 5(-1) \text{ and negatives} \\
\text{orbit of } 02: & 02, 6(-2), 6(-4) \text{ and negatives} \\
\text{orbit of } 30: & 30, 3(-3), 6(-3) \text{ and negatives} \\
\text{orbit of } 11: & 11, 1(-2), 5(-2), 5(-3), 4(-3), 4(-1) \text{ and negatives} \\
\text{orbit of } 20: & 20, 2(-2), 4(-2) \text{ and negatives} \\
\text{orbit of } 01: & 01, 3(-1), 3(-2) \text{ and negatives} \\
\text{orbit of } 10: & 10, 1(-1), 2(-1) \text{ and negatives}
\end{array}
$$

The dimension of $C_V(t)$ is equal to the number of weights appearing in V with first coordinate even. Hence we find $\dim C_V(t) = 124$. But $C_G(t)$ is D_8 or A_1E_7, whence $\dim C_{L(G)}(t) = \dim C_G(t) = 120$ or 136, a contradiction. This completes the proof of the lemma. □

Lemma 7.7. *Theorem 4 holds if $G = F_4$.*

Proof. Let $G = F_4$. By 7.3, 7.4 and 7.6, $\mathrm{rk}(X) \geq 3$. First suppose that $q > 2$. Then by [LST, Theorem 2] there is a connected simple subgroup $\bar{X}$ of G containing X and of the same type as X. As X fixes $L(\bar{X})$, we deduce from 1.10 that $L(\bar{X})$ has all its $\bar{X}$-composition factors of dimension $52 - 26\delta_{p,2}$. By 1.10, there is no such group $\bar{X}$.

Now let $q = 2$. The only possible subgroups X having an irreducible module of dimension 26 are $D_4^\epsilon(2)$ and $B_4(2) \cong C_4(2)$, the unique such module being $V_X(\lambda_2)$. It is enough to rule out $X = D_4^\epsilon(2)$ (since $B_4(2)$ contains $D_4(2)$ acting irreducibly

on $V(\lambda_2)$). So suppose $X = D_4^\epsilon(2)$ and pick a parabolic subgroup P_X of X of type $Q_X D_3^\epsilon(2)$ if $\epsilon = \pm$, and of type $Q_X L_2(8)$ if $\epsilon = 3$. By [BT], $P_X < QL$, a parabolic of G. If $\epsilon = \pm$, this must be a B_3- or C_3-parabolic. Moreover, by 7.2, L must act irreducibly and nontrivially on $C_V(Q) = C_V(Q_X)$, for $V = V_G(\lambda_i)$, $i = 1, 4$. But this is not the case, as L in fact acts trivially on $C_V(Q)$ for one of these values of i.

Finally, suppose $\epsilon = 3$. Then the factor $L_2(8)$ acts trivially on $C_V(Q_X)$ (where $V = V_X(\lambda_2)$), so QL is a B_2- or A_1-parabolic as it must be trivial on both 26-dimensional composition factors of $L(G)$. The 8-dimensional module $V_X(\lambda_1)$ restricts to $L_2(8)$ as $1^{(2)} \otimes 1^{(4)}/1/1$, hence $V_X(\lambda_2) \downarrow L_2(8)$ has a composition factor $1 \otimes 1^{(2)} \otimes 1^{(4)}$. But on restriction to QL, the modules $V_G(\lambda_i)$ $(i = 1, 4)$ have all composition factors of dimension at most 5, which is a contradiction. $\square$

From now on, we assume that $G = E_6, E_7$ or E_8. Also, by 7.3–7.6, $\mathrm{rk}(X) \geq 3$.

Lemma 7.8. *Let L_0 be a connected simple group of type A_r $(r \leq 7)$, D_4, D_5, D_5, D_7 or E_6, and let $W = V_{L_0}(\mu)$, where $\mu = \lambda_1, \lambda_1, \lambda_1, \lambda_4, \lambda_1$ or λ_1, respectively. Suppose that $Y = Y(q)$ is a quasisimple subgroup of L_0 such that $\mathrm{rk}(Y) \geq 2$, $Y \neq G_2^\epsilon(q)$ and Y is irreducible on W. Then one of the following holds (where $V \downarrow Y$ is given up to twists):*

Y	ν, where $V \downarrow Y = V_Y(\nu)$	L_0
$A_2^\epsilon(q)$	$10, 01, 20, 02, 11, 22,$	$A_2, A_2, A_5, A_5, A_{7-\delta_{p,3}}$
		or D_4, E_6,
	$\alpha \otimes \beta^{(p^i)} \otimes \gamma^{(p^j)}$	E_6 *(resp.)*
	$(\alpha, \beta, \gamma \in \{10, 01\})$	
$B_2^\epsilon(q)$	$10, 01, 02, 20, 11, 01 \otimes 01^{(p^i)},$	$A_{4-\delta_{p,2}}, A_3, D_5, D_7, D_5, D_5,$
	$10 \otimes 10^{(2^i)}, 10 \otimes 01^{(2^i)}(p = 2)$	D_5, D_5 *(resp.)*
$A_3^\epsilon(q)$	$100, 001, 010$	A_3, A_3, A_5
$B_3(q)$	$100, 200(p \neq 7), 001$	$A_{6-\delta_{p,2}}, E_6, A_7$ *or* D_4
$C_3(q)$ $(q$ *odd*$)$	$100, 010(p \neq 3)$	A_5, D_7
$B_4(q)$ $(q$ *odd*$)$	0001	D_5
$C_4(q)$	$1000, 0100, 0001(p = 2)$	A_7, E_6, D_5
$D_4^\epsilon(q)$	$1000, 0010, 0001$	A_7 *or* D_4
same type as L_0	μ	L_0

Proof. The dimension of W is $r + 1$, 8, 10, 16, 14 or 27 in the respective cases for L_0. The Y-modules of dimension up to 10 (self-dual if the dimension is 9 or 10) are well known (cf. [Li, 1.1]), and all possibilities are in the table. So assume $\dim W = 14, 16$ or 27. Let $W = V_Y(\lambda)$. If this is a nontrivial tensor product, then the tensor factors have dimension at most 9 and either W is in the table or $Y = A_3^\epsilon(q)$, $L_0 = D_5$, $\lambda = \lambda_4$ and $W \downarrow Y = \alpha \otimes \beta^{(p^i)}$ with $\alpha, \beta \in \{100, 001\}$; however, in the latter case, $V_{D_5}(\lambda_1) \downarrow Y$ must be either $100/001/000^2$ or $010/000^4$, and in neither case does the spin module $V_{D_5}(\lambda_4)$ restrict irreducibly to Y, so this case does not occur. Now assume $V_Y(\lambda)$ is not a nontrivial tensor product; then we may assume λ is restricted.

Suppose Y is not of type L_0. When $Y = A_2^\epsilon(q)$, 1.8 shows that the only possibility is $\lambda = 22$ (giving $\dim V_Y(\lambda) = 27$). For other types, we argue by counting conjugates of subdominant weights appearing in $V_Y(\lambda)$ that the only possibilities for (Y, λ),

with $V_Y(\lambda)$ having dimension between 11 and 27, are among the following:

Y	λ
$B_2(q)$	$20, 30, 03, 11, 12$
$B_3(q)$	$200, 002$
$C_3(q)$	$200, 010, 001$
$B_4(q)$	$0100, 0001$
$C_4(q)$	0100

For $Y = B_2(q)$ the tables in the proof of 7.6 exclude all but $\lambda = 20, 11$; for the other
cases, [BW] excludes all but $(Y, \lambda) = (B_3(q), 200), (C_3(q), 010$ or $001), (B_4(q), 0001)$
and $(C_4(q), 0100)$. All of these except $(C_3(q), 001)$ (q odd) are in the conclusion;
in the exceptional case, the 14-dimensional module $V_{C_3}(001)$ is symplectic rather
than orthogonal. Note also that $p \neq 7, 3$ for the $B_3(q)$, $C_3(q)$ cases in the table,
respectively, as otherwise $V_Y(\lambda)$ does not have the correct dimension. $\qquad\square$

Lemma 7.9. *We have $V \cong V_X(\delta)$, where δ is as in Table 6 (given up to twists).*

TABLE 6

X	δ
$A_3(q)$	$111, 202, 222$
$^2A_3(q)$	$0a0 \, (a \leq 7), 1a1 \, (a \leq 5)$
$^2A_4(q)$	$0110, 1111$
$B_3(q)$	$011, 020, 110, 111, 102, 202, 220$
$C_3(q)$	$002, 020, 011, 101, 111, 202, 220, 110(p=2)$
$^2D_4(q)$	$2010, 2001, 1100$
$^3D_4(q)$	$a000, 0b00, 1c00 \, (a \leq 4, b \leq 3, c \leq 3), 2100,$
	$1010, 1110, 2010, 3010, 3110,$
	$1d10, 1d11, 2d10 \, (p \leq 3, d \leq p-1)$
$F_4(q)$	$2000, 0010$
$^2F_4(q)$	$0100, 1100$

Proof. Recall that V is a self-dual G-module. We show first that if $V \downarrow X = V_X(\delta)$,
then the weight δ is restricted. If not, then $V \downarrow X$ is a nontrivial tensor product. As
$\dim V$ is $78 - \delta_{p,3}, 133 - \delta_{p,2}$ or 248, we see that one of the tensor factors must have
dimension $x \leq 8$. Thus X has a self-dual irreducible $V_X(\chi)$ of dimension x, from
which we deduce that $(X, \chi, x) = (A_3^\epsilon(q), 010, 6), (B_3(q), 100, 7), (B_3(q), 001, 8),$
$(C_3(q), 100, 6) \, (C_4(q), \lambda_1, 8)$ or $(D_4^\epsilon(q), \lambda_i, 8) \, (i = 1, 3, 4)$. We have $V \downarrow X = \chi \otimes \gamma^{(r)}$,
where $y = \dim V_X(\gamma)$ is 13 or 22, 11 or 19, 31, 13 or 22, 31, or 31, respectively.
Excluding the cases $(X, y) = (A_3^\epsilon(q), 22)$ or $(B_3(q), 31)$, we see in the usual way
(counting conjugates of subdominant weights, etc.) that the only possibility for a
self dual X-module of dimension y is $X = C_3(q), \gamma = \lambda_2$ with $p = 3$ and $y = 13$;
but then $xy = 78$, whereas $\dim V = 77$ for $G = E_6, p = 3$. For the excluded case
with $X = A_3^\epsilon(q)$, we have $xy = 132$, so $G = E_7, p = 2$. The only possibility for
γ is 111; but it is easy to see by counting conjugates of subdominant weights that
$V_X(111)$ has dimension greater than 22. The other excluded case with $X = B_3(q)$
is handled by counting conjugates of dominant weights in the usual way. Thus the
weight δ is restricted.

Consider first $X = A_3(q)$, and let $\delta = aba$. Choose a parabolic $P_X = Q_X L_X$ of X such that $L'_X \cong A_2(q)$ and $C_V(Q_X) \downarrow L_X \cong V_{L_X}(ab)$. Then from 7.2 and 7.8, we see that ab is as in the conclusion of 7.8 for $A_2(q)$, and so

$$ab = 00, 10, 01, 20, 02, 11, \text{ or } 22.$$

Therefore $\delta = 101, 010, 202, 020, 111$ or 222. Clearly $101, 010$ and 020 have dimension less than 77, leaving the possibilities in Table 6 for $X = A_3(q)$.

For $X = A_r(q)$ with $r \geq 4$, we first use the same argument with an A_{r-1}-parabolic: if $\delta = a_1 \ldots a_r$, we see that $a_1 \ldots a_{r-1}$ is equal to $0 \ldots 0, 10 \ldots 0$ or $0 \ldots 01$ (or, if $r = 4$, to 010). As $V \downarrow X$ is self-dual, we conclude that $a_1 \ldots a_r = 10 \ldots 01$. Now choose another parabolic $Q_X^{(1)} L_X^{(1)}$ of type $A_2 \times A_{r-3}$; then $C_V(Q_X^{(1)})$ has dimension $3(r - 2)$, and we deduce using 7.2 that $3(r - 2) \leq 8$. Hence $r = 4$ and $V \downarrow X = V_X(1001)$; but $\dim V_X(1001) \leq 24$, which is a contradiction.

Next let $X = B_3(q)$, with $\delta = abc$. From an A_2-parabolic and 7.8 we get $ab = 00, 10, 01, 20, 02, 11$ or 22; and from a B_2-parabolic we get $bc = 00, 10, 01, 20, 02$ or 11. Hence δ is in the following list:

$$001, 002, 100, 101, 102, 010, 011, 200, 201, 202, 020, 110, 111, 220.$$

If $\delta = 001, 002, 100, 101, 010$ or 200, then $V_X(\delta)$ appears in a tensor product $V_X(\lambda_i) \otimes V_X(\lambda_j)$ with $i, j \in \{1, 3\}$, hence has dimension less than 77; and if $\delta = 201$ then $p \neq 2$ and the acting group is the simply connected group $Spin_7(q)$, with centre of order 2, contrary to the fact that $Z(X) = 1$. All other possibilities for δ are in Table 6.

Now consider $X = B_r(q)$ with $r \geq 4$ and $\delta = a_1 \ldots a_r$. Observe that $r \leq 8$, as can be seen from an easy inductive argument (embed a B_{r-1}-parabolic of X in a parabolic of G, and repeat). Working with parabolics of types A_{r-1} and B_{r-1} as above, we find $a_1 \ldots a_{r-1} = 00...0, 10...0$ or $0...01$ (or 010 with $r = 4$), and $a_2 \ldots a_r = 00...0$ (or $100, 200, 001$ with $r = 4$, or 0001 with $r = 5$). Hence δ is one of the following:

$$10...0, 0001, 00001, 1001, 10001, 0100.$$

The first three cases have dimension $2r + 1$ or $2r$, 16 and 32 respectively, so are impossible; and the last has dimension less than 77. If $\delta = 10001$, then an $A_2 B_2$-parabolic acts on the fixed space of its radical as $10 \otimes 01$, so this fixed space has dimension 12, contrary to 7.2. This leaves only $\delta = 1001$; here also $p = 2$, since otherwise the acting group has centre of order 2 again. But when $p = 2$, $V_{B_4}(1001) \cong V_{B_4}(1000) \otimes V_{B_4}(0001)$ (see [Bor2, 7.6]), of dimension 8.16, so this case cannot occur either.

The cases $X = C_r(q)$ ($r \geq 3$) or $D_r(q)$ ($r \geq 4$) are handled in just the same way as $B_r(q)$ (for $D_r(q)$ we use two A_{r-1}-parabolics).

For $X = F_4(q)$ with $\delta = abcd$, use of a B_3-parabolic and 7.2 gives $abc = 000, 100, 200$ or 001, and use of a C_3-parabolic gives $bcd = 000, 100, 010$ or $001(p = 2)$. Hence $\delta = 1000, 1001(p = 2), 2000, 0010$ or $0001(p = 2)$. Of these, the first and last have dimension less than 77, and $1001 \cong 1000 \otimes 0001$ ([Bor2, 7.6]), of dimension 26.26; the rest are in Table 6.

When $X = E_6(q), E_7(q)$ or $E_8(q)$, use of suitable parabolics forces $\delta = \lambda_2, \lambda_1$ or λ_8 respectively, whence X has the same type as G, contrary to our original assumption.

This completes the case where X is of untwisted type.

Now suppose $X = {}^2A_3(q)$ and $\delta = aba$. Choose a parabolic $P_X = Q_X L_X$ of X with $L'_X \cong A_1(q^2)$. Then $C_V(Q_X) \downarrow L_X = a \otimes a^{(q)}$. Also Q_X is non-abelian, so we deduce from 7.2 that $\dim C_V(Q_X) = (a+1)^2 \leq 8$, whence $a \leq 1$. The usual argument using an $A_1(q)$-parabolic shows that $b \leq 7$. Now choose a subgroup $B \cong B_2(q)$ of X. Then $V \downarrow B$ has a composition factor $V_B(b, 2a)$. Using the list of B_2-modules given in the proof of 7.6, we see that either $a = 0, b \leq 7$ or $a = 1, b \leq 5$, as in Table 6.

Now consider $X = {}^2A_4(q)$, $\delta = abba$. There is a parabolic $Q_X L_X$ with $L_X \cong {}^2A_2(q)$ and $C_V(Q_X) \downarrow L_X \cong V_{L_X}(bb)$. Hence by 7.8, we have $b \leq 2$; moreover, if $b = 2$ then $\dim C_V(Q_X) = 27$; but this is impossible by 7.2, as Q_X is non-abelian. Thus $b \leq 1$. As in the previous paragraph, we see that $a \leq 1$. Clearly 0000 and 1001 have dimension less than 77, leaving 0110 and 1111, as in Table 6. For $X = {}^2A_r(q)$ with $r \geq 5$ and $\delta = a_1 \ldots a_r$, an $A_2(q^2)$ parabolic $Q_X L_X$ has $C_V(Q_X) \downarrow L_X = a_1 a_2 \otimes a_2 a_1^{(q)}$, whence $a_1 = a_2 = 0$, and a ${}^2A_{r-2}(q)$ parabolic now gives either $a_i = 0$ for all i, or $r = 5$, $\delta = 00100$; in the latter case, $V_X(\delta)$ has dimension 20, a contradiction.

Next let $X = {}^2D_r(q)$ ($r \geq 4$). Note again that $r \leq 9$. Use of 7.8 with parabolics of types ${}^2D_{r-1}$ and A_{r-2} yield the following possibilities for δ:

$$\lambda_1, \lambda_{r-1}, \lambda_r, \lambda_1 + \lambda_{r-1}, \lambda_1 + \lambda_r, 01000, 2000, 2010, 2001, 0100, 1100.$$

The first three possibilities have dimensions $2r, 2^{r-1}, 2^{r-1}$ respectively, so these cannot hold. For the next two cases, when $r \geq 6$ an $A_{r-4}(q) \times {}^2A_3(q)$ parabolic rules them out; for $r = 5$ they are not self dual; and for $r = 4$ they have dimension less than 77. Finally, the cases with $\delta = 01000, 2000, 0100$ have dimension less than 77 also. This leaves just the possibilities in Table 6.

Similarly, for $X = {}^2E_6(q)$ we use parabolics of types ${}^2A_5$ and ${}^2D_4$ to force $\delta = \lambda_1, \lambda_6, \lambda_2, \lambda_1 + \lambda_2$ or $\lambda_6 + \lambda_2$. Of these, only λ_2 is self dual. But this is the high weight of the adjoint module for X, which means that X and G must have the same type, contrary to assumption.

It remains to deal with $X = {}^2F_4(q)$ or ${}^3D_4(q)$. In the first case, $V \downarrow X = V_X(abcd) \cong V_X(ab00) \otimes V_X(00cd)$. Each tensor factor is either trivial or has dimension at least 26, so one of them, say $00cd$, is trivial. Thus $V \downarrow X = 1000, 0100$ or 1100; the first has dimension 26, and the others are in Table 6.

Finally, let $X = {}^3D_4(q)$, and suppose $\delta = abcd$. Pick a parabolic $Q_X L_X$ with $L'_X \cong A_1(q^3)$. Then $C_V(Q_X) \downarrow L_X = a \otimes c^{(q)} \otimes d^{(q^2)}$. Moreover Q_X is non-abelian, so we deduce from 7.2 and 1.1 that $(a+1)(c+1)(d+1) \leq 8$. Therefore, taking $a \geq c \geq d$, we have

$$(a, c, d) = (a, 0, 0)\,(a \leq 7), \ (a, 1, 0)\,(a \leq 3) \text{ or } (1, 1, 1).$$

Now take a subgroup $C = G_2(q)$ of X, so that $V \downarrow C$ has a composition factor $V_C(a + c + d, b)$. This module is restricted unless $acd = 210, 110$ or 111 and $p \leq 3$. These cases are included in Table 6 of the statement. In the restricted case, we see from the list of G_2-modules given in the proof of 7.6 that

$$a + c + d, b = 40, 30, 21, 20, 13, 12, 11, 10, 03, 02, \text{ or } 01.$$

Hence $abcd$ is as in Table 6. $\square$

The proof of Theorem 4 is completed by the next lemma.

Lemma 7.10. *Let X and δ be as in Table 6. Then $\dim V_X(\delta)$ is not equal to $77 - \delta_{p,3}$, $133 - \delta_{p,2}$ or 248.*

Proof. For many of the possibilities in Table 6, the dimension of $V_X(\delta)$ is given by [BW] or [GS]. For several of the remaining cases, counting conjugates of subdominant weights in $V_X(\delta)$ shows that $\dim V_X(\delta) > 248$. The final surviving cases are handled using the programme described in [GS] to determine the precise dimension of $V_X(\delta)$. $\square$

This completes the proof of Theorem 4.

8. Proof of Theorem 6 and corollaries

We begin with the proof of Theorem 6. As in the hypothesis of Theorem 6, let G be a simple adjoint algebraic group of exceptional type in characteristic p, and σ be a Frobenius morphism such that $L = O^{p'}(G_\sigma)$ is a finite simple group of exceptional Lie type. Let L_1 be a finite group with socle L (i.e. $L \leq L_1 \leq \operatorname{Aut} L$). Suppose that H is a maximal subgroup of L_1 such that $F^*(H) = X = X(q)$, a simple group of Lie type over $\mathbb{F}_q$, with q as in the hypothesis of Theorem 1. Assume further that X is not of the same type as G (otherwise (i) of Theorem 6 holds).

By Corollary 5, there is a proper connected subgroup $\bar{X}$ of G containing X such that X and $\bar{X}$ fix the same subspaces of $L(G)$.

Let $\mathcal{M}$ be the set of all X-invariant subspaces of all G-composition factors of $L(G)$, and define

$$Y = \bigcap_{W \in \mathcal{M}} G_W.$$

As every automorphism of L extends to a morphism $G \to G$, it follows from 1.12(iii) that Y is $H\langle\sigma\rangle$-stable. Moreover, by Theorem 4, Y is a proper closed subgroup of G, and $X < \bar{X} \leq Y$. Let Z be a maximal connected $H\langle\sigma\rangle$-invariant proper subgroup of G containing Y^0. By the maximality of H we have $H = N_{L_1}(Z)$. Clearly Z is reductive (otherwise H normalizes $R_u(Z)_\sigma$, hence is parabolic). Hence $X = O^{p'}(Z_\sigma)$. Consequently (ii) of Theorem 6 holds.

This completes the proof of Theorem 6.

Remark. As remarked after Theorem 6 in the Introduction, the above proof shows, more generally, that if H is any (not necessarily maximal) subgroup of L_1 such that $F^*(H) = X(q)$ with q as in Theorem 1, then $H \leq N(\bar{X}_\sigma)$ for some $\bar{X}$ as in conclusion (ii) of Theorem 6.

Proof of Corollary 7. Assume the hypotheses of Corollary 7, so that H is a maximal subgroup of L_1 as above, and $F^*(H) = X = X(q)$ with q as in Theorem 1. We may take it that $K = \bar{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$, since the conclusions of Corollary 7 concern finite groups only. (This allows us to apply 1.13 at a suitable point in the argument.)

Assume that H is not a subgroup of maximal rank, and that $X(q)$ is not of the same type as G. By Theorem 6, we have $X = O^{p'}(\bar{X}_\sigma)$, where $\bar{X}$ is a maximal closed connected reductive $H\langle\sigma\rangle$-stable subgroup of G. By assumption, $\bar{X}$ does not contain a maximal torus of G. Moreover, $Z(\bar{X})^0 = 1$, since otherwise $H \leq N(C(Z(\bar{X})^0))$, a group of maximal rank, which is impossible as H is not of maximal rank by assumption. Therefore $\bar{X}$ is semisimple. If $\bar{X}$ is not simple, then, by [LS1, Theorem

1], $\bar{X}$ is as in the second column of [LS1, Table II]; from this we see that $O^{p'}(\bar{X}_\sigma)$ is not simple, a contradiction.

Hence $\bar{X}$ is simple. Therefore, since $X = X(q) = O^{p'}(\bar{X}_\sigma)$, by 1.13 we have $q = q_1$ (where $G_\sigma = G(q_1)$), as in (iii) of Corollary 7. Moreover, if $p > N(X, G)$, then the possibilities for $\bar{X}$ are given by [LS1, Theorem 1]; note also from [LS1, Theorem 1] that these maximal $H\langle\sigma\rangle$-stable subgroups $\bar{X}$ are also maximal σ-stable.

There remain two points to establish: first, that the fixed point groups $X(q) = O^{p'}(\bar{X}_\sigma)$ are as in conclusion (iii) of Corollary 7; and second, the conjugacy statement at the end of Corollary 7.

The conjugacy statement is straightforward: first, the assertion of the uniqueness of $\mathrm{Aut}(G)$-classes of maximal connected subgroups $\bar{X}$ is given by [Se2, Theorem 1]. And second, by [SS, I,2.7], the G_σ-classes contained in a given G-class of subgroups $\bar{X}$ correspond to classes in the group $H^1(\sigma, N_G(\bar{X})/\bar{X})$; however, clearly $N_G(\bar{X})/\bar{X}$ has order 1 unless $\bar{X} = A_2$, in which case it has order 2 by Lemma 8.1 below, so the conclusion is immediate.

The statement that the fixed point groups $X(q) = O^{p'}(\bar{X}_\sigma)$ are as in conclusion (iii) of Corollary 7 is clear, except when $\bar{X} = A_2$, in which case we must establish that both the fixed point groups $A_2(q)$ and $^2A_2(q)$ arise. This follows from the next lemma.

Lemma 8.1. *Let $\bar{X} \cong A_2$ be a maximal connected subgroup of $G = E_6$ or E_7. Then $N_G(\bar{X})$ induces a nontrivial graph automorphism on $\bar{X}$ (and hence $N_G(\bar{X})/\bar{X}$ has order 2).*

Proof. We may assume that G is of adjoint type. When $G = E_6$, the conclusion is immediate from [Te2, Claim on p. 314]. So assume from now on that $G = E_7$. The class of maximal subgroups A_2 in E_7 for $p \geq 7$ is constructed in [Se2, 5.8].

Choose a parabolic subgroup $QJ_\alpha T_\alpha$ of $\bar{X}$, where J_α is a fundamental SL_2, T_α a rank 1 torus and Q the unipotent radical. Then by [Se2, p. 83], $C_G(T_\alpha) = T_2A_1A_4$, J_α projects onto the factor A_1, and J_α projects onto a regular A_1 in the factor A_4.

We claim that there is an involution $z \in N_G(T_2A_1A_4)$ normalizing J_α and inverting T_2. To see this, we work first in E_8, and observe that there is an involution u normalizing a maximal rank subgroup A_4A_4 of E_8, inducing a graph automorphism on each factor. Then u normalizes a fundamental A_1 in one of the factors A_4, inducing a nontrivial inner automorphism on this A_1. Therefore $u \in A_1E_7$ and is a product yz of elements y, z of order 4, where $y \in A_1$, $z \in E_7$, and $y^2 = z^2$ is a generator of $Z(E_7)$. Then z projects to an involution in the adjoint group of type E_7. Now $C_{A_4A_4}(A_1) = T_1A_2A_4$; therefore u, hence z, normalizes this group, acting as a graph automorphism on the A_2, A_4 factors and inverting the T_1. Now z centralizes a subgroup B_2 of the A_4 factor, and a subgroup PSL_2 therein. Adjusting z by a suitable involution of this PSL_2, we obtain an element z normalizing a subgroup $T_2A_1A_4$ of $T_1A_2A_4$, acting in the fashion claimed at the beginning of this paragraph.

Let t be the central involution in J_α. Then $C_G(t) = A_1D_6$, and $L(G) \downarrow A_1D_6 = L(A_1D_6) \oplus (V_{A_1}(1) \otimes V_{D_6}(\lambda_5))$. The restriction of $V_{D_6}(\lambda_5)$ to the projection of J_α in A_4 (which lies in D_6) is $6^2/4^2/2^2/0^2$; tensoring with $V_{J_\alpha}(1)$, we obtain

$$V_{A_1}(1) \otimes V_{D_6}(\lambda_5) \downarrow J_\alpha = 7^2/5^4/3^4/1^4.$$

We claim that there is an 8-dimensional J_α-submodule isomorphic to 1^4. Two of the required four summands are provided by $L(Q)$ and $L(Q^-)$. Now consider

the action of $T = T_\alpha$; say it acts with weight β on $L(Q)$, hence with weight $-\beta$ on $L(Q^-)$. Now T_2 centralizes J_α, so acts on the homogeneous component A of the socle of $L(G) \downarrow J_\alpha$ corresponding to high weight 1. It also centralizes T, hence acts on the β and $-\beta$ weight spaces in A; and z interchanges these. If the β weight space is just 2-dimensional (i.e. equal to $L(Q)$), then so is the $-\beta$ weight space, and conversely. But then the Lie algebra span of this part of the socle is just $\langle L(Q), L(Q^-) \rangle = L(\bar{X})$. However, this is not T_2-invariant, so this is a contradiction. Therefore the β and $-\beta$ weight spaces of A have dimension 4, and $A \cong 1^4$, proving the claim.

Let $W_\beta, W_{-\beta}$ be the $\pm\beta$ weight spaces in A. These are fixed by T_2 and interchanged by z. If $C_{T_2}(W_\beta)^0 \neq 1$, then there is a 1-dimensional torus acting trivially on W_β, hence also on $W_{-\beta}$ (as z inverts T_2). But this torus then acts trivially on $\langle L(Q), L(Q^-) \rangle = L(\bar{X})$ (Lie algebra span), which is impossible (as $\bar{X}$ is maximal). Therefore $C_{T_2}(W_\beta)^0 = 1$. Consequently $W_\beta = W_1 \oplus W_2$, a sum of two 2-dimensional T_2-weight spaces with different T_2-kernels; similarly $W_{-\beta} = W_3 \oplus W_4$.

Now none of the W_i is equal to $L(Q)$ or $L(Q^-)$ (otherwise we would get a 1-dimensional torus acting trivially on $L(\bar{X})$ as above). Hence $L(Q)$ is diagonal in $W_1 \oplus W_2$. As a $J_\alpha T_2$-module, we have $W_\beta \cong M \otimes N$, where M is a natural 2-dimensional module for J_α and N is a direct sum of two 1-dimensional T_2-modules with T_2-kernels having different connected components. As T_2 has three orbits on the nonzero vectors of N, it follows that T_2 has three orbits on the set of 2-dimensional J_α-invariant subspaces of W_β. Since $L(Q^-)^z$ is such a subspace, and is diagonal in $W_1 \oplus W_2$, there is therefore an element $a \in T_2$ such that $L(Q^-)^{za} = L(Q)$. Then za is an involution normalizing the Lie algebra span $\langle L(Q), L(Q^-) \rangle = L(\bar{X})$. Consequently za normalizes $\bar{X}$, inducing a graph automorphism, and this completes the proof of the lemma. $\square$

Proof of Corollary 8. Let L_1 be as in the first paragraph of this section, and let H be a maximal subgroup of L_1. Assume first that H is not almost simple. Then by [LS1, Theorem 2], one of the following holds:

(i) $H = N(\bar{X}_\sigma)$, where $\bar{X}$ is a maximal connected $H\langle\sigma\rangle$-stable subgroup of G;

(ii) H is the normalizer of a subgroup of the same type as G;

(iii) H one of the local subgroups $2^3.L_3(2)$, $3^3.L_3(3)$, $3^{3+3}.L_3(3)$, $2^{5+10}.L_5(2)$, $5^3.L_3(5)$ (in G_2, F_4, E_6, E_8, E_8 respectively), given in [LSS, Theorem 1(II)];

(iv) $F^*(H) = \text{Alt}_5 \times \text{Alt}_6$.

Taking $c > 2^{15}|L_5(2)|$, we see that (i) or (ii) of Corollary 8 holds for $|H| > c$.

Now assume that H is almost simple, and let $X = F^*(H)$. If X is alternating, sporadic, or of Lie type in p'-characteristic, then H has bounded order (see [LSS, §4] for instance); so we may choose c so that this does not occur for $|H| > c$. Finally, let $X = X(q)$ with $q = p^e$. The BN-rank of X is at most that of G (since a parabolic of X must lie in a parabolic of G), so again choosing c appropriately, we may assume that q satisfies the hypothesis of Theorem 1. Now the conclusion follows from Theorem 6. $\square$

9. Proofs of Theorems 9 and 10

For the proof of Theorem 9, we shall need the information on first cohomology groups given in the next two results. The first is taken from [Ja, 2.3].

Proposition 9.1 (Jantzen). *Let Y be a simply connected simple algebraic group over $\bar{\mathbb{F}}_p$, and let σ be a Frobenius morphism of Y such that Y_σ is not of type $^2B_2$,*

$^2G_2$ or $^2F_4$. Let $Y_\sigma = Y(q)$, a finite group of Lie type over $\mathbb{F}_q$, with $q = p^r$, and let α_0 be the highest short root of the root system of Y. Suppose that λ is a dominant weight for Y satisfying

$$\langle \lambda, \alpha_0^{\mathrm{v}} \rangle \leq \begin{cases} p^r - 2p^{r-1} - 3, & \text{if } Y \neq G_2, \\ p^r - 3p^{r-1} - 3, & \text{if } Y = G_2, \end{cases}$$

and let $V = V_Y(\lambda)$. Then the natural restriction map

$$H^1(Y, V) \rightarrow H^1(Y_\sigma, V)$$

is an isomorphism.

Proposition 9.2. Let Y be a simple algebraic group in characteristic p, let $Y(q) = Y_\sigma$ be a quasisimple group of Lie type over $\mathbb{F}_q$, $q = p^e$, and let $V = V_{Y(q)}(\lambda)$. Then $H^1(Y(q), V) = 0$ for all λ in Table 7.

TABLE 7

$Y(q)$	λ	conditions on p, q
$A_r^\epsilon(q)$	$\lambda_1, \lambda_2, \lambda_3$	$q > 3$, and $p > 2$ if $r = 1$
$B_r(q)\,(r \geq 2)$	λ_1, λ_r	$p > 2$ if $\lambda = \lambda_1$ or $r = 2$,
		$Y(q) \neq B_2(3)$
$D_r^\epsilon(q)\,(r \geq 4)$	$\lambda_1, \lambda_{r-1}, \lambda_r$	
$G_2^\epsilon(q)$	10	$p > 2, q > 3$
$C_4(q)$	λ_1, λ_3	$p > 2$
$F_4^\epsilon(q)$	λ_4	$p \neq 3, q > 8$
$E_6^\epsilon(q)$	λ_1	
$E_7(q)$	λ_7	
$A_2^\epsilon(q)$	$20, 30, 60, 11, 21, 22,$	$p \geq 5\,(p \geq 7 \text{ for } 60), q \geq 11$
	$10 \otimes 10^{(p^i)}, 10 \otimes 01^{(p^i)},$	
	$10 \otimes 11^{(p^i)}, 10 \otimes 20^{(p^i)},$	
	$10 \otimes 02^{(p^i)}$	
$B_2(q)$	$02, 03, 20, 11, 13,$	$p \geq 5, q \geq 11\,(p \geq 7 \text{ for } 20,$
	$10 \otimes 10^{(p^i)}, 10 \otimes 01^{(p^i)},$	$p = 3 \text{ allowed for } \lambda = 02)$
	$01 \otimes 01^{(p^i)}, 01 \otimes 02^{(p^i)}$	
$G_2(q)$	$20, 01, 11, 10 \otimes 10^{(p^i)}$	$p \geq 11$
$A_3^\epsilon(q)$	$101, 110, 200, 100 \otimes 100^{(p^i)},$	$p > 2, q \geq 11$
	$100 \otimes 001^{(p^i)}, 100 \otimes 010^{(p^i)}$	
$B_3(q)$	$010, 002, 101, 001 \otimes 001^{(p^i)}$	$p > 2, q \geq 11$
$C_3(q)$	$100, 010, 001, 110$	$p > 2\,(p > 3 \text{ for } \lambda = 010), q \geq 11$
$C_4(q)$	$0100, 0010$	$p > 2, q \geq 11$
$D_4^\epsilon(q)$	$0100, 0011$	$p > 2, q \geq 11$

Proof. For $Y(q)$ as in the first eight rows of the table (i.e. $Y(q) = A_r^\epsilon(q), \ldots,$ $E_7(q)$), the result follows immediately from [JP] and [LS3, 1.8]. For the other cases, our assumptions on p, q imply that $H^1(Y(q), V) \cong H^1(Y, V)$ by 9.1. (For cases where λ is non-restricted, we twist V by an automorphism of $Y(q)$ to take $i \leq r/2$.) It follows from [LS2, 1.7 and 1.9–1.15] that $H^1(Y, V) = 0$ in all cases. $\square$

We shall also need some elementary lower bounds for the numbers $t(G)$ defined in the Introduction.

Proposition 9.3. *We have* $t(G_2) \geq 12$, *and* $t(\Sigma(G)) \geq 16$ *for* $G = F_4, E_6, E_7, E_8$.

Proof. For $G = G_2$, take simple roots α (short) and β (long), and set

$$y = (3\alpha + 2\beta) - (-3\alpha - \beta) = 6\alpha + 3\beta, \qquad z = \beta - (-\beta) = 2\beta$$

(the middle terms in these equations are written simply to demonstrate that y and z are root differences); then $12\alpha = 2y - 3z$, but 6α and 4α are not in $\mathbb{Z}y + \mathbb{Z}z$. Hence $t(\Sigma(G)) \geq 12$.

For $G = F_4$, take simple roots $\alpha, \beta, \gamma, \delta$ with α and β long and β joined to γ. Set $x = (\alpha + 2\beta + 2\gamma) - (-\alpha - \beta - 2\gamma) = 2\alpha + 3\beta + 4\gamma$, $y = (\alpha + \beta) - (-\beta) = \alpha + 2\beta$, $z = \alpha - (-\alpha) = 2\alpha$; then $16\gamma = 4x - 6y - z$, but 8γ is not in $\mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z$. Hence $t(\Sigma(G)) \geq 16$.

For $G = E_6, E_7$ or E_8 take four simple roots $\alpha, \beta, \gamma, \delta$ forming an A_4 subsystem. Set $w = (\alpha + \beta) - (-\alpha) = 2\alpha + \beta$, $x = (\beta + \gamma) - (-\beta) = 2\beta + \gamma$, $y = (\gamma + \delta) - (-\gamma) = 2\gamma + \delta$, $z = \delta - (-\delta) = 2\delta$; then $16\alpha = 8w - 4x + 2y - z$, but 8α is not in $\mathbb{Z}w + \mathbb{Z}x + \mathbb{Z}y + \mathbb{Z}z$. Hence $t(\Sigma(G)) \geq 16$. $\square$

Proof of Theorem 9. Let G be a simple exceptional adjoint algebraic group over the algebraically closed field K of characteristic p, let $X = X(q) < G$ with q as in the hypotheses of Theorem 1, and assume that $p > N'(X, G)$ (as defined before the statement of Theorem 9). Suppose that $X < P = QL$, a parabolic subgroup of G with unipotent radical Q and Levi subgroup L. We aim to show that X lies in a Q-conjugate of L.

We follow closely the proof of [LS2, Theorem 1], which appears in [LS2, §3]. Assume first that P is minimal subject to containing X, and choose a subgroup X_1 of L such that $QX = QX_1$.

Suppose that $X \not\cong A_1(q)$. By Theorem 1, $X_1 < \bar{X} \leq L$, where $\bar{X}$ is connected and X_1 and $\bar{X}$ stabilize the same subspaces of the Lie algebra of L. Since $\bar{X}$ lies in no parabolic subgroup of L, it is semisimple, say $\bar{X} = \bar{X}_1 \ldots \bar{X}_r$ with all $\bar{X}_i$ simple. The fact that X_1 is irreducible on each $\bar{X}$-composition factor of $L(\bar{X}_i)$ implies that all the $\bar{X}_i$ are of the same type as X. Therefore X lies in a connected simple subgroup $\bar{X}_0$ of L of the same type (a diagonal subgroup of $\bar{X}_1 \ldots \bar{X}_r$). Hence the possible embeddings of X_1 in L are given by [LS2, 3.3] (with the same proof). The L'-composition factors within Q have the structure of rational KL'-modules, with high weights given by [LS2, 3.1]. We deduce as in the proof of [LS2, 3.4] that the restrictions to X_1 of such modules have composition factors with high weights λ all twists of those occurring in the table of Proposition 9.2. Moreover, the conditions on p, q given in the table are satisfied (since we are assuming $p > N'(X, G)$ and $q > 9$), except in the following cases:

    (1) $X_1 \cong B_r(q)$ $(r \geq 5)$, $p = 2$, $\lambda = \lambda_1$, $L' = D_{r+1}$
    (2) $X_1 \cong C_3(q)$, $p = 3$, $\lambda = \lambda_2$, $L' = A_5$
    (3) $X_1 \cong F_4(q)$, $p = 3$, $\lambda = \lambda_4$, $L' = E_6$.

Therefore, excluding these cases, the group QX_1 has just one Q-conjugacy class of complements to Q, and so X is equal to a Q-conjugate of X_1, as required. Finally, cases (1),(2),(3) above are dealt with exactly as in the proof of [LS2, 3.6], using 9.1.

Now let $X \cong A_1(q)$. First assume that $q = p$. By hypothesis, we have $q > 2t(\Sigma(G))$; and by 9.3, $t(\Sigma(G)) \geq 12$ if $G = G_2$, $t(\Sigma(G)) \geq 16$ otherwise; so $q = p > 24$ if $G = G_2$, and $q = p > 32$ otherwise. By 1.2, we have $H^1(X, V_X(r)) = 0$ when

$r < p - 3$. Now [LS2, 2.4], together with the first row of the table of [LS2, 2.13] shows that all X-composition factors within Q have high weights at most 27 (at most 4 if $G = G_2$). Hence $H^1(X, V) = 0$ for all such composition factors V, and we deduce as before that X is a Q-conjugate of X_1.

Finally, suppose $q = p^e$ with $e \geq 2$. By 1.2, if $H^1(X, V_X(r)) \neq 0$ then $V_X(r)$ must be a twist of the module $V_X(p-2) \otimes V_X(1)^{(p)}$. Now the proof of [LS2, 3.5] shows that such a module can appear as an X-composition factor within Q only if

$$p = 11 \text{ or } 17, \ L' = D_7 \text{ (and } G = E_8\text{)}.$$

The embedding of XQ/Q in D_7 is such that the action on the natural module is $10 \oplus 2^{(p)}$, and the composition factor $(p-2) \otimes 1^{(p)}$ occurs within the module Q/Q' with multiplicity 1; moreover, no nontrivial twist of this composition factor occurs within Q/Q'.

Suppose X does not lie in a Q-conjugate of L. By 1.3, the restriction map $H^1(A_1(K), (p-2) \otimes 1^{(p)}) \to H^1(X, (p-2) \otimes 1^{(p)})$ is injective. Hence there is a connected subgroup F of QL containing XQ' such that F/Q' is of type A_1. A similar analysis took place in [LS2, 3.7], except that there we were dealing with a connected subgroup of type A_1 instead of the finite group X. In [LS2, 3.7] it is shown that this connected A_1 subgroup lies in a Q-conjugate of L. This implies that the group F does not contain a closed complement to Q'. Now $Q' \downarrow X = 10 \oplus 2^{(p)}$. Since F contains X, it follows that the restriction map $H^2(A_1(K), 10 \oplus 2^{(p)}) \to H^2(X, 10 \oplus 2^{(p)})$ is not injective. However, this map is in fact injective: to see this, we verify the injectivity condition (5.4) of [CPSK] for $n = 2$, and the isomorphism condition (5.5) for $n = 1$, upon which the assertion follows from [CPSK, 5.1].

Therefore X lies in a Q-conjugate of L.

We have now proved Theorem 9 (except for the last sentence concerning σ-stability), in the case where $P = QL$ is minimal subject to containing X. The proof for non-minimal parabolics containing X follows as in [LS2, p. 43].

We finally deal with the σ-stability statement in the last sentence of Theorem 9. Suppose then that $X < G_\sigma$ and $X < P = QL$, a σ-stable parabolic. By what we have already proved, we can assume that $X < L$. Define

$$\Delta = \{L^u : u \in Q, X < L^u\}.$$

Observe first that if $X < L^u$ $(u \in Q)$, then $X^{u^{-1}} < L \cap QX = X$, whence $u \in N_Q(X) = C_Q(X)$. Consequently $C_Q(X)$ acts transitively on Δ. Write $C = C_Q(X)$.

We claim now that C is connected. To see this, let $T = Z(L)^0$. Then $X < C_G(T)$, so T normalizes C. Obviously T acts trivially on the finite group C/C^0, so if this group is nontrivial, then T fixes a nonzero vector in some composition factor of $Q \downarrow T$. However $C_Q(T) = 1$, so this is impossible. Hence C is connected, as claimed.

Thus the connected group C acts transitively on the σ-stable set Δ. It now follows from [SS, I,2.7] that Δ contains an element fixed by σ. In other words, X lies in a σ-stable Q-conjugate of L. This completes the proof of Theorem 9. $\qquad\square$

Proof of Theorem 10. As in the hypothesis of Theorem 10, assume that $X = X(q) < G$, with q as in Theorem 1 and $p > N'(X, G)$. We first aim to prove part (i) of Theorem 10.

By Corollary 5, there is a proper closed connected subgroup $\bar{X}$ of G containing X and fixing the same subspaces of $L(G)$ as X. Take $\bar{X}$ to be minimal subject to these conditions.

Lemma 9.4. $\bar{X}$ *is reductive.*

Proof. Suppose false, so $R_u(\bar{X}) \neq 1$. By [BT], $\bar{X}$ lies in a parabolic subgroup $P = QR$ of G (with unipotent radical Q and Levi subgroup R) such that $R_u(\bar{X}) \leq Q$. Then by Theorem 9, X lies in a Q-conjugate of R, so we may take it that $X < R$.

Now X fixes $L(R)$, and hence so does $\bar{X}$. But $N_{QR}(L(R))^0 = R$: for if not, $N_{QR}(L(R))^0 = Q_0 R$ with Q_0 a nontrivial connected subgroup of Q. Then $L(Q_0)$ commutes with $L(R)$. However, $C_{L(G)}(L(R)) \cap L(Q) = 0$, since if T is a maximal torus of R, then $C_{L(G)}(L(R)) \subseteq C_{L(G)}(L(T)) = L(T)$ by 1.11; this contradicts the previous sentence. Therefore $\bar{X} \leq N_{QR}(L(R))^0 = R$. But this means that $R_u(\bar{X}) \leq R$, which is absurd. □

By the minimality of $\bar{X}$ we have $\bar{X}' = \bar{X}$, and hence $\bar{X}$ is semisimple. Write $\bar{X} = \bar{X}_1 \ldots \bar{X}_t$, a commuting product of connected simple groups $\bar{X}_i$.

Lemma 9.5. *All the $\bar{X}_i$ are of the same type as X.*

Proof. Since X and $\bar{X}$ fix the same subspaces of $L(G)$, X must be irreducible on each $\bar{X}_i$-composition factor of $L(\bar{X}_i)$ (for all i). Hence by Theorem 4, for each i, either X and $\bar{X}_i$ are of the same type, or $(\bar{X}_i, X, p)$ is $(B_n$ or $C_n, D_n^\epsilon(q), 2)$ or $(B_3$ or $C_3, G_2(q), 2)$. In each of these exceptional cases, the projection of X in $\bar{X}_i$ lies in a connected simple subgroup of $\bar{X}_i$ of the same type as X (as this projection is determined up to conjugacy in $\bar{X}_i$). Hence the result follows by the minimality of $\bar{X}$. □

We now have

$$X < \bar{X}_1 \ldots \bar{X}_t$$

with each $\bar{X}_i$ connected and simple of the same type as X.

We now argue that $X < \tilde{X} \leq \bar{X}_1 \ldots \bar{X}_t$, where $\tilde{X}$ is connected and simple of the same type as X. This is clear if $t = 1$, so suppose $t > 1$. Let $\phi_i : X \to \bar{X}_i/Z(\bar{X}_i)$ be the ith projection map.

Let Y be a simple algebraic group of the same type as X, and containing X. If Y is classical, then [ST1] implies that each ϕ_i extends to a morphism $\hat{\phi}_i : Y \to \bar{X}_i/Z(\bar{X}_i)$; and if not, then as $t > 1$, we have $X = G_2(q)$, and the same conclusion follows from [LS3, 5.1]. Now define

$$\hat{X} = \{y\hat{\phi}_1 \ldots y\hat{\phi}_t : y \in Y\},$$

and take $\tilde{X}$ to be the connected preimage of $\hat{X}$ in $X < \bar{X}_1 \ldots \bar{X}_t$. Then $\tilde{X}$ is a closed, connected, simple subgroup of G of the same type as X, and containing X. Moreover $\tilde{X} \leq \bar{X}$, so by minimality, $\tilde{X} = \bar{X}$. This completes the proof of the first part of Theorem 10(i).

For the last part of Theorem 10(i), we must show that $C_G(X)^0 = C_G(\tilde{X})^0$. This is obvious if $C_G(X)^0 = 1$, so assume $C_G(X)^0 \neq 1$.

We now argue that $C_G(X)^0$ is reductive. The proof of this follows closely that of [LS2, Theorem 4.1]. Let $C = C_G(X)^0$, and suppose $U = R_u(C) \neq 1$. Then XC lies in a parabolic subgroup $P = QL$, with $U \leq Q$. Certainly X fixes $L(P)$, hence so does $\tilde{X}$, and therefore $\tilde{X} < P$. Then by Theorem 9, we may assume

that $\tilde{X} \leq L$. Now Lemmas 4.2–4.4 of [LS2] show that there exists $l \in L$ such that $\bar{w}_0 l$ normalizes $\tilde{X}$ (where $\bar{w}_0$ is an automorphism of G normalizing a maximal torus of L and inducing -1 on $\Sigma(G)$). Using [LS3, 5.1], we may take it that $\bar{w}_0 l$ also normalizes X. But now we have $1 \neq |C_Q(X)| = |C_{Q^{\bar{w}_0 l}}(X^{\bar{w}_0 l})| = |C_{Q^-}(X)|$, where Q^- is the unipotent radical of the parabolic opposite to P. This gives $1 \neq C_{Q^-}(X) \leq Q^- \cap P = 1$, a contradiction.

Thus $C_G(X)^0$ is reductive. Let T be a torus in $C_G(X)^0$. Then $R = C_G(T)$ is a Levi subgroup of G containing X. Now $R = P \cap P^-$, the intersection of a parabolic P and its opposite P^-. Hence X fixes $L(P)$ and $L(P^-)$, and therefore so does $\tilde{X}$. Since $N_G(L(P)) = P$ and $N_G(L(P^-)) = P^-$, we deduce that $\tilde{X} \leq P \cap P^- = R$. Thus $\tilde{X}$ centralizes T. The reductive group $C_G(X)^0$ is generated by tori, so we conclude that $C_G(X)^0 = C_G(\tilde{X})^0$, as required. This completes the proof of Theorem 10(i).

Now we prove part (ii) of Theorem 10. Suppose that $X = X(q) < G$, with q as in Theorem 1. By Theorem 1, we can choose a σ-stable connected subgroup $\bar{X}$ of G, minimal subject to containing X and fixing the same subspaces as X of every G-composition factor of $L(G)$.

We claim that $\bar{X}$ is reductive. Suppose false, so that by [BT], $\bar{X}$ lies in a σ-stable parabolic $P = QR$ of G, with $R_u(\bar{X}) \leq Q$; then by Theorem 9, X lies in a σ-stable Q-conjugate of R. If $L(G)$ is G-irreducible, the proof of Lemma 9.4 shows that $R_u(\bar{X}) \leq R$, a contradiction. Otherwise, by 1.10, we have $(G, p) = (E_7, 2)$, $(E_6, 3)$, $(F_4, 2)$ or $(G_2, 3)$. In the first two cases we may assume that G is simply connected, so that G is irreducible on $L(G)/Z(L(G))$ and $\bar{X}$ fixes each X-invariant subspace of $L(G)/Z(L(G))$. As $L(R) > L(T) > Z(L(G))$ (where T is a maximal torus of R), we conclude that X fixes $L(R)$, and at this point the proof of 9.4 works as before to give the claim. For the F_4 and G_2 cases, by the prime restrictions $p > N'(X, G)$, the only possibility is that $(G, p, X) = (F_4, 2, A_3^\epsilon(q))$. Then $QL = QB_3$ or QC_3. Let I be the 26-dimensional ideal of $L(G)$ generated by short root elements. We may take it that $X < L$; hence X fixes $L(B_3) \cap I$ or $L(C_3) \cap I$. Therefore $\bar{X}$ lies in the stabilizer of one of these subspaces, hence lies in a subgroup $B_3 \tilde{A}_1$ or $C_3 A_1$ of F_4. Moreover, X, hence $\bar{X}$, fixes every subspace of $L(\tilde{A}_1)$ or $L(A_1)$, from which it follows that $\bar{X} < B_3 T_1$ or $C_3 T_1$, whence $\bar{X} < L$.

Thus $\bar{X}$ is reductive. By minimality, $\bar{X}$ is semisimple, say $\bar{X} = \bar{X}_1 \ldots \bar{X}_t$ with each $\bar{X}_i$ simple. We claim also that each $\bar{X}_i$ is of the same type as X. As in the proof of Lemma 9.5, this follows from Theorem 4, unless for some i, $(\bar{X}_i, X, p)$ is $(B_n$ or $C_n, D_n^\epsilon(q), 2)$ or $(B_3$ or $C_3, G_2(q), 2)$. Assume then that one of the latter two cases holds. The fact that $p > N'(X, G)$ implies that the first case holds with $t = 1$, $n \geq 4$. But then X lies in a σ-stable subgroup D_n of $\bar{X}$, contrary to the minimality of $\bar{X}$. This proves our claim.

To finish the proof of Theorem 10(ii), we need to show that $C_G(X)^0 = C_G(\bar{X})^0$. We showed for part (i) of Theorem 10 that $C_G(X)^0 = C_G(\tilde{X})^0$, where $\tilde{X}$ is a suitable simple connected diagonal subgroup of $\bar{X}$ of type X. Define $CC^0(X) = C_G(C_G(X)^0)^0$. This group contains $\tilde{X}$ and is σ-stable. If

$$Y = \langle \tilde{X}^{\sigma^i} : \text{ all } i \rangle,$$

then Y lies in $\bar{X}$ and is σ-stable; hence $Y = \bar{X}$ by the minimality of $\bar{X}$, and consequently $CC^0(X)$ contains $\bar{X}$. It follows that $C_G(X)^0 \leq C_G(\bar{X})^0$. The reverse inclusion is trivial, so $C_G(X)^0 = C_G(\bar{X})^0$, as required.

10. Proof of Theorem 11

In this final section we prove Theorem 11. Let C be a finite classical group in characteristic p with usual module V, and suppose X, Y are simple groups of Lie type in characteristic p, not of the same type, such that $X < Y < C$ and X is absolutely irreducible on V. Suppose also that if Y is of exceptional type, then $X = X(q)$ with q as in the hypothesis of Theorem 11.

First observe that by [ST1], the embedding $Y < C$ lifts to an embedding $\bar{Y} \le \bar{C}$ of corresponding simple algebraic groups. Let $\bar{V}$ be the usual module for $\bar{C}$. If $\bar{Y}$ is classical, then again by [ST1], $X < Y$ lifts to an embedding $\bar{X} < \bar{Y}$, giving the required conclusion.

Thus we assume that $\bar{Y}$ is exceptional. By Theorem 1, $X < Z \le \bar{Y}$ for some closed connected subgroup Z of $\bar{Y}$ fixing the same subspaces of $L(\bar{Y})$ as X. By Theorem 4, X is reducible on some $\bar{Y}$-composition factor of $L(\bar{Y})$, and so $Z < \bar{Y}$. Now X, and hence Z, is irreducible on $\bar{V}$, so Z is semisimple.

At this point we have

$$X < Z < \bar{Y} < \bar{C}.$$

Now we apply [Se1, Theorem 1]; this says that the triple $(Z, \bar{Y}, \bar{C})$ is one of those in Table 1 of [Se1]. Inspection of that table (recalling that $\bar{Y}$ is exceptional) shows that Z must be simple. If Z is classical, then [ST1] again implies that $X < \bar{X} \le Z$ for some simple connected group $\bar{X}$ of the same type as X, as required. If Z is exceptional and of the same type as X, we take $\bar{X} = Z$. Finally, if Z is exceptional and not of the same type as X, we replace $\bar{Y}$ with Z and repeat the argument; eventually we obtain a suitable group $\bar{X}$. This completes the proof of Theorem 11.

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