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Research Article

On the Sum of Reciprocal Generalized Fibonacci Numbers

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We consider infinite sums derived from the reciprocals of the generalized Fibonacci numbers. We obtain some new and interesting identities for the generalized Fibonacci numbers.

1. Introduction

For any integer $n \geq 0$, the famous Fibonacci numbers F_n and Pell numbers are defined by the second-order linear recurrence sequences

$$\begin{gathered} F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0,\ F_1 = 1, \\ P_{n+2} = 2P_{n+1} + P_n, \quad P_0 = 0,\ P_1 = 1. \end{gathered} \tag{1}$$

There are many interesting results on the properties of these two sequences; see [1–9]. In 2009, Ohtsuka and Nakamura [5] studied the properties of the Fibonacci numbers and proved the following two interesting identities:

$$\begin{gathered} \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \\ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-1}F_n, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \end{gathered} \tag{2}$$

where $\lfloor x \rfloor$ is the floor function; that is, it denotes the greatest integer less than or equal to x. Recently, Holliday and Komatsu [1] (Theorems 3 and 4) and Xu and Wang [7] proved the following interesting identities for the Pell numbers:

$$\begin{gathered} \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_{n-1} + P_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ P_{n-1} + P_{n-2} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \\ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k^2} \right)^{-1} \right\rfloor = \begin{cases} 2P_{n-1} + P_n - 1, & \text{if } n \text{ is even and } n \geq 2; \\ 2P_{n-1} + P_n, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \\ \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k^3} \right)^{-1} \right\rfloor = \begin{cases} P_n^2 P_{n-1} + 3P_n P_{n-1}^2 + \left\lfloor -\dfrac{61}{82} P_n - \dfrac{91}{82} P_{n-1} \right\rfloor, & \\ \qquad \text{if } n \text{ is even and } n \geq 2; & \\ P_n^2 P_{n-1} + 3P_n P_{n-1}^2 + \left\lfloor \dfrac{61}{82} P_n + \dfrac{91}{82} P_{n-1} \right\rfloor, & \\ \qquad \text{if } n \text{ is odd and } n \geq 1, & \end{cases} \end{gathered} \tag{3}$$

where providing $P_{-1} = P_1 = 1$. In [7, 8], the authors asked whether there exists a computational formula for

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k^s} \right)^{-1} \right\rfloor, \tag{4}$$

where $s \geq 4$ is a positive integer.

Let p and q be integers such that $p^2 + 4q > 0$. Define the generalized Fibonacci sequence $\{U_n(p,q)\}$, briefly $\{U_n\}$, as shown: for $n \geq 2$

$$U_n = pU_{n-1} + qU_{n-2}, \tag{5}$$

where $U_0 = 0, U_1 = 1$. The Binet formula for $\{U_n\}$ is

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{6}$$

where $\alpha, \beta = (p \pm \sqrt{p^2 + 4q})/2$.

The main purpose of this paper related to the computing problem of

$$U(s,n) = \left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k^s} \right)^{-1} \right\rfloor \tag{7}$$

for $s = 3$ and $q = -1$. For easy computation, we assume that $p = a$ is a positive integer and $q = -1$ throughout the paper. We have the following.

Theorem 1. *Let $a \geq 3$ be a positive integer, and let G_n be defined by the second-order linear recurrence sequence $G_{n+2} = aG_{n+1} - G_n, G_0 = 0, G_1 = 1$. Then for all $n \geq 2$ one has*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k^3} \right)^{-1} \right\rfloor = \begin{cases} G_n^3 - G_{n-1}^3 - 3 \displaystyle\sum_{k=0}^{\lfloor (n-4)/5 \rfloor} G_{n-3-5k} - 2, & a = 3,\ n \equiv 3 \pmod 5; \\ G_n^3 - G_{n-1}^3 - 3 \displaystyle\sum_{k=0}^{\lfloor (n-4)/5 \rfloor} G_{n-3-5k} - 1, & \textit{otherwise}. \end{cases} \tag{8}$$

2. Proof of the Main Result

In this section, we will prove our main result. We consider the case that $\alpha\beta = 1$ and $s = 3$.

Proof. From the Taylor series expansion of $(1-\varepsilon)^{-3}$ as $\varepsilon \to 0$, we have

$$\begin{aligned}(1-\varepsilon)^{-3} &= 1 + \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2} \varepsilon^n \\ &= 1 + 3\varepsilon + 6\varepsilon^2 + O(\varepsilon^3).\end{aligned} \tag{9}$$

Using (6), we have

$$\begin{aligned}\frac{1}{G_k^3} &= \frac{(\alpha-\beta)^3}{\alpha^{3k}} \left(1 - \frac{1}{\alpha^{2k}}\right)^{-3} \\ &= \frac{(\alpha-\beta)^3}{\alpha^{3k}} \frac{1}{(1 - 3/\alpha^{2k} + 3/\alpha^{4k} + 1/\alpha^{6k})} \\ &= \frac{(\alpha-\beta)^3}{\alpha^{3k}} \left[1 + \frac{3}{\alpha^{2k}} + \frac{6}{\alpha^{4k}} + \frac{10\alpha^{4k} - 15\alpha^{2k} + 6}{\alpha^{4k}(\alpha^{2k}-1)^3} \right] \\ &= (\alpha-\beta)^3 \left[\frac{1}{\alpha^{3k}} + \frac{3}{\alpha^{5k}} + \frac{6}{\alpha^{7k}} + \frac{10\alpha^{4k} - 15\alpha^{2k} + 6}{\alpha^{7k}(\alpha^{2k}-1)^3} \right].\end{aligned} \tag{10}$$

It is easy to check that

$$\frac{10}{\alpha^{9k}} < \frac{10\alpha^{4k} - 15\alpha^{2k} + 6}{\alpha^{7k}(\alpha^{2k}-1)^3} < \frac{11}{\alpha^{9k}} \tag{11}$$

holds for $a \geq 3$ and $k \geq 2$.

Thus

$$\begin{aligned}\sum_{k=n}^{\infty} \frac{1}{G_k^3} &= (\alpha-\beta)^3 \\ &\quad \times \left[\frac{1}{\alpha^{3n}} \cdot \frac{\alpha^3}{\alpha^3 - 1} + \frac{3}{\alpha^{5n}} \cdot \frac{\alpha^5}{\alpha^5 - 1} + \frac{6}{\alpha^{7n}} \cdot \frac{\alpha^7}{\alpha^7 - 1} \right. \\ &\quad \left. + \sum_{k=n}^{\infty} \frac{10\alpha^{4k} - 15\alpha^{2k} + 6}{\alpha^{7k}(\alpha^{2k}-1)^3} \right] \\ &= \frac{(\alpha-\beta)^3 \alpha^3}{\alpha^{3n}(\alpha^3 - 1)} \\ &\quad \times \left[1 + \frac{3}{\alpha^{2n}} \frac{\alpha^2(\alpha^3-1)}{\alpha^5 - 1} + \frac{6\alpha^4}{\alpha^{4n}} \frac{(\alpha^3 - 1)}{\alpha^7 - 1} + R_n \right],\end{aligned} \tag{12}$$

where

$$R_n = \frac{(\alpha^3 - 1)\alpha^{3n}}{\alpha^3} \sum_{k=n}^{\infty} \frac{10\alpha^{4k} - 15\alpha^{2k} + 6}{\alpha^{7k}(\alpha^{2k}-1)^3}. \tag{13}$$

Since $\sum_{k=n}^{\infty}(1/\alpha^{9k}) = \alpha^9/\alpha^{9n}(\alpha^9 - 1)$, we have

$$\frac{10\alpha^6}{\alpha^{6n}(\alpha^6 + \alpha^3 + 1)} < R_n < \frac{11\alpha^6}{\alpha^{6n}(\alpha^6 + \alpha^3 + 1)} \tag{14}$$

holds for $a \geq 3$ and $k \geq 2$.

Taking reciprocal, we get

$$\left(\sum_{k=n}^{\infty}\frac{1}{G_k^3}\right)^{-1} = \frac{(\alpha^3-1)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3}\left(1\times\left(1+\frac{3}{\alpha^{2n}}\frac{\alpha^2(\alpha^3-1)}{\alpha^5-1} + \frac{6\alpha^4}{\alpha^{4n}}\frac{(\alpha^3-1)}{\alpha^7-1}+R_n\right)^{-1}\right)$$
$$< \frac{(\alpha^3-1)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3}\left(1\times\left(1+\frac{3}{\alpha^{2n}}\frac{\alpha^2(\alpha^3-1)}{\alpha^5-1}+\frac{6\alpha^4}{\alpha^{4n}}\frac{(\alpha^3-1)}{\alpha^7-1} + \frac{10\alpha^6}{\alpha^{6n}(\alpha^6+\alpha^3+1)}\right)^{-1}\right)$$
$$< \frac{(\alpha^3-1)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3} - \frac{3(\alpha^3-1)^2\alpha^n}{(\alpha-\beta)^3\alpha(\alpha^5-1)}+\delta_1, \tag{15}$$

where

$$\delta_1 = -\frac{6\alpha(\alpha^3-1)^2}{\alpha^n(\alpha-\beta)^3(\alpha^7-1)} - \frac{10\alpha^3(\alpha^3-1)}{\alpha^{3n}(\alpha-\beta)^3(\alpha^6+\alpha^3+1)} + \frac{9\alpha(\alpha^3-1)^3}{\alpha^n(\alpha-\beta)^3(\alpha^5-1)^2} + \frac{36\alpha^3(\alpha^3-1)^3}{\alpha^{3n}(\alpha-\beta)^3(\alpha^5-1)(\alpha^7-1)} + \frac{36\alpha^5(\alpha^3-1)^3}{\alpha^{5n}(\alpha-\beta)^3(\alpha^7-1)^2} + \frac{60\alpha^5(\alpha^3-1)^2}{\alpha^{5n}(\alpha-\beta)^3(\alpha^5-1)(\alpha^6+\alpha^3+1)} + \frac{120\alpha^7(\alpha^3-1)^2}{\alpha^{7n}(\alpha-\beta)^3(\alpha^7-1)(\alpha^6+\alpha^3+1)} + \frac{100\alpha^9(\alpha^3-1)}{\alpha^{9n}(\alpha-\beta)^3(\alpha^6+\alpha^3+1)^2} \tag{16}$$

since

$$\frac{1}{1+\varepsilon} = 1-\varepsilon+\varepsilon^2-\frac{\varepsilon^3}{1+\varepsilon}. \tag{17}$$

An easy calculation shows that $\delta_1 \le 4/\alpha^{n+3}$ holds for $a \ge 3$ and $k \ge 2$. Therefore,

$$\left(\sum_{k=n}^{\infty}\frac{1}{G_k^3}\right)^{-1} < \frac{(\alpha^3-1)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3} - \frac{3(\alpha^3-1)^2\alpha^n}{(\alpha-\beta)^3\alpha(\alpha^5-1)}+\delta_1$$
$$\le \frac{\alpha^{3n}-\alpha^{3n-3}}{(\alpha-\beta)^3} - \frac{3(\alpha^3-1)^2\alpha^n}{(\alpha-\beta)^3\alpha(\alpha^5-1)} + \frac{4}{\alpha^{n+3}}$$
$$= G_n^3 - G_{n-1}^3 - \frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha(\alpha^5-1)} + \frac{3(\alpha-1)}{(\alpha-\beta)^3\alpha^n} - \frac{\alpha^3-1}{(\alpha-\beta)^3\alpha^{3n}} + \frac{4}{\alpha^{n+3}}$$
$$< G_n^3 - G_{n-1}^3 - \frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha(\alpha^5-1)} + \frac{3(\alpha-1)}{(\alpha-\beta)^3\alpha^n} + \frac{4}{\alpha^{n+3}}$$
$$= G_n^3 - G_{n-1}^3 - \frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha(\alpha^5-1)} + \lambda_1, \tag{18}$$

where

$$\lambda_1 = \frac{3(\alpha-1)}{(\alpha-\beta)^3\alpha^n} + \frac{4}{\alpha^{n+3}} < 0.1681 \tag{19}$$

for $a \ge 3$ and $n \ge 2$.

Similarly, we have

$$\left(\sum_{k=n}^{\infty}\frac{1}{G_k^3}\right)^{-1} = \frac{(\alpha^3-1)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3}\left(1\times\left(1+\frac{3}{\alpha^{2n}}\frac{\alpha^2(\alpha^3-1)}{\alpha^5-1} + \frac{6\alpha^4}{\alpha^{4n}}\frac{(\alpha^3-1)}{\alpha^7-1}+R_n\right)^{-1}\right)$$
$$> \frac{(\alpha^3-1)\alpha^{3n}}{(\alpha-\beta)^3\alpha^3}\left(1\times\left(1+\frac{3}{\alpha^{2n}}\frac{\alpha^2(\alpha^3-1)}{\alpha^5-1}+\frac{6\alpha^4}{\alpha^{4n}}\frac{(\alpha^3-1)}{\alpha^7-1} + \frac{11\alpha^6}{\alpha^{6n}(\alpha^6+\alpha^3+1)}\right)^{-1}\right)$$
$$> G_n^3 - G_{n-1}^3 - \frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha(\alpha^5-1)} + \lambda_2. \tag{20}$$

Since

$$\frac{1}{1+\varepsilon} = 1-\varepsilon+\varepsilon^2-\varepsilon^3+\frac{\varepsilon^4}{1+\varepsilon}, \tag{21}$$

and $\varepsilon = (3/\alpha^{2n})(\alpha^2(\alpha^3-1)/(\alpha^5-1)) + (6\alpha^4/\alpha^{4n})((\alpha^3-1)/(\alpha^7-1)) + 11\alpha^6/\alpha^{6n}(\alpha^6+\alpha^3+1) < 0.3$ for $a \geq 3$ and $n \geq 2$, we have $\varepsilon^2 - \varepsilon^3 > 0.7\varepsilon^2$, whence we can take

$$\lambda_2 = \frac{3(\alpha-1)}{(\alpha-\beta)^3\alpha^n} - \frac{\alpha^3-1}{(\alpha-\beta)^3\alpha^{3n}} - \frac{6\alpha(\alpha^3-1)^2}{\alpha^n(\alpha-\beta)^3(\alpha^7-1)} - \frac{11\alpha^3(\alpha^3-1)}{\alpha^{3n}(\alpha-\beta)^3(\alpha^6+\alpha^3+1)} + \frac{6.3\alpha(\alpha^3-1)^3}{\alpha^n(\alpha-\beta)^3(\alpha^5-1)^2} > 0 \quad (22)$$

for $a \geq 3$ and $n \geq 2$.

Consequently, we have shown that

$$G_n^3 - G_{n-1}^3 - \frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha(\alpha^5-1)} + \lambda_2 < \left(\sum_{k=n}^{\infty}\frac{1}{G_k^3}\right)^{-1} < G_n^3 - G_{n-1}^3 - \frac{3\alpha^{n+2}}{(\alpha-\beta)\alpha(\alpha^5-1)} + \lambda_1, \quad (23)$$

where $0 < \lambda_2 < \lambda_1 < 0.1681$ for $a \geq 3$ and $n \geq 2$, and $\lambda_1 < 0.0053$ for $a \geq 4$ and $n \geq 3$.

Now the calculations show that

$$\frac{\alpha^{n+2}}{(\alpha-\beta)\alpha(\alpha^5-1)} = \begin{cases} G_{n-3} + G_{n-8} + \cdots + G_7 + G_2 - \dfrac{\alpha^2+\alpha^3}{(\alpha-\beta)(\alpha^5-1)}, & n \equiv 0 \pmod 5; \\ G_{n-3} + G_{n-8} + \cdots + G_8 + G_3 - \dfrac{\alpha^2+\alpha^3}{(\alpha-\beta)(\alpha^5-1)}, & n \equiv 1 \pmod 5; \\ G_{n-3} + G_{n-8} + \cdots + G_9 + G_4 - \dfrac{\alpha+\alpha^4}{(\alpha-\beta)(\alpha^5-1)}, & n \equiv 2 \pmod 5; \\ G_{n-3} + G_{n-8} + \cdots + G_{10} + G_5 - \dfrac{1+\alpha^5}{(\alpha-\beta)(\alpha^5-1)}, & n \equiv 3 \pmod 5; \\ G_{n-3} + G_{n-8} + \cdots + G_6 + G_1 - \dfrac{\alpha+\alpha^4}{(\alpha-\beta)(\alpha^5-1)}, & n \equiv 4 \pmod 5. \end{cases} \quad (24)$$

The calculations also show that $3(\alpha^2+\alpha^3)/(\alpha-\beta)(\alpha^5-1) > \lambda_1$ for $a \geq 3$ and $n \geq 2$; $3(\alpha+\alpha^4)/(\alpha-\beta)(\alpha^5-1) > \lambda_1$ for $a \geq 3$ and $n \geq 2$; and $3(1+\alpha^5)/(\alpha-\beta)(\alpha^5-1) > \lambda_1 + 1$ for $a = 3$ and $n \geq 3$; $0.87 < 3(1+\alpha^5)/(\alpha-\beta)(\alpha^5-1) < 1$ for $a > 3$ and $n \geq 3$. Combining the calculations and (23), we obtain

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{G_k^3}\right)^{-1} \right\rfloor = \begin{cases} G_n^3 - G_{n-1}^3 - 3\displaystyle\sum_{k=0}^{\lfloor (n-4)/5 \rfloor} G_{n-3-5k} - 2, & a = 3,\ n \equiv 3 \pmod 5; \\ G_n^3 - G_{n-1}^3 - 3\displaystyle\sum_{k=0}^{\lfloor (n-4)/5 \rfloor} G_{n-3-5k} - 1, & \text{otherwise.} \end{cases} \quad (25)$$

Therefore we have proved Theorem 1. □

Remark 2. We can also compute the cases $s > 3$ or $q = 1$; however, the computations are much more complicated. So we stop here.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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