ON THE SUMMATION OF MULTIPLE FOURIER SERIES[1)]

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1. Generalities. Let $f(x_1, \ldots\ldots, x_k) = f(x)$ be a real valued integrable function periodic with period 2π in $0 \leqq x_i \leqq 2\pi$, $i = 1, 2, \ldots\ldots k$. Following S. Bochner [1] and K.Chandrasekharan [2], we define the *'spherical means'* $f(x, t)$ of a function $f(x)$ at a point $x = (x_1 \ldots\ldots, x_k)$, for $t > 0$,

$$(1.\ 1) \qquad f(x, t) = \frac{\Gamma(k/2)}{2(\pi)^{k/2}} \int_\sigma f(x_1 + t\xi_1, \ldots\ldots, x_k + t\xi_k)\, d\sigma_\xi,$$

where σ is the sphere $\xi_1^2 + \ldots\ldots + \xi_k^2 = 1$ and $d\sigma_\xi$ is its $(k-1)$ – dimentional volume element. $f(x, t)$ considered as a function of the single variable t exists for almost all t, and integrable in every finite t-interval.

If $p > 0$, we define

$$(1.\ 2) \qquad f_p(x, t) = \frac{2}{B(p, k/2)\ t^{2p+k-2}} \int_0^t (t^2 - s^2)^{p-1} s^{k-1} f(x, s)\, ds,$$

which called the spherical mean of order p of the function $f(x)$. At a point x, we write $f_p(x, t) = f_p(t)$ for $p \geqq 0$, where we assume that $f_0(x, t) = f(x, t)$. The following properties of $f_p(t)$ are known [2].

$$(1.\ 3) \qquad \int_0^u t^{k-1} |f(x, t)|\, dt = O(u^k), \qquad \text{as } u \to \infty.$$

$$(1.\ 4) \qquad \int_0^u t^{k-1} |f(x, t)|\, dt = o(1), \qquad \text{as } u \to 0.$$

$$(1.\ 5) \qquad f_p(u) = O(1), \quad \text{for } p \geqq 1, \qquad \text{as } u \to \infty.$$

Further, if we define, for $p \geqq 0$ [2],

$$(1.\ 6) \qquad \varphi_p(t) = t^{2p+k-2} f_p(t)\ B(p, k/2)/2^p\Gamma(p).$$

then we have, for $p + q \geqq 1$,

$$(1.\ 7) \qquad \varphi_{p+q}(t) = \frac{1}{2^{q-1}\Gamma(q)} \int_0^t (t^2 - s^2)^{q-1} s\, \varphi_p(s)\, ds.$$

It is clear for (1. 7) that if $p \geqq 1$ then $\varphi_p(t)$ is absolutely continuous in every finite interval excluding the origin.

Next, let us write the Fourier series of $f(x)$ in the form,

1) The problem considered here was suggested by Professor G. Sunouchi.

(1. 8) $$f(x) \sim \sum a_{n_1 \ldots n_k} e^{i(n_1 x_1 + \ldots + n_k x_k)}$$

where

$$a_{n_1 \ldots n_k} = \frac{1}{(2\pi)^k} \int_{-\pi}^{\pi} \ldots\ldots \int_{-\pi}^{\pi} f(x)\, e^{i(n_1 x_1 + \ldots + n_k x_k)}\, dx_1 \ldots\ldots dx_k.$$

Define, for $\delta \geqq 0$,

(1. 9) $$S_R^\delta(x) = \sum_{n \leqq R^2} \left(1 - \frac{n}{R^2}\right)^\delta a_{n_1 \ldots n_k}\, e^{i(n_1 x_1 + \ldots + n_k x_k)}$$

where

$$n = n_1^2 + \ldots\ldots + n_k^2.$$

At a fixed point x, we may write $S_R^\delta(x) = S^\delta(R)$; $S^\delta(R)$ is the Riesz mean of order δ of the series (1. 8), when summed 'spherically'. If we write

(1. 10) $$A_n = \sum_{n = n_1^2 + \ldots + n_k^2} a_{n_1 \ldots n_k}\, e^{i(n_1 x_1 + \ldots + n_k x_k)}$$

with the convention that $A_n(x) \equiv 0$ if n cannot be represented as the sum of k squeres,

$$S^\delta(R) = \sum_{n \leqq R^2 < n+1} \left(1 - \frac{n}{R^2}\right)^\delta A_n$$

We write $S^\delta(R) = T^\delta(R)\, R^{-2\delta}$ so that $S^0(R) = S(R) = T^0(R) = T(R)$. We have the analogue of (1.7)

(1. 11) $$T^{p+q}(R) = \frac{2\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q)} \int_0^R (R^2 - t^2)^{q-1}\, t\, T^p(t)\, dt.$$

If $J_\mu(t)$ denote the Bessel function of order μ, it is well-known that [10]

(1. 12) $$\frac{d}{dt}\left(\frac{J_\mu(t)}{t^\mu}\right) = \frac{d}{dt} V_\mu(t) = -\, t\, V_{\mu+1}(t)$$

(1. 13), (1. 14) $$V_\mu(t) = \begin{cases} O(1), & \text{as } t \to 0, \\ O(t^{-\mu-1/2}), & \text{as } t \to \infty, \end{cases}$$

and

(1. 15) $$\int_z^\infty t\, V_\mu(at)\, (t^2 - z^2)^\rho\, dt = c\, a^{-2\rho-2}\, V_{\mu-\rho-1}(az),$$

for $a > 0$, $\mu - 1/2 \geqq 2\rho + 2 > 0$, where c is a unspecipied numerical constant (here and elesewhere in this paper).

Then we know that

(1. 16) $$S^\delta(R) = cR^k \int_0^\infty t^{k-1} f(t) V_{\delta+k/2}(tR)\, dt.$$

At last, if $D(n)$ denotes the number of solutions in integers of

$$n \geqq n_1^2 + \ldots\ldots + n_k^2,$$

and $d(n)$ denotes the number of solutions in integers of the equation

$$n = n_1^2 + \ldots\ldots + n_k^2,$$

then

$$D(n) - D(n-1) = d(n) \tag{1. 17}$$

and

$$D(n) = O(n^{k/2}). \tag{1. 18}$$

2. K.Chandrasekharan [4] have proved the following theorems.

THEOREM A. *If $p > 0$, h is the greatest integer less than p, and $\alpha > 0$, then*

$$f^p(t) = o(t^\alpha) \qquad as \qquad t \to 0 \tag{2. 1}$$

implies

$$S^\delta(R) = o(1) \qquad as \qquad R \to \infty, \tag{2. 2}$$

where

$$\delta = p + \frac{k-1}{2} - \theta \qquad and \qquad \theta = \frac{\alpha(p-h)}{1+h+\alpha}$$

THEOREM B. *If $0 < \alpha < 1$ and $\alpha < \delta$ then*

$$S^\delta(R) = o(R^{-\alpha}) \qquad as\ R \to \infty \tag{2. 3}$$

implies

$$f_p(t) = o(1) \qquad as\ t \to 0$$

for

$$p = \delta - \frac{1}{2}(k-3) - \theta,$$

where

$$\theta = \alpha\left(1 + \frac{\delta - h}{1+h+\alpha}\right)$$

h being the greatest integer less than δ provided that

$$p \geqq \frac{k+1}{2} + k\left(\frac{\theta-\alpha}{\theta+\alpha}\right).$$

Above theorems are quite questional to us compared with the theorems of Fourier series of one variable. Especially the estimation (3. 23) and (4. 10) of Chandrasekharan's [4] seems to be incorrect.

Concerning these theorems we obtain the following theorems:

THEOREM 1. *If* $p>0,\ \alpha>0$

(2. 1) $$f_p(t)=o(t^{\alpha}) \qquad as \qquad t\to 0$$

implies

(2. 2) $$S^{\delta}(R)=o(1) \qquad as \qquad R\to\infty,$$

where

(2. 3) $$\delta=\frac{p(1+2\tau)}{1+2\tau+\alpha}+\tau \ and \ \tau=\frac{k-1}{2}.$$

THEOREM 2. *If* $\alpha>0$

(2. 4) $$S^{\delta}(R)=o(R^{-\alpha}) \qquad as \qquad R\to\infty$$

implies

$$f_p(t)=o(1) \qquad as \qquad t\to 0,$$

where

(2. 5) $$p=\frac{(\delta+1)(1+2\tau)}{1+2\tau+\alpha}-\tau,\ \tau=\frac{k-1}{2} \ and \ \delta>2\tau+\alpha.$$

THEOREM 3. *If* $\alpha>0,\ 1>\mu>0,\ p>0$

(2. 6) $$f_0(t)=O(t^{-2\tau-\mu}) \qquad as \qquad t\to 0$$

and

(2. 1) $$f_p(t)=o(t^{\alpha}) \qquad as \qquad t\to 0$$

implies

(2. 2) $$S^{\delta}(R)=o(1) \qquad as \qquad R\to\infty,$$

where

(2. 7) $$\delta=\frac{p(\mu+2\tau)}{\mu+2\tau+\alpha}+\tau \qquad and \qquad \tau=\frac{k-1}{2}.$$

THEOREM 4. *If* $\mu>0,\ \delta>0,\ \alpha>0$

(2. 8) $$a_{n_1\ldots n_k}=O\{(n_1^2+\ldots\ldots+n_k^2)^{-\mu/2}\}$$

and

(2. 3) $$S^{\delta}(R)=o(R^{-\alpha}) \qquad as \qquad R\to\infty$$

implies

$$f_p(t)=o(1) \qquad as \qquad t\to 0,$$

for

$$p=\frac{(\delta+1)(1+2\tau-\mu)}{1+2\tau+\alpha-\mu}-\tau$$

where

$$\delta > 2\tau + \alpha - \mu > -1 \quad and \quad \tau = \frac{k-1}{2}.$$

For the case $k = 1$ S. Isumi [7], G. Sunouchi [9] and the present author [8] has obtained the theorems of similar type.

3. Proof of thcorem 1. Since $\delta > \tau$ we can appeal to the formula

$$(1.\ 16) \qquad S^{\delta}(R) = cR^{k} \int_{0}^{\infty} t^{k-1} f_0(t)\, V_{\delta+k/2}(tR)\, dt$$

$$(3.\ 1) \qquad = cR^{k} \left[\int_{0}^{\eta} + \int_{\eta}^{\infty} \right] t^{k-1} f_0(t)\, V_{\delta+k/2}(tR)\, dt = I + J,$$

say, where η be chosen sufficiently small and kept fixed. Using the formula (1. 3) and (1. 13) we get

$$J = O\left\{ R^{k-(\delta+\frac{k+1}{2})} \int_{\eta}^{\infty} t^{k-1} f_0(t)\, t^{-(\delta+\frac{k+1}{2})}\, dt \right\}$$

$$= O\left\{ R^{(k-1)/2-\delta} \int_{\eta}^{\infty} \frac{dF(t)}{t^{\delta+(k+1)/2}}\, dt \right\},\ F(t) = \int_{0}^{t} s^{k-1} |f_0(s)|\, ds,$$

$$= O\left\{ R^{(k-1)/2-\delta} \left(\Big[F(t)\, t^{-(\delta+(k+1)/2)} \Big]_{\eta}^{\infty} + \int_{\eta}^{\infty} \frac{F(t)}{t^{\delta+(k+1)/2+1}}\, dt \right) \right\}$$

$$= O\left\{ R^{(k-1)/2-\delta} \Big[t^{-\delta+(k-1)/2} \Big]_{\eta}^{\infty} \right\}$$

$$(3.\ 2) \qquad = o(1) \qquad \text{as } R \to \infty, \text{ by integration by part.}$$

$$(3.\ 3) \qquad I = CR^{k} \left[\int_{0}^{CR^{-\rho}} + \int_{CR^{-\rho}}^{\eta} \right] t^{k-1} f_0(t)\, V_{\delta+k/2}(tR)\, dt = I_1 + I_2,$$

say, where C is a sufficiently large constant and

$$(3.\ 4) \qquad \rho = \frac{\delta - \tau}{\delta + \tau + 1} < 1.$$

$$I_2 = CR^{k} \int_{CR^{-\rho}}^{\eta} t^{k-1} f_0(t) V_{\delta+k/2}(tR)\, dt$$

$$= O\left\{ R^{(k-1)/2-\delta} \int_{CR^{-\rho}}^{\eta} t^{k-1} f_0(t)\, t^{-\delta-(k-1)/2}\, dt \right\}$$

$$= O\left\{ R^{(k-1)/2-\delta} R^{\rho(\delta+\overline{k+1}/2)} C^{-(\delta+\overline{k+1}/2)} \int_{CR^{-\rho}}^{\eta} t^{k-1} |f_0(t)|\, dt \right\}$$

$$= O\left\{ R^{\tau-\delta+\rho(\delta+\tau+1)} C^{-(\delta+\tau+1)} \right\}$$

$$(3.\ 5) \qquad = O\{ C^{-(\delta+\tau+1)} \} = o(1),$$

by (1.3), (1. 13) and (3. 4),

We may assume that p is not an integer. For the case that p is an integer we can easily deduced the theorem by the familiar argument. Let h be the greatest integer less than p. By $(h+1)$-times applications of integration by parts, and noting (1. 6), (1. 7) and (1. 12) the integral I_1, becomes

$$I_1 = CR^k \int_0^{CR^{-\rho}} t^{k-1} f_0(t)\, V_{\delta+k/2}(tR)\, dt$$

$$= \left[\sum_{s=0}^{h} c_s R^{k+2s} \varphi_{s+1}(t)\, V_{\delta+k/2+s}(Rt) \right]_0^{CR^{-\rho}}$$

$$+ CR^{k+2h+2} \int^{CR^{-\rho}} \varphi_{h+1}(t)\, t\, V_{\delta+k/2+h+1}(Rt)\, dt$$

$$(3.\ 6) = \sum_{s=0}^{h} K_s + K,$$

say, where $s = 0$ if $p < 1$ and $s = 0, 1, 2, \ldots\ldots h$ if $p \geqq 1$.

Now, by K. Chandrasekharan and O. Szász [6],

$$\phi_p(t) = t^{2p+k-2} f_p(t) = c \int_0^t (t^2 - s^2)^{p-1} s\, \varphi_0(s)\, ds = o(t^{2p+k-2+\alpha})$$

is equivalent to

$$\varphi_p^*(t) = c \int_0^t (t) - s^{p-1} s\, \varphi_0(s)\, ds = o(t^{p+k-1+\alpha}).$$

Therefore, according to $\varphi_1^*(t) = \varphi_1(t) = \int_0^t s\, \varphi_0(s)\, ds = o(1)$ and $\varphi_p^*(t) = o(t^{p+k-1+\alpha})$, applying M. Riesz's convexity theorem we have

$$\varphi_s^*(t) = o(t^{(s-1)(p+k-1+\alpha)/(p-1)}), \quad 1 \leqq s \leqq h,$$

and

$$\varphi_{h+1}^*(t) = o(t^{h+k+\alpha}).$$

That is,

$$\varphi_s(t) = o(t^{(s-1)(p+k-1+\alpha)+s-1}), \quad 1 \leqq s \leqq h,$$

and

$$\varphi_{h+1}(t) = o(t^{2h+k+\alpha}).$$

Hence, we obtain

$$\sum_{s=0}^{h-1} K_h = \sum_{s=1}^{h-1} o\left[R^{k+2s} R^{-(\delta+s+k/2+1/2)}\, t^{s(p+k-1+\alpha)/(p-1)+s}\, t^{-(\delta+s+k/2+1/2)} \right]_0^{CR^{-\rho}}$$

$$= \sum_{s=0}^{h-1} o\left[R^{(k-1)/2+s-\delta} R^{-\rho\{s(p+k+\alpha-1)/(p-1)-\delta-(k+1)/2\}} \right]_0^{CR^{-\rho}}$$

The exponent of R in the bracket is

$$(k-1)/2+s-\delta-\rho\,\{s(p+k+\alpha-1)/(p-1)-\delta-(k+1)/2\}$$

$$=\tau+s-\delta-\frac{\delta-\tau}{\delta+\tau+1}\,\{s(p+2\tau+\alpha)/(p-1)-(\delta+\tau+1)\}$$

$$=\tau+s-\delta-\frac{ps(p+2\tau+\alpha)}{(p+2\tau+\alpha+1)(p-1)}+\delta-\tau$$

$$=-(2\tau+\alpha+1)s/(p-1)(p+1+2\tau+\alpha)\leqq 0$$

And

$$K_h=\Big[c_h\,R^{k+2h}\,\varphi_{h+1}(t)\;V_{\delta+k/2+h}(Rt)\Big]_0^{CR^{-\rho}}$$

$$=o\Big[R^{k+2h}\,R^{-(\delta+h+k/2+1/2)}\,t^{h+k+\alpha}t^{-(\delta+h+k/2+1.2)}\Big]_0^{CR^{-\rho}}$$

$$=o\,[R^{(k-1)/2+h-\delta-\rho\{(k-1)/2+h+\alpha-\delta\}}].$$

The exponent of R in the last bracket is equal to

$$h+\tau-\delta-\rho(h+\tau-\delta+\alpha)=h+\tau-\delta-\frac{\delta-\tau}{\delta+\tau+1}(h+\tau-\delta+\alpha)$$

$$=\{(h+\tau-\delta)(2\tau+1)+(\tau-\delta)\alpha\}/(\delta+\tau+1)$$

$$=\{h(2\tau+1)-(\delta-\tau)(1+2\tau+\alpha)\}/(\delta+\tau+1)$$

$$=\{h(2\tau+1)-p(1+2\tau)\}/(\delta+\tau+1)$$

$$=(2\tau+1)(h-p)/(\delta+\tau+1)<0,$$

for $\delta-\tau=p(1+2\tau)/(1+2\tau+\alpha)$ and $h<p$.

Thus we have

$$\text{(3. 7)}\qquad \sum_{s=0}^{h}K_s=o(1)\qquad \text{as}\qquad R\to\infty.$$

Let us estimate K. For the sake of completeness, we reproduce the same method to theorem 1 of K. Chandrasekharan [4]. Using (1. 7) we get

$$K=cR^{k+2h+2}\int_0^{CR^{-\rho}}t\,V_{\delta+k/2+h+1}(Rt)\,dt\int_0^{t}(t^2-s^2)^{h-p}\,s\,\varphi_p(s)\,ds$$

$$=cR^{k+2h+2}\int_0^{CR^{-\rho}}s\,\varphi_p(s)\,ds\int_s^{CR^{-\rho}}(t^2-s^2)^{h-p}\,t\,V_{\delta+k/2+h+1}(Rt)\,dt$$

$$\text{(3.8)}=cR^{k+2h+2}\int_0^{CR^{-\rho}}s\,\varphi_p(s)\,\psi(s,\;R)\,ds,\qquad\text{say.}$$

The interchange in the order being justified by the succeeding argument. We may write, by (1. 15),

(3. 9) $$\psi(s,R) = \left(\int_s^\infty - \int_{CR^{-\rho}}^\infty\right)(t^2 - s^2)^{h-p}\, t\, V_{\delta+k/2+h+1}(Rt)\, dt$$

$$= R^{2p-2h-2}\, V_{\delta+p+k/2}(Rs) - \int_{CR^{-\rho}}^\infty (t^2 - s^2)^{h-p}\, t\, V_{\delta+k/2+h+1}(Rt)\, dt,$$

where

$$\int_{CR^{-\rho}}^\infty (t^2 - s^2)^{h-p}\, t\, V_{\delta+k/2+h+1}(Rt)\, dt$$

$$= (C^2R^{-2\rho} - s^2)^{h-p} \int_{CR^{-\rho}}^{\xi} t\, V_{\delta+k/2+h+1}(Rt)\, dt, \qquad CR^{-\rho} < \xi < \infty,$$

$$= (C^2R^{-2\rho} - s^2)^{h-p}\, R^{-2} \int_{CR^{1-\rho}}^{\xi R} s\, V_{\delta+k/2+h+1}(s)\, ds$$

$$= (C^2R^{-2\rho} - s^2)^{h-p}\, R^{-2} \Big[V_{\delta+k/2+h}(s) \Big]_{CR^{1-\delta}}^{\xi R}$$

(3. 10) $= O\{(R^{-2\rho} - s^2)^{h-p} R^{-2} R^{-(1-\rho)(\delta+k/2+1/2+h)}\}$, by (1. 12) and (1. 14).

Using (3. 9) and (3. 10) in (3. 8) we obtain

$$K = cR^{k+2p} \int_0^{CR^{-\rho}} s\, \varphi_p(s)\, V_{\delta+k/2+p}(Rs)\, ds$$

(3. 11) $$+ O\left\{R^{k+2h+(\rho-1)(\delta+k/2+1/2+h)} \int_0^{CR^{-\rho}} (R^{-2\rho} - s^2)^{h-p}\, s\, |\varphi_p(s)|\, ds\right\}.$$

The first term is

$$cR^{k+2p}\left(\int_0^{1/R} + \int_{1/R}^{CR^{-\rho}}\right) s\, \varphi_p(s)\, V_{\delta+k/2+p}(Rs)\, ds = L_1 + L_2, \text{ say.}$$

By (1. 6), (1. 13) and (2. 1), we get

(3. 12) $$L_1 = o\{R^{k+2p} \int_0^{1/R} s^{2p+k+\alpha-1}\, ds\} = o(R^{-\alpha}) = o(1) \text{ as } R \to \infty,$$

and in addition, by (1. 14),

$$L_2 = o\left\{R^{k+2p} \int_{1/R}^{CR^{-\rho}} s^{2p+k+\alpha-1}\, (sR)^{-\delta-(k+1)/2-p}\, ds\right\}$$

$$= o\left\{R^{k+p-\delta-(k+1)/2} \Big[s^{p+(k-1)/2-\delta+\alpha} \Big]_{1/R}^{CR^{-\rho}}\right\}$$

(3. 13) $= o\{R^{p+\tau-\delta-\rho(p+\tau-\delta+\alpha)}\}$,

for $p + \tau - \delta + \alpha = p\alpha/(1 + 2\tau + \alpha) + \alpha = \alpha(p+1+2\tau+\alpha)/(1+2\tau+\alpha) < 0$.

The exponent of R is $\quad p + \tau - \delta - \rho(p + \tau - \delta + \alpha) = 0,$

because $\quad p + \tau - \delta = p\alpha/(1 + 2\tau + \alpha)$

and $\quad \rho = p/(p + 1 + 2\tau + \alpha).$

Since $\varphi_p(t) = o(t^{2p+k-2+\alpha})$ by hypothesis, the second term is

$$o\left\{R^{k+2h+(\rho-1)\{\delta+(k+1)/2+h\}}\int_0^{CR^{-\rho}}(R^{-\rho}-s)^{h-p}(R^{-\rho}+s)^{h-p}s^{2p+k+\alpha-1}\,ds\right\}$$

$$= o\left\{R^{k+2h+(\rho-1)\{\delta+(k+1)/2+h\}}R^{-\rho(h-p)-\rho(2p+k+\alpha-1)}\int_0^{R^{-\rho}}(R^{-\rho}-s)^{h-p}\,ds\right\}$$

$$(3.\ 14) \qquad = o\{R^{k+2h+(\rho-1)\{\delta+(k+1)/2+h\}}R^{-\rho(2h+k+\alpha)}\}.$$

The exponent of R is

$$k+2h-\delta-\frac{1}{2}(k-1)-h-\rho\{h+(k-1)/2+\alpha-\delta\}$$

$$= -\delta+\tau+h-\rho(h+\tau-\delta+\alpha)$$

$$= (h+\tau-\delta)(1-\rho)-\alpha p < (p+\tau-\alpha)(1-\rho)-\alpha p$$

$$= \frac{p\alpha}{1+2\tau+\alpha}\cdot\frac{1+2\tau+\alpha}{p+1+2\tau+\alpha}-\frac{\alpha p}{p+1+2\tau+\alpha}=0.$$

Therefore, we obtain

$$(3.\ 15) \qquad K = o(1) \qquad \text{as} \qquad R\to\infty.$$

Summing up (3. 1), (3. 2), (3. 3), (3. 5), (3. 6), (3. 7) and (3. 15) we have

$$S^{\delta}(R) = o(1) \qquad \text{as} \qquad R\to\infty,$$

which is the required.

4. proof of theorem 2. We need the following lemma.

LEMMA. *Let $W(x)$ be a positive non-decreasing function of x, $V(x)$ any positive function of x, both defined for $x>0$, $A(t)$ a function of t which is of bounded variation in every finite interval, and*

$$A_k(t) = k\int_0^t (t-u)^{k-1}A(u)\,du.$$

Then

$$A(x+t)-A(x) = O(t^{\gamma}V(x)),\ o<t=O[\{W/V\}^{1/(k+\gamma)}],\ \gamma>0,$$

and

$$A_k(x) = o[W(x)], \qquad k>0$$

where

$$0 < W(x')/W(x) < H < \infty,\ \text{for } 0<x'-x=O(W/V)^{1/(k+\gamma)},$$

together imply

$$A(x) = o[V^{k/(k+\gamma)}W^{\gamma/(k+\gamma)}].$$

If further $V^{k/(k+\gamma)}W^{\gamma/(k+\gamma)}$ is non-decreasing, then

$$A_r(x) = o[V^{(k-\gamma)/(k+\gamma)}W^{(\gamma+r)/(k+\gamma)}],\ 0\leqq r\leqq k.$$

(See, for example [5, p. 20].)

We know that

$$(4.\ 1)\qquad f_p(t)\sim c\sum_{n=0}^{\infty} A_n\ V_{p+(k-2)/2}(\sqrt{n}\,t).$$

(see [3]). Let us put $m=[t]^{-\rho}$, where

$$(4.\ 2)\qquad \rho=2(p+\tau)/(p-\tau-1)>0,$$

for $\qquad p-\tau-1=(1+2\tau)(\delta-2\tau-\alpha)/(1+2\tau+\alpha)>0.$

Then we have, since $a_{n_1 n_k\cdots}\to 0$ and (1. 18),

$$\sum_{n=m+1}^{\infty} A_n\ V_{p+\tau-1/2}(\sqrt{n}\,t)=o\Big(\sum_{m=1}^{\infty}\frac{d(n)}{n^{(p+\tau)/2}\,t^{p+\tau}}\Big)$$

$$=o\Big(t^{-p-\tau}\int_{m+1}^{\infty}\frac{dD(x)}{x^{(p+\tau)/2}}\Big)=o\Big(t^{-p-\tau}\int_{m+1}^{\infty}\frac{dx}{x^{(p+\tau)/2+1-k/2}}\Big)$$

$$(4.\ 3)\qquad =o(t^{-p-\tau}\,m^{-(p+\tau)/2+\tau+1/2})=o(t^{-(p+\tau)}\,m^{-(p-\tau-1)/2})=o(1).$$

Since $p-\tau-1>0$, the "$\sim$" in (4. 1) can be replaced by equality.

Let h be the greatest integer less than δ, for the case δ is an integer we can deduced by the following argument, then by partial integration $(h+1)$-times, we obtain

$$\sum_{n=0}^{m} A_n\ V_{p+\tau-1/2}(\sqrt{n}\,t)$$

$$=\sum_{r=0}^{h+1} c_r\,t^{2r}\,T^r(\sqrt{m})\,V_{p+\tau-1/2+r}(\sqrt{m}\,t)+t^{2h+4}\int_0^{\sqrt{m}} S^{h+1}(R)\,R^{2h+3}\,V_{p+\tau+h+3/2}(Rt)\,dR$$

$$=\sum_{r=0}^{h}\psi_r(t)+\psi_{h+1}(t)+\psi(t),\ \text{say}.$$

For $t=O(R)$ we get

$$|S\{(R+t)^{1/2}\}-S(R^{1/2})|\leqq\sum_{R<n\leqq R+t}|A_n(t)|$$

$$=\sum_{R<n\leqq R+t}|a_{n_1\ldots n_k}|=o\Big(\sum_{R<n\leqq R+t}d(n)\Big)$$

$$=o\Big(\int_R^{R+t}dD(x)\Big)=o(t\,R^{k/2-1}),\ \text{by (2. 18)}.$$

Since $S^\delta(R)=o(n^{-\alpha})$ by hypothesis, we obtain by Lemma

$$S^r(R)=o\Big[R^{\frac{2}{\delta+1}\{\delta r+(\delta-r)k/2+r(1-\alpha/2)-\alpha/2\}-2r}\Big],\ 0\leqq r\leqq h.$$

Thus we get

$$(4.\ 4)\qquad T^r(R)=o\Big[R^{\frac{2}{\delta+1}\{\delta r+(\tau+1/2)\delta-r(\tau+1/2)+r(1-\alpha/2)-\alpha/2\}}\Big]$$

$$= o\left[R^{\frac{1}{\delta+1}\{r(2\delta-2\tau+1-\alpha)+(2\tau+1)\delta-\alpha\}} \right],\ 0 \leqq r \leqq h.$$

And by hypothesis (2. 4), we obtain

$$T^{h+1}(R) = o(R^{2h+2-\alpha}). \tag{4. 5}$$

Substituting (4. 4), we have

$$\sum_{r=0}^{h} \psi_r(t) = o\left[\sum_{r=0}^{h} t^{2r}\, t^{-(p+\tau+r)}\, m^{\frac{1}{2(\delta+1)}\{r(2\delta-2\tau+1-\alpha)+(2\tau+1)\delta-\alpha\}-\frac{1}{2}(p+\tau+r)} \right]$$

$$= o\left[\sum_{r=0}^{h} t^{r-p-\tau} t^{-\frac{\rho}{2}[\{r(2\delta-2\tau+1-\alpha)+(2\tau+1)\delta-\alpha\}/(\delta+1)-(p+\tau+r)]} \right]$$

The exponent of t in the last bracket is

$$2r - (r+p+\tau) - \frac{\delta+1}{\delta-2\tau-\alpha}\,\frac{1}{\delta+1}\,\{2(\delta-2\tau-\alpha)+(1+2\tau+\alpha)r$$

$$+ (1+2\tau)\delta - \alpha\} + \frac{\delta+1}{\delta-2\tau-\alpha}(p+\tau+r)$$

$$= -\frac{r(1+2\tau+\alpha)+\delta(1+2\tau)-\alpha}{\delta-2\tau-\alpha} + \frac{p+\tau+r}{\delta-2\tau-\alpha}(1+2\tau+\alpha)$$

$$= \{(p+\tau)(1+2\tau+\alpha) - \delta(1+2\tau+\alpha\}/(\delta-2\tau-\alpha)$$

$$= \{(\delta+1)(1+2\tau) - \delta(1+2\tau)+\alpha\}/(\delta-2\tau-\alpha)$$

$$= (1+2\tau+\alpha)/(\delta-2\tau-\alpha) > 0$$

for $\qquad p+\tau = (\delta+1)(1+2\tau)/(1+2\tau+\alpha)$

and $p-\tau-1 = (\delta-2\tau-\alpha)(1+2\tau)/(1+2\tau+\alpha)$. Thus, we obtain

$$\sum_{r=0}^{h} \psi_r(t) = o(1) \qquad \text{as } t \to 0. \tag{4. 7}$$

From (4. 5), we have

$$\psi_{h+1}(t) = o\{t^{2(h+1)}\, t^{-(p+\tau+h+1)} m^{\frac{1}{2}(2h+2-\alpha)} m^{-\frac{1}{2}(p+\tau+h+1)}\}$$

$$= o\{t^{h+1-p-\tau} m^{(h+1-\alpha-p-\tau)/2}\} = o\{t^{h+1-q-\tau-\rho(h+1-\alpha-p-\tau)/2}\}. \tag{4. 8}$$

The exponent of t is

$$h+1-p-\tau + \frac{p+\tau}{p-\tau-1}(\alpha+p+\tau-h-1)$$

$$= \{(h+1-p-\tau)(p-\tau-1) + (p+\tau)(\alpha+p+\tau-h-1)\}/(p-\tau-1)$$

$$= \{(p+\tau)(1+2\tau+\alpha) - (h+1)(1+2\tau)\}/(p-\tau-1)$$

$$= \{(\delta+1)(1+2\tau) - (1+2\tau)(h+1)\}/(p-\tau-1)$$

$$= (1+2\tau)(\delta-h)/(p-\tau-1) > 0.$$

Hence, we have

$$(4.\ 9) \qquad \psi_{h+1}(t) = o(1) \qquad \text{as} \qquad t \to 0.$$

Now we consider the integral $\psi(t)$. By the same reason as in theorem 1, we repreat the argument of [4].

$$\psi(t) = ct^{2h+4} \int_0^{\sqrt{m}} R\, V_{p+\tau+h+3/2}(Rt)\, dR \int_0^R (R^2 - s^2)^{h-\delta} s\, T^\delta(s)\, ds$$

$$(4.\ 10) \quad = ct^{2h+4} \int_0^{\sqrt{m}} s\, T^\delta(s)\, ds \int_s^{\sqrt{m}} R\, V_{p+\tau+h+3/2}(Rt)\, (R^2 - s^2)^{h-\delta}\, dR$$

The interchange of integration being justified by the succeeding argument. (4. 10) may be written as

$$ct^{2h+4} \Big[\int_0^{\sqrt{m}} s\, T^\delta(s)\, ds \int_s^\infty R\, V_{p+\tau+h+3/2}(Rt)\, (R^2 - s^2)^{h-\delta}\, dR$$

$$- \int_0^{\sqrt{m}} s\, T^\delta(s)\, ds \int_{\sqrt{m}}^\infty R\, V_{p+\tau+h+3/2}(Rt)\, (R^2 - s^2)^{h-\delta}\, dR \Big]$$

$$= ct^{2h+4} t^{-2h+2\delta-2} \int_0^{\sqrt{m}} s\, T^\delta(s)\, V_{p+\tau+h+3/2-h+\delta+1}(st)\, ds$$

$$- ct^{2h+4} \int_0^{\sqrt{m}} s\, T^\delta(s)\, ds \int_{\sqrt{m}}^\infty R\, V_{p+\tau+h+3/2}(Rt)\, (R^2 - s^2)^{h-\delta}\, dR$$

(4. 11) $= \chi_1(t) + \chi_2(t)$, say. And

$$\left| \int_{\sqrt{m}}^\infty R\, V_{p+\tau+3/2}(Rt)\, (R^2 - s^2)^{h-\delta}\, dR \right| \leqq (m - s^2)^{h-\delta} \max_{\sqrt{m'} > \sqrt{m}} \left| \int_{\sqrt{m}}^{\sqrt{m'}} R\, V_{p+\tau+h+3/2}(Rt)\, dR \right|$$

$$= (m - s^2)^{h-\delta} \max_{\sqrt{m'} > \sqrt{m}} t^{-2} \Big[V_{p+\tau+h+1/2}(Rt) \Big]_{\sqrt{m}}^{\sqrt{m'}}$$

$$= O\{(m - s^2)^{h-\delta}\, t^{-2}\, t^{-(p+\tau+h+1)}\, m^{-(p+\tau+h+1)/2}\}.$$

Thus we obtain, by $\quad T^\delta(R) = R^{2\delta} S^\delta(R) = o(R^{2\delta-\alpha})$,

$$\chi_2(t) \quad = O\left\{ t^{2h+1-p-\tau-h}\, m^{-(p+\tau+h+1)/2} \int_0^{\sqrt{m}} s\, |T^\delta(s)|\, (m - s^2)^{h-\delta}\, ds \right\}$$

$$= o\left\{ \{t^{h-p-\tau+1}\, m^{-(p+\tau+h+1)} \int_0^{\sqrt{m}} s^{2\delta+1-\alpha}\, (m - s^2)^{h-\delta}\, ds \right\}$$

$$= o\,\{t^{h-p-\tau+1}\, m^{-(p+\tau+h+1)/2}\, m^{(2\delta-\alpha)/2+h+1-\delta}\}$$

$$(4.\ 12) \quad = o\,\{t^{h-p-\tau+1}\, m^{-(p+\tau-h+\alpha-1)/2}\} = o(1),$$

by the same reasoning as in (4. 8).

And at last

$$\chi_1(t) = t^{2\delta+2}\left(\int_0^{1/t} + \int_{1/t}^{\sqrt{m}}\right) s\, T^{\delta}(s)\, V_{p+\tau+\delta+1/2}(st)\, ds$$

$$= o\left\{ t^{2\delta+2}\int_0^{1/t} s^{2\delta+1-\alpha}\, ds\right\} + o\left\{ t^{2\delta+2}\int_{1/t}^{\sqrt{m}} s^{1+2\delta-\alpha}\, s^{-(p+\tau+\delta+1)}\, t^{-(p+\tau+\delta+1)}\, ds\right\}$$

$$= o\,(t^{\alpha}) + o\,\{t^{2\delta+1-p-\tau-\delta}\, m^{(\delta+1-\alpha-p-\tau)/2}\}$$

$$= o(1) + o\,(t^{\delta+1-p-\tau-\rho(\delta+1-\alpha-p-\tau)/2}),$$

for $\delta+1-\alpha-p-\tau = \delta+1-\alpha-(\delta+1)(1+2\tau)/(1+2\tau+\alpha)$

$= \alpha(\delta-2\tau-\alpha)/(1+2\tau+\alpha) > 0.$

The exponent of t of the second term is

$$\delta + 1 - p - \tau - \frac{p+\tau}{p-\tau-1}(\delta+1-\alpha-p-\tau)$$

$$= \{p(1+2\tau+\alpha) - 2\tau(\delta+1) - (\delta+1) + \tau(1+2\tau+\alpha)\}/(p-\tau-1)$$

$$= \{(p+\tau)(1+2\tau+\alpha) - (1+2\tau)(\delta+1)\}/(p-\tau-1) = 0,$$

for $\quad p+\tau = (\delta+1)(1+2\tau)/(1+2\tau+\alpha).$

Therefore, we get

(4. 13) $\qquad \chi_1(t) = o(1) \qquad$ as $\quad t \to 0.$

On account of (4. 1), (4. 3), (4. 7), (4. 9), (4. 11), (4. 12) and (4. 13) we have

$$f_p(t) = o(1) \qquad \text{as} \quad t \to 0.$$

Thus the proof is completed.

5. Proof of Theorem 3. The argument closely resembles that of Theorem 1. And so, we omit the detailed calculation. Since

$\delta = p(2\tau+\mu)/(\mu+2\tau+\alpha) + \tau > \tau$, we have

(5. 1) $$S^{\delta}(R) = cR^k \int_0^{\eta} t^{k-1} f_0(t)\, V_{\delta+k/2}(t\,R)\, dt + o(1) = I + o(1),$$

say, as $R \to \infty$.

(5. 2) $$I = cR^k\left[\int_0^{CR^{-\rho}} + \int_{CR^{-\rho}}^{\eta}\right] t^{k-1} f_0(t)\, V_{\delta+k/2}(Rt)\, dt = I_1 + I_2,$$

say, where C is a sufficiently large constant and

(5. 3) $\quad \rho = (\delta-\tau)/(\mu+\delta+\tau) < 1.$

$$I_2 = cR^k \int_{CR^{-\rho}}^{\eta} t^{k-1} f_0(t)\, V_{\delta+k/2}(tR)\, dt$$

$$= O\left\{ R^{k-(\delta+k/2+1/2)} \int_{CR^{-\rho}}^{\eta} t^{-\mu-\delta-(k+1)/2}\, dt\right\}, \text{ by (1. 14) and (2. 6),}$$

$$= O\{R^{(k-1)/2-\delta} C^{-(\mu+\delta+k/2-1/2)} R^{\rho(\mu+\delta+k/2-1/2)}\}$$
$$= O\{R^{\tau-\delta+\rho(\mu+\delta+\tau)} C^{-(\mu+\delta+\tau)}\}$$
$$(5.\ 4) \quad = O\{C^{-(\mu+\delta+\tau)}\} = o(1), \text{ by } (5.\ 3).$$

Now we consider I_1. Let h be the greatest integer less than p. By $(h+1)$-times applications of integration by parts, we have

$$I_1 = \Big[\sum_{s=0}^{h} c_s R^{k+2s} \varphi_{s+1}(t) V_{\delta+k/2+s}(tR)\Big]_0^{CR^{-\rho}} + cR^{k+2h+2}\int_0^{CR^{-\rho}} \varphi_{h+1}(t)\, t\, V_{\delta+k/2+h+1}(tR)\, dt$$

$$(5.\ 5) \quad = \sum_{s=0}^{h} K_s + K, \qquad \text{say.}$$

Applying similar method to that of Theorem 1, by (2. 6) and (2. 1), we get

$$\varphi_s(t) = o(t^{-\mu+s-1+s(p+\mu+2\tau+\alpha)/p}), \qquad 0 \leqq s \leqq h,$$
$$\varphi_{h+1}(t) = o(t^{2h+2\tau+\alpha+1}).$$

Hence, we obtain

$$\sum_{s=0}^{h-1} K_s = \sum_{s=0}^{h-1} o\Big[R^{k+2s} R^{-(\delta+k/2+s+1/2)} t^{-\mu+(s+1)(p+2\tau+\alpha+\mu)/p+s} t^{-(\delta+k/2+s+1/2)}\Big]_0^{CR^{-\rho}}$$
$$= \sum_{s=0}^{h-1} o\,[R^{(k-1)/2+s-\delta-\rho\{(s+1)(p+2\tau+\alpha+\mu)/p-(\mu+\delta+k/2+p/2)\}}]$$

The exponent of R in the bracket is

$$\tau + s - \delta - \rho\{(s+1)(p+2\tau+\alpha+\mu)/p - (\mu+\delta+\tau+1)\}$$
$$= \tau + s - \delta - \frac{\delta-\tau}{\mu+\delta+\tau}\{(s+1)(p+2\tau+\alpha+\mu)/p - 1\} + \delta - \tau$$
$$= s - \frac{(s+1)(p+2\tau+\alpha+\mu)-p}{\mu+2\tau+\alpha+\mu} = -\frac{\mu+2\tau+\alpha}{\mu+2\tau+\alpha+p} < 0$$

for $\quad \rho = (\delta-\tau)/(\mu+\delta+\tau) = p/(\mu+2\tau+\alpha+p)$.

Thus we have

$$(5.\ 6) \qquad \sum_{s=0}^{h-1} K_s = o(1) \qquad \text{as} \quad R \to \infty.$$

$$K_h = \Big[c_h R^{k+2h} \varphi_{h+1}(t) V_{\delta+k/2+h}(Rt)\Big]_0^{CR^{-\rho}}$$
$$= o\Big[R^{k+2h-\delta-k/2-1/2-h} t^{2h+2\tau+\alpha+1-(\delta+k/2+1/2+h)}\Big]_0^{CR^{-\rho}}$$
$$= o\,[R^{\tau+h-\delta-\rho(\tau+h+\alpha-\delta)}].$$

The exponent of R is

$$\tau + h - \delta - \rho(\tau + h - \delta + \alpha)$$

$$= \tau + h - \delta - \frac{\delta - \tau}{\mu + \delta + \tau}(\tau + h - \delta + \alpha)$$
$$= \{(h + \tau - \delta)(\mu + 2\tau) - (\delta - \tau)\alpha\}/(\mu + \delta + \tau)$$
$$= \{h(\mu + 2\tau) - (\delta - \tau)(\alpha + 2\tau + \mu)\}/(\mu + \delta + \tau)$$
$$= \{h(\mu + 2\tau) - p(\mu + 2\tau)\}/(\mu + \delta + \tau)$$
$$= (h - p)(\mu + 2\tau)/(\mu + \delta + \tau) < 0,$$

for $\delta - \tau = p(\mu + 2\tau)/(\mu + 2\tau + \alpha)$ and $h < p$.

Hence we get

(5. 7) $K_h = o(1)$ as $R \to \infty$.

Next we have, in the similar way as (3. 11),

$$K = c\, R^{1+2\tau+2p} \int_0^{CR^{-\rho}} s\, \varphi_p(s)\, V_{\delta+k/2+p}(sR)\, ds$$
$$+ cR^{1+2\tau+2h+\rho-1)(\delta+\tau+1+h)} \int_0^{CR^{-\rho}} (R^{-2\rho} - s^2)^{h-p} s\, |\varphi_p(s)|\, ds.$$

We may write the first term as

$$cR^{1+2\tau-2p}\left(\int_0^{1/R} + \int_{1/R}^{CR^{-\rho}}\right) s\, \varphi_p(s)\, V_{\delta+k/2+p}(sR)\, ds = L_1 + L_2, \qquad \text{say.}$$

$L_1 = o(R^{-\alpha}) = o(1)$,

by (3. 12), as $R \to \infty$. By (3. 13)

$L_2 = o\{R^{p+\tau-\delta-\rho(p+\tau-\delta+\alpha)}\} = o(1)$ as $R \to \infty$,

for $(p + \tau - \delta)/(p + \tau - \delta + \alpha) = p/(p + \alpha + 2\tau + \mu) = \rho$.

The second term is, by (3. 14),

$o\{R^{h+\tau-\delta-\rho(h+\tau+\alpha-\delta)}\} = o(1)$ as $R \to \infty$,

for $h + \tau - \delta - \rho(h + \tau + \alpha - \delta) < (p + \tau - \delta)(1 - \rho) - \alpha\rho$

$$= \frac{p\alpha}{\alpha + 2\tau + \mu} \cdot \frac{\alpha + 2\tau + \mu}{p + \alpha + 2\tau + \mu} - \frac{p\alpha}{p + \alpha + 2\tau + \mu} = 0$$

Hence we obtain

(5. 8) $K = o(1)$ as $R \to \infty$.

Summing up (5. 1), (5. 2), (5. 4), (5. 5), (5. 6), (5. 7) and (5. 8) we have

$S^{\delta}(R) = o(1)$ as $R \to \infty$,

which is the required.

6. Proof of Theorem 4. By the same reasoning as in Theorem 3, we omit the detailed calculation. We know that

$$(6.\ 1)\qquad f_p(t) \sim c \sum_{n=0}^{\infty} A_n V_{p+(k-2)/2}(\sqrt{n}\, t).$$

Let us put $m = [\varepsilon t]^{-\rho}$, where

$$(6.\ 2)\qquad \rho = 2(p+\tau)/(p+\mu-\tau-1) > 0, \text{ because } p+\mu-\tau-1$$
$$= (1+2\tau-\mu)(\delta+1)/(1+2\tau+\alpha-\mu)+\mu-2\tau-1$$
$$= (1+2\tau-\mu)(\delta+\mu-2\tau-\alpha)/(1+2\tau+\alpha-\mu) > 0,$$

and ε is sufficiently small positive number.

Then, by hypothesis (2. 8), we get

$$\sum_{n=m+1}^{\infty} A_n V_{p+\tau-1/2}(\sqrt{n}\, t) = O\left\{ \sum_{n=m+1}^{\infty} \frac{n^{-\mu/2} d(n)}{n^{(p+\tau)/2} t^{p+\tau}} \right\}$$
$$= O\left\{ t^{-p-\tau} \int_{m+1}^{\infty} \frac{dD(x)}{x^{(p+\tau+\mu)/2}} \right\} = O\{t^{-(p+\tau)}\, t^{\rho(p+\mu-\tau-1)/2}\, \varepsilon^{\rho(p+\mu-\tau-1)/2}\}$$
$$(6.\ 3)\qquad = O(\varepsilon^{(p+\tau)}) = o(1), \text{ by } (6.\ 2)$$

Since $p+\mu-\tau-1>0$, the "$\sim$" in (6. 1) can be replaced by equality.

If h is the greatest integer less than δ, then by partial integration $(h+1)$ times, we get

$$\sum_{n=0}^{m} A_n V_{p+\tau-1/2}(\sqrt{n}\, t) = \sum_{r=0}^{h+1} c_r t^{2r} T^r(\sqrt{m})\, V_{p+\tau-1/2+r}(\sqrt{m}\, t)$$
$$+ ct^{2h+4} \int_0^{\sqrt{m}} S^{h+1}(R)\, R^{2h+3}\, V_{p+\tau+h+3/2}(Rt)\, dR$$
$$(6.\ 4)\qquad = \sum_{r=0}^{h} \psi_r(t) + \psi_{h+1}(t) + \psi(t), \qquad \text{say.}$$

For $t = O(R)$, by hypothesis, we have

$$|S\{(R+t)^{1/2}\} - S(R^{1/2})| \leqq \sum_{R<n\leqq R+t} |A_n| = \sum_{R<n\leqq R+t} |a_{n_1\cdots n_k}|$$
$$= O\{\Sigma\{(n_1^2 + \cdots\cdots + n_k^2)^{-\mu/2}\} = O\left\{ \sum_{R<n\leqq R+t} d(n) n^{-\mu/2} \right\}$$
$$= O\left\{ \int_R^{R+t} x^{-\mu/2}\, dD(x) \right\} = O(tR^{(k+\mu)/2-1}).$$

Therefore we obtain, by Lemma,

$$S^r(R) = o[R^{\frac{2}{\delta+1}\{\delta r+(\delta-r)(k-\mu)/2+r(1-\alpha/2)-\alpha/2\}-2r}],$$

that is,

$$T^r(R) = o[R^{\frac{2}{\delta+1}\{\delta r+(\delta-r)(\tau+1/2-\mu/2)+\mu(1-\alpha/2)-\alpha/2\}}]$$

$$= o[R^{\frac{1}{\delta+1}\{r(2\delta-2\tau+1+\mu-\alpha)+\delta(1+2\tau-\mu)-\alpha\}}]$$

Moreover, it is easy to see that

$$T^{h+1}(R) = o(R^{2h+2-\alpha}).$$

Hence we get

$$\sum_{r=0}^{h} \psi_r(t) = \sum_{r=0}^{h} o\,[t^{2r-(p+\tau+r)} m^{-(p+\tau+r)/2+\{r(2\delta-2\tau+1+\mu-\alpha)+\delta(1+2\tau-\mu)-\alpha\}/2(\delta+1)}]$$

$$= \sum_{r=0}^{h} o\,[t^{r-p-\tau-(\rho/2)[\{r(2\delta-2\tau+1+\mu-\alpha)+\delta(1+2\tau-\mu)-\alpha\}/(\delta+1)-(p+\tau+r)]}]$$

The exponent of t in the bracket is

$$r-p-\tau- \frac{\delta+1}{\delta-2\tau-\alpha+\mu}[\{r(2\delta-2\tau+1+\mu-\alpha)+\delta(1+2\tau-\mu)-\alpha\}/(\delta+1)-(p+\tau+r)]$$

$$=2r-(r+p+\tau)- \frac{1}{\delta-2\tau-\alpha+\mu}\{2(\delta-2\tau-\alpha+\mu)r+(1+2\tau+\alpha-\mu)r+\delta(1+2\tau-\mu)-\alpha\}$$
$$+\frac{(\delta+1)(p+\tau+r)}{\delta-2\tau-\alpha+\mu}$$
$$=\frac{(1+2\tau+\alpha-\mu)(p+\tau+r)}{\delta-2\tau-\alpha+\mu}-\frac{1}{\delta-2\tau-\alpha+\mu}\{(1+2\tau+\alpha-\mu)r+\delta(1+2\tau-\mu)-\alpha\}$$
$$=\{(2\tau+\alpha+1-\mu)(p+\tau)-\delta(1+2\tau-\mu)+\alpha\}/(\delta-2\tau-\alpha+\mu)$$
$$=\{(\delta+1)(1+2\tau-\mu)-\delta(1+2\tau-\mu)+\alpha\}/(\delta-2\tau-\alpha+\mu)$$
$$=(1+2\tau-\mu+\alpha)/(\mu+\delta-2\tau-\alpha)>0,$$

for $p+\tau=(1+2\tau-\mu)(\delta+1)/(1+2\tau+\alpha-\mu)$ and $\rho=2(\delta+1)/(\delta-2\tau-\alpha+\mu)$.

Thus we have

$$(6.\ 5) \qquad \sum_{r=0}^{h} \psi_r(t) = o(1) \qquad \text{as} \quad t \to 0.$$

By the same reasoning as in (4. 8) we have,

$$\psi_{h+1}(t) = o\{t^{h+1-p-\tau-\rho(h+1-\alpha-p-\tau)/2}\}.$$

The exponent of t is

$$h+1-p-\tau+\frac{p+\tau}{p+\mu-\tau-1}(\alpha+p+\tau-h-1)$$
$$=\{(1+2\tau+\alpha-\mu)(p+\tau)-(1+2\tau-\mu)(h+1)\}/(p+\mu-\tau-1)$$
$$=\{(\delta+1)(1+2\tau-\mu)-(h+1)(1+2\tau-\mu)\}/(p+\mu-\tau-1)$$
$$(6.\ 6) \qquad =(1+2\tau-\mu)(\delta-h)/(p+\mu-\tau-1)>0, \quad \text{for } \delta>h.$$

Thus we get

$$(6.\ 7) \qquad \psi_{h+1}(t) = o(1) \qquad \text{as} \quad t \to 0.$$

By the similar calculation to that of Theorem 2, we obtain

$$\psi(t) = ct^{2\delta+2}\left(\int_0^{1/t} + \int_{1/t}^{\sqrt{m}}\right) s\, T^{\delta}(s)\, V_{p+\tau+\delta+1/2}(st)\, ds$$

$$+ct^{2h+4}\int_0^{\sqrt{m}} sT^{\delta}(s)\, ds \int_{\sqrt{m}}^{\infty} R\, V_{p+\tau+3/2+h}(Rt)(R^2-s^2)^{h-\delta} dR$$

$$= o(t^{\alpha}) + o\{t^{\delta+1-p-\tau-\rho(\delta+1-\alpha-p-\tau)/2}\} + o\{t^{h+1-p-\tau-\rho(h+1-\alpha-p\tau)/2}\}$$

$$\delta+1-p-\tau-\frac{p+\tau}{p+\mu-\tau-1}(\delta+1-\alpha-p-\tau)$$

$$=\{(\delta+1)(\mu-2\tau-1)-(p+\tau)(\mu-2\tau-\alpha-1)\}/(p+\mu-\tau-1)=0.$$

In addition, by (6. 6), we have

$$h+1-p-\tau-\rho(h+1-\alpha-p-\tau)/2>0.$$

Hence, we have

$$(6.\ 3) \qquad \psi(t) = o(1) \qquad \text{as} \quad t\to 0.$$

From (6. 1), (6. 3), (6. 4), (6. 5), (6. 7) and (6. 8) we obtain

$$f_p(t) = o(1) \qquad \text{as} \quad t\to 0,$$

which is the required.

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