ON THE SUM OF TWO BOREL SETS

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ABSTRACT. It is shown that the linear sum of two Borel subsets of the real line need not be Borel, even if one of them is compact and the other is G_δ. This result is extended to a fairly wide class of connected topological groups.

1. **Introduction.** If C and D are Borel subsets of the real line R, need $C+D$ be Borel?[2] Here $C+D$ denotes the set $\{x+y \mid x\in C, y\in D\}$. In the simplest cases the answer is obviously "yes"; for example if at least one of C, D is countable or open, or if both are F_σ sets. We shall show that in the next simplest case, in which C is compact and D is G_δ, the answer is "no"; $C+D$ need not be Borel.[3] (It will, of course, be analytic; in fact the sum of two analytic sets is analytic, being a continuous image of their product.)

The answer to the corresponding question about the plane (with $+$ denoting vector sum) has been known for some time, though it does not appear to be in the literature. The present construction imitates the plane counterexample in the space $A\times B$, where A, B are suitable additive subgroups of R, and then transfers it to $A+B\subset R$. The axiom of choice is not required.

2. **The subgroups.** As was shown by von Neumann [**3**], if we put (1) $f(x)=\sum_{n=1}^{\infty} p(p([nx]))/p(p(n^2))$, where $p(a)=2^a$, then the numbers $f(x)$, $x>0$, are algebraically independent. Clearly f is strictly increasing, and is continuous at each irrational x; hence, if P^+ denotes the set of positive irrationals, $f(P^+)$ is homeomorphic to P^+ and therefore contains a Cantor set K.[4] In turn, K clearly contains two (in fact, c) disjoint Cantor sets K_1, K_2. We let A, B denote the additive subgroups of R generated by K_1, K_2 respectively. Thus

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[2] We are indebted to Mr. B. V. Rao for calling our attention to this problem.

[3] A closely related result has been obtained independently, by a different method, by C. A. Rogers [**4**].

[4] By "Cantor set" we mean any space homeomorphic to the usual Cantor ternary set; that is, a compact, zero-dimensional dense-in-itself metric space. In particular, the Cantor subsets of R are just the nonempty bounded perfect nowhere dense sets.

(2) A and B are σ-compact and contain Cantor sets, and

$$A \cap B = \{0\}.$$

3. **The sets.**

THEOREM. *There exist a Cantor set $C \subset R$, and a G_δ subset D of R, such that $C+D$ is not Borel.*

PROOF. The subgroup A contains K_1 which contains a homeomorph P_1 of the space of irrational numbers. Take a non-Borel analytic subset E of the Cantor set K_2 (cf. [**1**, p. 368]). There is a continuous map g of P_1 onto E; let G be its graph, a subset of $P_1 \times K_2 \subset A \times B$. As in [**1**, pp. 366, 367], G is closed in $P_1 \times B$; and P_1 is an absolute G_δ. Thus G is G_δ in $A \times B$, and therefore

(3) $$(A \times B) \backslash G \text{ is } \sigma\text{-compact.}$$

Let $F = A \times \{0\}$. Note that $F+G$ (where $+$ here refers to the group operation in the direct product $A \times B$) is not Borel in $A \times B$, because its intersection with $\{0\} \times B$ is the non-Borel set $\pi_2(G) = E$.

Now consider the homomorphism $\phi: A \times B \to R$ given by $\phi(a, b) = a+b$. Clearly ϕ is continuous and (by choice of A and B) one-to-one. We note that $\phi(F+G)$ is not Borel in R, since otherwise the continuity of ϕ would show that $\phi^{-1}(\phi(F+G))$ would be Borel in $A \times B$; but this set is $F+G$. Thus

(4) $\phi(F)+\phi(G)$ is not Borel in R.

We have, however,

(5) $\phi(F) = A = \bigcup_{m=1}^{\infty} A_m$ where each A_m is a Cantor set.

For we may take A_m = set of all numbers of the form $a_1+a_2+ \cdots +a_m$ where $\pm a_i \in K_1$ $(i=1, 2, \cdots, m)$. This is a Cantor set because it is clearly compact and perfect, and also nowhere dense (since otherwise $A=R$, contradicting (4)).

Again, $\phi(G)$ is G_δ in $A+B$, for (since ϕ is 1-1) its complement $(A+B)\backslash\phi(G)$ is the image under ϕ of $(A \times B)\backslash G$, and is therefore σ-compact, by (3). But $A+B$ is F_σ in R; hence $\phi(G)$ is $G_{\delta\sigma}$ in R, and we may write $\phi(G) = \bigcup_{n=1}^{\infty} G_n$ where each G_n is a G_δ in R. Now (4) and (5) show that $\bigcup_{m,n} (A_m+G_n)$ is non-Borel; hence there exist m, n such that A_m+G_n is non-Borel, and we merely take $C=A_m$, $D=G_n$.

4. **Remarks.**

Mr. Rao has called to our attention that, starting from the above theorem, L. A. Rubel's method [**5**] will produce pathological Borel measurable functions on the real line. For instance, if $\phi(x) = \sup_{-\infty<t<\infty} |f(x+t)-f(x-t)|$, then the Borel measurability of f does not imply that of ϕ.

It may also be worth remarking that not every analytic subset of R is expressible as the sum of two (or more) Borel sets. For example, if H is an arbitrary non-Borel analytic subset of $[0, 1]$, and $L = H \cup \{3\}$, then L is not expressible in the form $X+Y$ for any nondegenerate sets X, Y. For otherwise it is easy to see that, for some $\lambda \neq 0$, $L \cap (L+\lambda)$ contains a translate of X (take $\lambda = y_1 - y_2$ where y_1, $y_2 \in Y$), and thus that diam $X < 1$. Similarly diam $Y < 1$ and so diam $(X+Y) < 2$, contradicting $X+Y=L$.

5. **More general groups.** Mycielski [2] has generalized von Neumann's construction, showing in particular that every connected topological group with a complete metric, which is either locally compact or abelian, contains an independent Cantor subset. The foregoing arguments apply virtually unchanged[5] to show that every such group (written additively) contains two Borel sets (in fact a compact set and a G_δ) whose sum is not Borel. It would be interesting to know whether this remains true if "connected" is weakened to "nondiscrete".

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[5] In the nonabelian case, $A+B$ need not be a group, and ϕ need not be a homomorphism; however, we still have $\phi(F+G) = \phi(F) + \phi(G)$ because of the special nature of F.