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On the Symplectic Structure of General Relativity

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Abstract. The relation between the symplectic structures on the canonical and radiative phase spaces of general relativity is exhibited.

1. Introduction

There are available in the literature, two Hamiltonian descriptions of general relativity. The first and the more established one is based on spacelike hypersurfaces and uses the initial value formulation of general relativity and the Dirac theory of constrained systems [1, 2]. Over the years, this formulation has been systematically developed and refined by several authors and has shed considerable light on the structure of Einstein's theory. (See, e.g., [3].) In particular, these investigations have brought out the role of the Arnowitt–Deser–Misner [4] energy-momentum as the generator of space-time translations [5] and have paved the way for canonical quantization of gravity [3]. The second Hamiltonian description became available more recently [6]. It is based on null infinity [7] and uses techniques from the gravitational radiation theory in exact general relativity. (See especially, [8] and [9].) Here the focus is on the radiative aspects of the gravitational field; the phase space is the space of radiative modes. This description has also given one new insight. In particular, fluxes of energy-momentum and angular momentum carried away by gravitational waves have been shown to be the generators of the Bondi–Metzner–Sachs (BMS) group, the asymptotic symmetry group at null infinity [10]. More importantly, the formulation has enabled one to carry out the asymptotic quantization of the non-linear gravitational field [6, 11].

In view of this situation, it is natural to ask for the relation between the two descriptions. Apart from its intrinsic interest, such an analysis would clarify several issues which arise in the two frameworks separately. For example, since the radiative

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phase space is not constructed from a cotangent bundle over a configuration space, the symplectic tensor field thereon had to be simply postulated [10]. Here, one was guided by general considerations such as the requirement that the Poisson bracket between the basic variables should have the dimensions of action, that one should obtain the correct results in the weak field limit, and that the expression of the symplectic structure should fit in the pattern suggested by the spin zero and one fields. However, one could not show that these considerations suffice to determine the symplectic tensor field uniquely. It is therefore desirable to have as strong an evidence as possible supporting the choice that was made. A strong—perhaps the strongest possible—evidence would be that the chosen symplectic structure is, in an appropriate sense, the same as the one on the canonical phase space. The canonical approach would also be enriched from the analysis of its relation to the radiative framework. For example, the canonical quantization programme has met with severe difficulties in the construction of a Hilbert space of states (or a substitute thereof). In the approach based on the radiative phase space, on the other hand, these difficulties do not arise: one can readily construct not only the Fock spaces of asymptotic gravitons but also the Hilbert spaces required to handle the infrared problems, i.e., which are analogous to the charged sectors in quantum electrodynamics [12]. Therefore, an understanding of the relation between the two phase spaces may give one considerable insight in the Hilbert space problem of canonical quantization. In particular, the analysis may shed light on the nature of the (canonical) quantum vacuum, which, one now suspects, may not be simply a gaussian peaked at the flat metric.

The purpose of this paper is to provide the first steps towards establishing the relation between the two phase spaces. At an intuitive level, one may divide the problem into two parts: differential geometric issues and functional analytic difficulties. In a broad sense, this paper resolves the first part. More precisely, we shall assume that globally hyperbolic, vacuum, asymptotically flat, horizon-free space-times exist and show that each such space-time leads to a natural *symplectic structure preserving* identification of a point of the canonical phase space with a point of the radiative phase space.

The main obstacle in relating the two phase spaces is, of course, that whereas the canonical phase space is constructed from initial data sets on *space-like* surfaces, the radiative phase space consists of certain equivalence classes of connections on null infinity, $\mathscr{I}$.[1] Therefore, to exhibit the relation between the two, we shall introduce a structure which can interpolate between the two regimes: the symplectic vector space of *linearized* gravitational fields on a globally hyperbolic asymptotically flat, vacuum space-time without horizons. This introduction serves the following purpose. A linearized solution induces on any Cauchy surface a set of linearized Cauchy data and may be therefore regarded as a tangent vector at a point of the canonical phase space. As one might expect, this identification preserves the symplectic structure: We shall show explicitly that the symplectic structure on the space of linearized solutions (off a fixed background) reduces to the symplectic structure evaluated at the tangent space of any point of the canonical phase space

1 Throughout this paper, the symbol $\mathscr{I}$ will stand for future *or* past null infinity

corresponding to the given background, when the linearized fields are identified with their initial data.[2] On the other hand, if the background is asymptotically flat at null infinity, each linearized solution also defines a linearized connection on $\mathscr{I}$, and therefore, a tangent vector to the phase space of radiative modes at the point corresponding to the given background. We show that this identification is also symplectic structure preserving, thereby exhibiting the equality between the canonical and the radiative symplectic structures. To summarize, the difficulty in relating the two frameworks is overcome by first recognizing that a linearized solution to Einstein's equation defines a tangent vector at suitable points of both phase spaces and that the symplectic structure, being a tensor field, is completely determined by its action on the tangent vectors, and then letting the linearized solutions do the desired interpolation between the space-like surface and $\mathscr{I}$.

2. Preliminaries

This section is divided into four parts. The first summarizes the usual Hamiltonian formulation of general relativity; the second outlines the structure available on the space of radiative modes in exact general relativity; the third describes the phase space of linearized gravitational fields on a vacuum, globally hyperbolic background space-time and the fourth recalls certain results on the asymptotic behavior of these linearized solutions.

2.1 The Hamiltonian Formulation of General Relativity

Fix a C^∞ 3-manifold Σ and consider thereon pairs $(q_{ab}, p^{ab}{}_{mnr})$ consisting of C^∞ positive-definite metrics q_{ab} and C^∞ tensor fields $p^{ab}{}_{mnr}$ such that $p^{ab}{}_{mnr} = p^{(ab)}{}_{[mnr]}$.[3] We shall assume that either Σ is compact or the pairs (q, p) are asymptotically flat in a suitable sense. The space $\tilde{\Gamma}$ of these pairs has the structure of a cotangent bundle. It therefore possesses a natural symplectic tensor field $\tilde{\Omega}$:

$$\tilde{\Omega}|_{(q,p)}((\alpha, \beta); (\alpha', \beta')) = \int_\Sigma (\alpha_{ab}\beta'^{ab}{}_{mnr} - \alpha'_{ab}\beta^{ab}{}_{mnr})dS^{mnr}, \tag{1}$$

where (α, β) represents a tangent vector to $\tilde{\Gamma}$ at (q, p). (Thus, α_{ab} is a symmetric tensor field and $\beta^{ab}{}_{mnr}$ has the symmetries of $p^{ab}{}_{mnr}$). Denote by $\bar{\Gamma}$ the "constraint submanifold" of $\tilde{\Gamma}$ consisting of pairs (q, p) satisfying the following equations:

$$D_a p^{ab}{}_{mnr} = 0; \quad \text{and}, \tag{2.a}$$

$$\mathscr{R} - (1/6)p^{ab}{}_{mnr}p_{ab}{}^{mnr} + (1/12)p^a{}_{amnr}p^b{}_b{}^{mnr} = 0, \tag{2.b}$$

where D and $\mathscr{R}$ are, respectively, the derivative operator and the scalar curvature of q_{ab}, and where indices are raised and lowered by q_{ab}. Each point of $\bar{\Gamma}$ represents a permissible data for Einstein's vacuum equation: q_{ab} is the intrinsic metric on Σ and $\pi^{ab} := (1/6)(p^{ab}{}_{mnr} - \frac{1}{2}q^{ab}p^c{}_{cmnr})\varepsilon^{mnr}$, the extrinsic curvature, where ε^{abc} is the unique 3-form on Σ defined by $\varepsilon^{abc}\varepsilon_{abc} = 3!$. Denote by $\bar{\Omega}$ the pull-back to $\bar{\Gamma}$ of $\tilde{\Omega}$.

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2 Because of gauge problems, this result is not as straightforward as one might have expected

3 One often uses tensor densities of weight one in place of tensor fields $p^{ab}{}_{mnr}$

Thus, we have: $\bar{\Omega}|_{(\bar{q},\bar{p})}((\bar{\alpha},\bar{\beta});(\bar{\alpha}',\bar{\beta}')) = \tilde{\Omega}|_{(\bar{q},\bar{p})}((\bar{\alpha},\bar{\beta});(\bar{\alpha}',\bar{\beta}'))$ for all vectors $(\bar{\alpha},\bar{\beta})$ and $(\bar{\alpha}',\bar{\beta}')$ tangential to $\bar{\Gamma}$ at $(\bar{q},\bar{p})$. This $\bar{\Omega}$ is, however, degenerate: $\bar{\Omega}((\bar{\alpha},\bar{\beta});(\bar{\alpha}',\bar{\beta}'))=0$ for all tangent vectors $(\bar{\alpha}',\bar{\beta}')$ to $\bar{\Gamma}$ if and only if $(\bar{\alpha},\bar{\beta})$ is the (restriction to the point $(\bar{q},\bar{p})$ of the) Hamiltonian vector field on $\tilde{\Gamma}$ generated by the constraint function

$$C_{NN^a}(q,p) := \int_{\Sigma} dS^{mnr}[N(\tfrac{1}{6}p^{ab}{}_{uvw}p_{ab}{}^{uvw} - \tfrac{1}{12}p^{s}{}_{suvw}p^{t}{}_{t}{}^{uvw} - \mathscr{R})\varepsilon_{mnr} - 2N_b D_a p^{ab}{}_{mnr}]. \quad (3)$$

Here, the lapse N and the shift N^a are C^∞ fields on Σ, which, in the asymptotically flat case, vanish at a suitable rate at infinity.[4] The reduced phase space of general relativity is the "manifold of orbits" of these constraint vector fields, restricted to $\bar{\Gamma}$. Denote it by $\hat{\Gamma}$. The tangent space $\hat{T}$ at any point $(\hat{q},\hat{p})$ of $\hat{\Gamma}$ can be identified with the quotient $\bar{T}/\bar{S}$ of the tangent space $\bar{T}$ at any point $(\bar{q},\bar{p})$ of $\bar{\Gamma}$ (which projects down to $(\hat{q},\hat{p})$ in $\hat{\Gamma}$) by its subspace $\bar{S}$ which is spanned by the constraint vector fields. Hence, $\hat{\Gamma}$ inherits from $\bar{\Gamma}$ a weakly non-degenerate symplectic structure $\hat{\Omega}$:

$$\hat{\Omega}|_{(\hat{q},\hat{p})}((\hat{\alpha},\hat{\beta});(\hat{\alpha}',\hat{\beta}')) = \bar{\Omega}|_{(\bar{q},\bar{p})}((\bar{\alpha},\bar{\beta});(\bar{\alpha}',\bar{\beta}')), \quad (4)$$

where $(\bar{\alpha},\bar{\beta})$ is any element of $\bar{T}$ which projects to $(\hat{\alpha},\hat{\beta})$ in $\hat{T}$. (For details, see, e.g. [13], [14], or [15].)

2.2. *The Phase Space of Radiative Modes in the Exact Theory*

Fix a 3-manifold $\mathscr{I}$, topologically $S^2 \times R$, equipped with a collection of pairs of C^∞ fields (q_{ab}, n^a), with q_{ab} symmetric, satisfying the following conditions: i) $q_{ab}V^b = 0$ if and only if V^a is proportional to n^a; ii) $\mathscr{L}_n q_{ab} = 0$; iii) (q,n) and $(\bar{q},\bar{n})$ are both in the collection if and only if there exists a function ω on $\mathscr{I}$ such that $\bar{q}_{ab} = \omega^2 q_{ab}, \bar{n}^a = \omega^{-1} n^a$ and $\mathscr{L}_n \omega = 0$; and, iv) n^a is a complete vector field and the space of its orbits is diffeomorphic to S^2. Thus $\mathscr{I}$ is equipped with the "universal structure" of Penrose's null infinity [16]. Fix a conformal frame—i.e. a pair (q,n) from the collection—on $\mathscr{I}$ and denote by $\mathscr{C}$ the affine space of torsion-free connections D on $\mathscr{I}$ satisfying

$$D_a q_{bc} = 0 \quad \text{and} \quad D_a n^b = 0. \quad (5.a)$$

Finally, introduce the following equivalence relation on $\mathscr{C}$:

$$D \sim D' \qquad \text{if and only if} \qquad (D_a - D'_a)K_b = f q_{ab} n^c K_c \quad (5.b)$$

for any function f on $\mathscr{I}$ (independent of the choice of K_a). Denote the space of equivalence classes $\{D\}$ by Γ. This is the required space of radiative modes of the non-linear gravitational field in exact general relativity. Let us examine the structure available on Γ. It is easy to show that connections D and $\bar{D}$ both belong to $\mathscr{C}$ if and only if there exists a symmetric tensor field Σ_{ab} with $\Sigma_{ab}n^b = 0$ such that $(D_a - \bar{D}_a)K_b = \Sigma_{ab}K_c n^c$ for all K_a on $\mathscr{I}$. Hence it follows that the difference between any two elements $\{D\}$ and $\{\bar{D}\}$ of Γ can be completely characterized by the trace-free part, γ_{ab}, of Σ_{ab}. Thus, Γ has the structure of an affine space; by fixing any one

4 N and N^a have to fall-off "at least as $1/r$" for C_{N,N^a} to be a C^1 function on $\tilde{\Gamma}$, i.e., to generate a Hamiltonian vector field. According to the Dirac theory of constrained systems, such vector fields generate gauge motions

point $\{D^0\}$ as the "origin," one can coordinatize Γ by tensor fields γ_{ab} satisfying $\gamma_{ab} = \gamma_{(ab)}, \gamma_{ab}n^b = 0$, and $\gamma_{ab}q^{ab} = 0$. (The two independent components of γ_{ab} represent the two radiative modes of the gravitational field.) Finally, the following symplectic tensor field has been introduced on Γ:

$$\Omega|_{\{D\}}(\gamma, \gamma') := \int_{\mathscr{I}} (\gamma_{ab}\mathscr{L}_n\gamma'_{cd} - \gamma'_{ab}\mathscr{L}_n\gamma_{cd})q^{ac}q^{bd}\varepsilon_{mnr}dS^{mnr}. \tag{6}$$

Here, q^{ab} is any "inverse" of q_{ab} and ε_{abc} is the unique 3-form on $\mathscr{I}$ satisfying $\varepsilon_{abc}\varepsilon^{abc} = 3!$ where ε^{abc} is defined by $\varepsilon^{abc}\varepsilon^{mnp}q_{am}q_{bn} = n^c n^p$ [17]. It is easy to verify that Ω is conformally invariant and weakly non-degenerate and has the dimensions of action. These properties, together with the pattern suggested by the symplectic structures of zero rest mass, spin zero and one fields, provided the original motivation behind this choice of Ω. Further evidence came from the fact that the action of the BMS group of $\mathscr{I}$ induces motions on Γ which preserve Ω and the Hamiltonians generating these canonical transformations on (Γ, Ω) provide the formulae for fluxes of energy-momentum, supermomentum and angular momentum carried away by the gravitational waves. (For details, see [10] and [11].)

2.3. The Symplectic Tensor of Linearized Gravitational Fields

Fix a globally hyperbolic, vacuum space-time $(\mathbf{M}, g_{ab})$. Let us suppose that this space-time admits a foliation by Cauchy surfaces which are either compact or asymptotically flat at spatial infinity in a suitable sense. Denote by h_{ab} a solution to the linearized vacuum equation:

$$\nabla^m\nabla_m h_{ab} + 2R_{ambn}h^{mn} - 2\nabla_{(a}\nabla_{|m|}(h_{b)}{}^m - \tfrac{1}{2}h\delta_{b)}{}^m) = 0, \tag{7}$$

where ∇ and $R_{abc}{}^d$ are respectively the derivative operator and the Riemann tensor on $(\mathbf{M}, g_{ab})$. It is easy to verify that $h_{ab} = \nabla_{(a}\zeta_{b)}$ satisfies Eq. (7) for arbitrary vector fields ζ^b. Such solutions represent "pure gauge" linearized fields. Denote by $\bar{V}$ the space of C^∞ solutions to Eq. (7) the intersection of whose support with any Cauchy surface is compact and by $\hat{V}$ the quotient of $\bar{V}$ by the subspace containing pure gauge fields $\nabla_{(a}\zeta_{b)}$, where the support of ζ^a has a compact intersection with any Cauchy surface. Consider the skew tensor $\bar{\omega}$ on $\bar{V}$, defined by:

$$\bar{\omega}(h, h') = 3\int_{\Sigma} \varepsilon^{mnp}{}_q(h_{ms}\nabla_n h'_{pr} - h'_{ms}\nabla_n h_{pr})dS^{qrs}. \tag{8}$$

It is easy to check, using Eq. (7), that the integrand is a curl-free 3-form whence the integral is independent of the choice of the Cauchy surface Σ. Using this property (or, by direct substitution in Eq. (8)) one can show [18] that $\bar{\omega}(h, h') = 0$ for all h'_{ab} in $\bar{V}$ if h_{ab} is a pure gauge field, i.e., if $h_{ab} = \nabla_{(a}\zeta_{b)}$ for some ζ_b. Hence $\bar{\omega}$ induces a skew tensor $\hat{\omega}$ on $\hat{V}$:

$$\hat{\omega}(\{h\}, \{h'\}) = \bar{\omega}(h, h'), \tag{9}$$

where $\{h\}$ in $\hat{V}$ denotes the equivalence class of elements of $\bar{V}$ to which h in $\bar{V}$ belongs. This $\hat{\omega}$ can be shown to be weakly non-degenerate[5]: $\hat{\omega}(\{h\}, \{h'\}) = 0$ for

5 This holds provided the background metric g_{ab} does not admit Killing fields near spatial infinity

all $\{h\}$ in $\hat{V}$ if and only if $\{h'\} = 0$. Thus, $(\hat{V}, \hat{\omega})$ is a symplectic vector space. This structure has been exploited in the construction of conserved quantities from linearized fields in the case when $(\mathbf{M}, g_{ab})$ admits a Killing field. (For details, see [18] and [19].)

2.4. Asymptotic Behavior of Linearized Fields

The connection between the symplectic structure $\hat{\Omega}$ of the canonical phase space and Ω of the radiative modes will be established in the next section using the symplectic vector space $(\hat{V}, \hat{\omega})$ of linearized fields. Therefore, we shall need information about the asymptotic behavior of the linearized fields h_{ab} at null infinity. Fortunately, an extensive analysis of this issue already exists in the literature. We shall therefore merely quote the required result:

Theorem [20]. *Let (M, g_{ab}) be an asymptotically flat and empty space-time in the sense of Geroch and Horowitz [21]. Denote by $(\hat{M} = M \cup \mathscr{I}, \hat{g}_{ab} = \alpha^2 g_{ab})$ one of its conformal completions in which $\mathscr{I}$ is divergence-free* (i.e., *in which α satisfies $\hat{\nabla}^m \hat{\nabla}_m \alpha = 0$ on $\mathscr{I}$). Let h'_{ab} be a C^∞ solution to the linearized vacuum equation in a neighborhood of $\mathscr{I}$, the intersection of whose support with some Cauchy surface is compact. Then, there exists a solution h_{ab}, related to h'_{ab} by a gauge transformation, such that $\mathbf{h}_{ab} := \alpha h_{ab}$, $h_{ab}n^b$ and $\alpha^{-1} h_{ab} n^a n^b$ are C^∞ fields on $\mathscr{I}$ and $\hat{g}^{ab} \mathscr{L}_n \mathbf{h}_{ab}$ vanishes on $\mathscr{I}$, where, $n^a = \hat{\nabla}^a \alpha$ is the null normal to $\mathscr{I}$.*

3. Relation between $\hat{\Omega}$ and Ω

This section is divided into two parts. In the first, we investigate the relation between the reduced phase-space $(\hat{\Gamma}, \hat{\Omega})$ and the symplectic vector space $(\hat{V}, \hat{\omega})$ constructed from the linearized solutions, and, in the second, that between $(\hat{V}, \hat{\omega})$ and the phase-space (Γ, Ω) of radiative modes.

3.1. Relation Between $(\hat{\Gamma}, \hat{\Omega})$ and $(\hat{V}, \hat{\omega})$

Consider a globally hyperbolic, vacuum space-time (M, g_{ab}). Fix a Cauchy surface Σ and denote by $(q_{ab}, p^{ab}{}_{mnr})$ the point of the constraint submanifold of the canonical space $\tilde{\Gamma}$ defined by the initial data of g_{ab} on Σ. We shall assume that either Σ is compact or that the pair $(q_{ab}, p^{ab}{}_{mnr})$ is asymptotically flat in a suitable sense and show that there exists a natural one to one mapping $\hat{\Psi}$ from $\hat{V}$ to the tangent space $\hat{T}_{(q,p)}$ of $\hat{\Gamma}$ at $(\hat{q}_{ab}, \hat{p}^{ab}{}_{mnr})$ and that $\hat{\Psi}$ maps $\hat{\Omega}$ to $\hat{\omega}$.

Lemma 1.1. *Given a linearized solution h'_{ab} in $\bar{V}$, there exists a gauge related solution h_{ab} in $\bar{V}$ satisfying $h_{ab}n^b \triangleq 0$, where n^b is the unit normal to Σ and $\triangleq$ denotes equality restricted to Σ.*

Proof. Consider a scalar field N and a vector field N^a on M satisfying

$$N \triangleq 0, N^a \triangleq 0, 2n^a \nabla_a N \triangleq n^a n^b h'_{ab} \quad \text{and} \quad q^{bm} n^a \nabla_a N_b \triangleq -n^a q^{bm} h'_{ab}.$$

Set $h_{ab} = h'_{ab} + 2\nabla_{(a}\xi_{b)}$ with $\xi^a = Nn^a + N^a$, where n^a now denotes an unit time-like, hypersurface orthogonal extension to M of the normal to Σ. This h_{ab} is again in $\bar{V}$ and satisfies $h_{ab}n^b \triangleq 0$. □

Denote by V the subspace of $\bar{V}$ consisting of linearized solutions h_{ab} satisfying $h_{ab}n^b \hat{=} 0$. By Lemma 1.1, the natural mapping, $h_{ab} \to \{h_{ab}\}$ from V to $\hat{V}$ is onto. (However, it is not one to one: given h_{ab} in $V, h_{ab} + 2\nabla_{(a}\xi_{b)}$ is again in V provided ξ^a satisfies $n^a\nabla_{(a}\xi_{b)} = 0$.) The gauge condition $h_{ab}n^b \hat{=} 0$ is introduced for convenience only; it simplifies the task of computing the first order changes in the intrinsic metric q_{ab} and the extrinsic curvature π^{ab} of Σ. A simple calculation yields the following result: $\delta q_{ab} \hat{=} h_{ab}$ and $\delta\pi^{ab} \hat{=} \frac{1}{2}q^a{}_c q^b{}_d \mathscr{L}_n(q^{mc}q^{nd}h_{mn})$. Since $p^{ab}{}_{mnr}$ is related to π^{ab} via

$$p^{ab}{}_{mnr} = (\pi^{ab} - \pi q^{ab})\varepsilon_{mnr},$$

one has,

$$\delta p^{ab}{}_{mnr} \hat{=} [\tfrac{1}{2}q^a{}_c q^b{}_d \mathscr{L}_n q^{sc}q^{td}h_{st} - \tfrac{1}{2}(\mathscr{L}_n q^{st}h_{st})q^{ab} + \pi h^{ab} + \tfrac{h}{2}(\pi^{ab} - \pi q^{ab})]\varepsilon_{mnr}. \quad (10)$$

Thus, the natural mapping Ψ from V to the tangent space $\bar{T}_{(q,p)}$ of the constraint sub-manifold $\bar{\Gamma}$ at the point $(q_{ab}, p^{ab}{}_{mnr})$ is given by

$$\Psi(h_{ab}) = (\alpha_{ab}, \beta^{ab}{}_{mnr}), \quad (11)$$

where $\alpha_{ab} \equiv \delta q_{ab} \hat{=} h_{ab}$ and $\beta^{ab}{}_{mnr} \equiv \delta p^{ab}{}_{mnr}$ has the expression given in Eq. (10). We now ask if Ψ gives rise, naturally, to a mapping $\hat{\Psi}$ from $\hat{V}$ to $\hat{T}_{(\hat{q},\hat{p})}$, i.e., if Ψ maps "pure gauge" linearized fields $h_{ab} \equiv 2\nabla_{(a}\xi_{b)}$ to a tangent vector in $\bar{\Gamma}$ representing an infinitesimal "gauge motion." An affirmative answer is given by the following result:

Lemma 1.2. *Let $h_{ab} = 2\nabla_{(a}\xi_{b)}$ be in V, where the intersection of the support of ξ_b with any Cauchy surface is compact. Then, $\Psi(h_{ab}) \equiv (\alpha_{ab}, \beta^{ab}{}_{mnr})$ is the restriction to $(q_{ab}, p^{ab}{}_{mnr})$ of the Hamiltonian vector field generated on $\bar{\Gamma}$ by the constraint function $C_{N,N^a}(q,p)$ (of Eq. (3)), where the lapse N and the shift N^a are given by $N \hat{=} -\xi_a n^a$ and $N^a \hat{=} q^a{}_m\xi^m$. (Here, as before, n^a is the unit (future-directed) normal to Σ and $q^a{}_m = \delta^a{}_m + n^a n_m$ is the projection operator associated with Σ).*

Proof. Since h_{ab} belongs to $V, n^a\nabla_{(a}\xi_{b)} \hat{=} 0$. This implies that $\nabla_a N \hat{=} 0$ and $\mathscr{L}_n N^a \hat{=} 0$. Using these properties, it is straightforward to show that

$$\Psi(h_{ab}) \hat{=} (2N\pi_{ab} + \mathscr{L}_N q_{ab}, N(-\mathscr{R}^{ab} + \mathscr{R}q^{ab} + \tfrac{1}{6}p^{ab}{}_{cdf}p^t{}_t{}^{cdf} - \tfrac{1}{3}p^a{}_{tcdf}p^{btcdf})\varepsilon_{mnr} + \mathscr{L}_N p^{ab}{}_{mnr}),$$

where π_{ab} is the extrinsic curvature of Σ and $\mathscr{R}^{ab}$ is the Ricci tensor of q_{ab}. On the other hand, using the fact that N and N^a have compact spatial support, one can show [14] that the restriction to $(q_{ab}, p^{ab}{}_{mnr})$ in $\bar{\Gamma}$ of the Hamiltonian vector field generated by $C_{N,N^a}(q,p)$ is given by:

$$(\varepsilon_{mnr}(\delta C/\delta p^{ab}{}_{mnr}), (-\delta C/\delta q_{ab})\varepsilon_{mnr}) = \Bigg(\frac{N}{3}(p_{ab}{}^{uvw} - \tfrac{1}{2}q_{ab}p^m{}_m{}^{uvw})\varepsilon_{uvw} + \mathscr{L}_N q_{ab},$$
$$N(-\mathscr{R}^{ab} + \tfrac{1}{2}\mathscr{R}q^{ab} + \tfrac{1}{6}p^{ab}{}_{uvw}p^c{}_c{}^{uvw} - \tfrac{1}{3}p^a{}_{cuvw}p^{bcuvw} + \tfrac{1}{12}p^{mn}{}_{uvw}p_{mn}{}^{uvw}q^{ab}$$
$$- \tfrac{1}{24}p^c{}_{cuvw}p^d{}_d{}^{uvw}q^{ab})\varepsilon_{mnr} + \mathscr{L}_N p^{ab}{}_{mnr}\Bigg).$$

Since $(q_{ab}, p^{ab}{}_{mnr})$ satisfy the constraint equation (2.b), we have

$$\Psi(h_{ab}) = ((\delta C/\delta p^{ab}{}_{mnr})\varepsilon_{mnr}, (-\delta C/\delta q_{ab})\varepsilon_{mnr}). \quad \square$$

Remarks. (i) Since h_{ab} is an element of V, it follows that $\nabla_{(a}\xi_{b)}$ has compact spatial support. Note, however, that ξ^a need not share this property: ξ^a may be a Killing field outside a bounded world tube. This is why an explicit condition on the support of ξ^a had to be imposed in the statement of Lemma 1.2. (ii) The requirements on the support of h_{ab} and ξ^a can be weakened substantially without altering the essence of the results contained in Lemmas 1.1 and 1.2. Let us suppose, for example, that (M, g_{ab}) is asymptotically flat at spatial infinity in the sense of [22] and denote by $(\hat{M} \equiv M \cup i^0, \hat{g}_{ab} \equiv \alpha^2 g_{ab})$, one of its conformal completions. Consider on (M, g_{ab}) linearized solutions h_{ab} which preserve the requirements of asymptotic flatness to first order. (Thus, h_{ab} need not have compact spatial support, it may fall off only "as $1/r$" at space-like infinity.) Denote this space by $\bar{V}'$. Consider as gauge those elements of $\bar{V}'$ which are of the form $\nabla_{(a}\xi_{b)}$ where, on $\hat{M}$, ξ^a falls-off as $\alpha^{3/2}$. (Thus the one parameter group of diffeomorphisms generated by ξ^a is asymptotically identity.) Denote by $\hat{V}'$ the quotient of $\bar{V}'$ by its gauge subspace. We could have used $\bar{V}'$ and $\hat{V}'$ in place of $\bar{V}$ and $\hat{V}$ in Lemmas 1.1 and 1.2 and the above enlarged class of vector fields ξ^a in Lemma 1.2. Note, incidentally, that if ξ^a generates (non-identity) spin symmetries [22], $h_{ab} = \nabla_{(a}\xi_{b)}$ belongs to $\bar{V}'$ but *not* to its gauge subspace. Hence $\Psi(h_{ab})$ is *not* the restriction to $(q_{ab}, p^{ab}{}_{mnr})$ of the Hamiltonian vector field generated by C_{N,N^a} for *any* choice of N and N^a: If ξ^a falls off slower than $\alpha^{3/2}$, C_{N,N^a} defined by $N \hat{=} \xi^a n_a$, $N^a \hat{=} q^a{}_b \xi^b$ fails to be C^1 on $\bar{\Gamma}$ and therefore cannot lead to a Hamiltonian vector field.

Lemmas 1.1 and 1.2 imply that $\hat{\Psi}$ given by $\hat{\Psi} \circ \{h\} \equiv \{\Psi \circ h\} (\in \bar{T}/\bar{S})$ is a well-defined mapping from $\hat{V}$ to the tangent space $\hat{T}_{(\hat{q},\hat{p})}$ of the reduced space $\hat{\Gamma}$. We now ask if this mapping preserves the symplectic structure. One has:

Theorem 1. *$\hat{\omega}(\{h\}, \{h'\}) = \hat{\Omega}(\hat{\Psi} \circ \{h\}, \hat{\Psi} \circ \{h'\})$ for all $\{h\}$ and $\{h'\}$ in $\hat{V}$.*

Proof. By Lemma 1.1, we can choose from the equivalence classes $\{h\}$ and $\{h'\}$ elements h_{ab} and h'_{ab} in V. Let us make such a choice. Then,

$$\begin{aligned}
\bar{\omega}(h, h') &\equiv 3 \int_\Sigma \varepsilon^{mnp}{}_q (h_{ms} \nabla_n h'_{pr} - h'_{ms} \nabla_n h_{pr}) dS^{qrs} \\
&= \tfrac{1}{2} \int_\Sigma \varepsilon^{mnp}{}_q (h_{ms} \nabla_n h'_{pr} - h'_{ms} \nabla_n h_{pr}) \varepsilon^{qrst} n_t dV_\Sigma \\
&= -\tfrac{1}{2} \int_\Sigma \{ h \mathscr{L}_n (h'_{ab} q^{ab}) - h^{ab} \mathscr{L}_n h'_{ab} + h h'_{ab} \pi^{ab} - (h \leftrightarrow h') \} dV_\Sigma,
\end{aligned}$$

where in the last step, we have used the gauge condition $h_{ab} n^b \hat{=} 0$ and $h'_{ab} n^b \hat{=} 0$. On the other hand, one has

$$\begin{aligned}
\bar{\Omega}(\Psi(h), \Psi(h')) &\equiv \int_\Sigma (\alpha_{ab} \beta'^{ab}{}_{mnr} - \alpha'_{ab} \beta^{ab}{}_{mnr}) dS^{mnr} \\
&= \int_\Sigma \{ h_{ab} [\tfrac{1}{2} \mathscr{L}_n h'_{mn} q^{am} q^{bn} - \tfrac{1}{2} q^{ab} \mathscr{L}_n (h'^{cd} q_{cd}) + \pi h'^{ab} \\
&\quad + \tfrac{1}{2} (h'^{cd} q_{cd})(\pi^{ab} - \pi q^{ab})] - (h \leftrightarrow h') \} dV_\Sigma
\end{aligned}$$

$$= -\tfrac{1}{2}\int_{\Sigma} \{h\mathscr{L}_n(h'^{cd}q_{cd}) - h^{ab}\mathscr{L}_n h'_{ab} + hh'_{ab}\pi^{ab} - (h \leftrightarrow h')\} dV_{\Sigma}.$$

Thus[6], $\bar{\omega}(h,h') = \bar{\Omega}(\Psi(h), \Psi(h'))$. Hence, by Eqs. (4) and (9), we have $\hat{\omega}(\{h\},\{h'\}) = \hat{\Omega}(\hat{\Psi}\circ\{h\}, \hat{\Psi}\circ\{h'\})$ for all $\{h\}$ and $\{h'\}$ in $\hat{V}$. □

Remarks. (i) Consider the case when $(\mathbf{M}, g_{ab})$ admits no Killing field in a "neighborhood of spatial infinity." Then, the symplectic structure $\hat{\omega}$ on $\hat{V}$ is weakly non-degenerate. From Theorem 1 it now follows that the mapping $\hat{\Psi}$ must be one to one. (ii) If we had enlarged $\bar{V}$ to $\bar{V}'$ as indicated in the remark following Lemma 1.2, the mapping $\hat{\Psi}$ would have been an isomorphism between $\hat{V}'$ and $\hat{\Gamma}$. Without this enlargement, however, $\hat{\Psi}$ is only an imbedding of $\hat{V}$ into $\hat{\Gamma}$.

3.2. Relation Between $(\hat{V}, \hat{\omega})$ and (Γ, Ω)

Let us now make further assumptions on the background space-time $(\mathbf{M}, g_{ab})$: let us suppose that $(\mathbf{M}, g_{ab})$ is asymptotically flat at null infinity and is free of horizons, i.e., has the property that the causal past of the future null infinity is all of $\mathbf{M}$. We shall now show that the natural mapping from $\hat{V}$ to the tangent space $T_{\{D\}}$ of Γ (at the point $\{D\}$ corresponding to the physical metric g_{ab}) sends Ω to $\hat{\omega}$.

Fix a conformal completion $(\hat{\mathbf{M}} = \mathbf{M} \cup \mathscr{I}, \hat{g}_{ab} = \alpha^2 g_{ab})$ of $(\mathbf{M}, g_{ab})$ in which $\mathscr{I}$ is divergence-free, i.e., in which α satisfies $\hat{\nabla}^a\hat{\nabla}_a\alpha \mathrel{\hat{=}} 0$ $(\Leftrightarrow \hat{\nabla}_a\hat{\nabla}_b\alpha \mathrel{\hat{=}} 0)$. (In this subsection, $\mathrel{\hat{=}}$ will denote equality restricted to points of $\mathscr{I}$.) Consider a linearized field h_{ab} in $\bar{V}$ satisfying the Geroch–Xanthopoulos [20] gauge conditions. Thus, $\alpha h_{ab}, h_{ab}n^b$, and $\alpha^{-1}h_{ab}n^a n^b$ have C^∞ limits on $\mathscr{I}$ and $\mathscr{L}_n h_{ab}q^{ab} \mathrel{\hat{=}} 0$, where n^a and q_{ab} are respectively the null normal and the degenerate intrinsic metric on $\mathscr{I}$ and q^{ab} is any "inverse" of q_{ab}. We wish to compute the linearized connection "$\delta\{D\}$" induced on $\mathscr{I}$ by h_{ab}.

Let us first recall how one obtains the equivalence class $\{D\}$ on $\mathscr{I}$ starting from the space-time metric g_{ab}. Given a covector field $\mathbf{K}_b$ on $\mathscr{I}$, set $D_a\mathbf{K}_b = \underleftarrow{\hat{\nabla}_a K_b}$, where K_b is any C^∞ extension of $\mathbf{K}_b$ to a neighbourhood of $\mathscr{I}$ in $\hat{\mathbf{M}}$, $\hat{\nabla}$ is the derivative operator compatible with $\hat{g}_{ab}$ on $\hat{\mathbf{M}}$ and where the arrow stands for "pull-back to $\mathscr{I}$." Since any two C^∞ extensions, K_b and K'_b of $\mathbf{K}_b$ are related by $K'_b - K_b = f\hat{\nabla}_b\alpha + \alpha V_b$, where f and V_b admit C^∞ limit to $\mathscr{I}$, and since $\alpha \mathrel{\hat{=}} 0$ and $\hat{\nabla}_a\hat{\nabla}_b\alpha \mathrel{\hat{=}} 0$, it follows that $D_a\mathbf{K}_b$ is independent of the choice of the extension of $\mathbf{K}_b$. By requiring that $D_a f$ be the gradient of f for all C^1 functions f on $\mathscr{I}$ and that D be linear and satisfy the Leibnitz rule with respect to outer product, one can extend the action of D_a uniquely to arbitrary tensor fields within $\mathscr{I}$. Thus, D may be thought of as the pull-back to $\mathscr{I}$ of $\hat{\nabla}$. It automatically satisfies Eq. (5.a). Under the permissible conformal rescalings of $\hat{g}_{ab}$, both $\hat{\nabla}$ and D change. One is therefore led to introduce the equivalence relation of Eq. (5.b): Whereas D refers to a specific conformal completion $(\hat{M}, \hat{g}_{ab})$ of (M, g_{ab}), $\{D\}$ refers to all permissible completions. More specifically, $\{D\}$ represents the radiative modes of the gravitational field associated with the physical metric g_{ab}. [11]

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6 In the light of this result, Lemma 1.2 may seem superfluous. However, it is not: we have not shown that Ψ is on to

Let us now consider a one-parameter family of vacuum metrics $g_{ab}(\lambda)$ such that $g_{ab}(0) = g_{ab}$, the given background, and $\frac{d}{d\lambda} g_{ab}(\lambda)|_{\lambda=0} = h_{ab}$, the given linearized metric. Denote by ${}^{\lambda}\nabla$ the one parameter family of connections associated with[7] $\hat{g}_{ab}(\lambda) = \alpha^2 g_{ab}(\lambda)$, and by ${}^{\lambda}D$ the corresponding family of connections on $\mathscr{I}$. It is easy to check that the first order changes in the connections $\hat{\nabla}$ and D induced by h_{ab} are given by:

$$(\delta\hat{\nabla})_a K_b := \frac{d}{d\lambda} {}^{\lambda}\nabla_a K_b|_{(\lambda=0)} = -\tfrac{1}{2}\hat{g}^{cd}(\hat{\nabla}_a \Omega^2 h_{bd} + \hat{\nabla}_b \Omega^2 h_{ad} - \hat{\nabla}_d \Omega^2 h_{ab})K_c,$$

and

$$(\delta D)_a \mathbf{K}_b := \frac{d}{d\lambda} {}^{\lambda}D_a \mathbf{K}_b|_{(\lambda=0)} = \mathrm{Lim}(-2\alpha) h_{ab}(\hat{g}^{cd}\hat{\nabla}_d \alpha) K_c \hat{=} -2\mathbf{h}_{ab} n^c K_c,$$

where K_b and $\mathbf{K}_b$ are, respectively, arbitrary covector fields on $\hat{M}$ and $\mathscr{I}$, n^a is the null normal to $\mathscr{I}$ and $\mathbf{h}_{ab} = \mathrm{Lim}_{\to\mathscr{I}}\, \alpha h_{ab}$. It now follows that the tangent vector to Γ at the point $\{D\}$, defined by the linearized perturbation h_{ab} is given by[8] the trace-free part of $\mathbf{h}_{ab}$:

$$\delta\{D\} \equiv \gamma_{ab} = \mathbf{h}_{ab} - \tfrac{1}{2}\mathbf{h}_{mn} q^{mn} q_{ab}. \tag{12}$$

Thus, we have a natural mapping Φ from the subspace $\tilde{V}$ of $\bar{V}$, consisting of linearized solutions satisfying the Geroch–Xanthopoulos gauge, to the tangent space $T_{\{D\}}$ of Γ at the point $\{D\}$:

$$\Phi(h_{ab}) = \mathbf{h}_{ab} - \tfrac{1}{2}\mathbf{h}_{mn} q^{mn} q_{ab}. \tag{13}$$

We can now ask if Φ gives rise to a symplectic structure preserving mapping $\hat{\Phi}$ from $\hat{V}$ to Γ. To answer this question, we first establish two results.

Lemma 2.1. *Let $h_{ab} = 2\nabla_{(a}\xi_{b)}$ where ξ^a admits a C^∞ limit to $\mathscr{I}$. Let, furthermore, the restriction to $\mathscr{I}$ of ξ^a be tangential to $\mathscr{I}$. Then,*

$$(\delta D)_a \mathbf{K}_b \equiv -2\mathbf{h}_{ab} n^c \mathbf{K}_c = (\mathscr{L}_\xi D_a - D_a \mathscr{L}_\xi)\mathbf{K}_b.$$

Proof. Since $h_{ab} = \mathscr{L}_\xi g_{ab}$, one can take for $g_{ab}(\lambda)$ the images of g_{ab} under the one parameter group of diffeomorphisms generated by ξ^a; these metrics automatically satisfy $g_{ab}(0) = g_{ab}$ and $\frac{d}{d\lambda} g_{ab}(\lambda)|_{\lambda=0} = h_{ab}$. With this choice, one has, for all covector fields K_b on $\hat{M}$,

$$(\delta\hat{\nabla})_a K_b = (\mathscr{L}_\xi \hat{\nabla}_a - \hat{\nabla}_a \mathscr{L}_\xi) K_b.$$

Hence, using the fact that ξ^a is tangential to $\mathscr{I}$, one has:

$$(\delta D)_a \mathbf{K}_b \hat{=} (\mathscr{L}_\xi \hat{\nabla}_a - \hat{\nabla}_a \mathscr{L}_\xi) K_b \hat{=} \mathscr{L}_\xi D_a \mathbf{K}_b - D_a \mathscr{L}_\xi \mathbf{K}_b,$$

where $\mathbf{K}_b$ is the pull-back to $\mathscr{I}$ of K_b. □

7 The dependence of α on λ can be removed by a gauge transformation

8 Recall that a tangent vector on Γ is represented by a symmetric trace-free tensor field γ_{ab} on $\mathscr{I}$, satisfying $\gamma_{ab} n^b \hat{=} 0$. By Geroch–Xanthopoulos gauge conditions, $\mathbf{h}_{ab}$ satisfies $\mathbf{h}_{ab} n^b \hat{=} 0$

Lemma 2.2. *Let* $h_{ab} = 2\nabla_{(a}\xi_{b)}$ *be in* $\tilde{V}$, i.e. *satisfy the Geroch–Xanthopoulos gauge conditions. Let, furthermore, the intersection of the support of* ξ^a *with any Cauchy surface be compact. Then* $\Phi(h_{ab}) = 0$.

Proof. A detailed examination of the procedure by which Geroch and Xanthopoulos impose their gauge conditions shows that ξ^a admits a C^∞ limit to $\mathscr{I}$. Hence $\mathscr{L}_\xi \hat{g}_{ab}$ is also C^∞ on $\mathscr{I}$. However, $\mathscr{L}_\xi \hat{g}_{ab} = (\alpha^{-1}\mathscr{L}_\xi\alpha)\hat{g}_{ab} + \alpha^2 \mathscr{L}_\xi g_{ab} \hat{=} (\alpha^{-1}\mathscr{L}_\xi\alpha)\hat{g}_{ab}$, since $\alpha^2 h_{ab} \hat{=} 0$. Thus, $\mathscr{L}_\xi \alpha \hat{=} \xi^m \hat{\nabla}_m \alpha \hat{=} \xi^m n_m \hat{=} 0$; ξ^a is tangential to $\mathscr{I}$. Next, the equation $\mathscr{L}_\xi \hat{g}_{ab} \hat{=} (\alpha^{-1}\mathscr{L}_\xi\alpha)\hat{g}_{ab}$ implies that $\xi|_{\mathscr{I}}$ is a BMS vector field. The condition on the support of ξ^a now yields $\xi^a \hat{=} 0$, whence, by Lemma 2.1, $\mathbf{h}_{ab} = 0$. Thus $\Phi(h_{ab}) = \mathbf{h}_{ab} - \frac{1}{2}\mathbf{h}_{mn}q^{mn}h_{ab} = 0$. □

Lemma 2.2 implies that Φ naturally defines a mapping $\hat{\Phi}$ from $\hat{V}$ into the tangent space $T_{\{D\}}$ of Γ: $\hat{\Phi}(\{h_{ab}\}) = \Phi(h_{ab})$, where h_{ab} is the unique element of $\{h_{ab}\}$ satisfying the Geroch–Xanthopoulous gauge. The question now is whether $\hat{\Phi}$ preserves the symplectic structure. We have:

Theorem 2. $\hat{\omega}(\{h\},\{h'\}) = \Omega(\hat{\Phi}\{h\},\hat{\Phi}\{h'\})$, *provided* $h_{ab}\in\{h\}$ *and* $h'_{ab}\in\{h'\}$ *satisfying the Geroch–Xanthopoulos gauge have the property that* $\mathbf{h}_{ab}$ *and* $\mathbf{h}'_{ab}$ *remain bounded at* $i^\pm$.

Proof. $\hat{\omega}(\{h\},\{h'\}) \equiv \bar{\omega}(h,h')$

$$= 3\int_\Sigma \varepsilon^{mnp}{}_q(h_{ms}\nabla_n h'_{pr} - h'_{ms}\nabla_n h_{pr})dS^{qrs},$$

where Σ is any Cauchy surface in $(\mathbf{M}, g_{ab})$. Using the fact that $\hat{\varepsilon}^{mnp}{}_q = \alpha^{-2}\varepsilon^{mnp}{}_q$ and $h_{ab} = \alpha\mathbf{h}_{ab}$, one has:

$$\hat{\omega}(\{h\},\{h'\}) = 3\int_\Sigma \hat{\varepsilon}^{mnp}{}_q(\mathbf{h}_{ms}\nabla_n\mathbf{h}'_{pr} - \mathbf{h}'_{ms}\nabla_n\mathbf{h}_{pr} - \alpha^{-1}(\nabla_n\alpha)\mathbf{h}_{ms}\mathbf{h}'_{pr}$$
$$+ \alpha^{-1}(\nabla_n\alpha)\mathbf{h}'_{ms}\mathbf{h}_{pr})dS^{qrs}.$$

Let us now use a conformal factor such that $\alpha = \text{const}$ surfaces are space-like Cauchy surfaces in a neighborhood of $\mathscr{I}$. Since the 3-form appearing in the integrands above is curl-free, we may choose for Σ a surface defined by $\alpha = \text{const}$. Then, substituting for ∇ in terms of $\hat{\nabla}$ one obtains

$$\hat{\omega}(\{h\},\{h'\}) = 3\int_\Sigma \hat{\varepsilon}^{mnp}{}_q[\mathbf{h}_{ms}\hat{\nabla}_n\mathbf{h}'_{pr} - \mathbf{h}'_{ms}\hat{\nabla}_n\mathbf{h}_{pr} + \alpha^{-1}\hat{\mathbf{g}}_{nr}n^a(\mathbf{h}_{ms}\mathbf{h}'_{pa} - \mathbf{h}'_{ms}\mathbf{h}_{pa})]\,dS^{qrs}.$$

Finally, we replace Σ by $\mathscr{I}$. Using the fact that $\mathbf{h}_{ab}n^b \hat{=} 0$ and $\mathscr{L}_n(\mathbf{h}_{ab}q^{ab}) \hat{=} 0$ one obtains,

$$\hat{\omega}(\{h\},\{h'\}) = \int_{\mathscr{I}} (\gamma_{ab}\mathscr{L}_n\gamma'_{cd} - \gamma'_{ab}\mathscr{L}_n\gamma_{cd})q^{ac}q^{bd}\,d\mathscr{I},$$

where

$$\gamma_{ab} \hat{=} \mathbf{h}_{ab} - \tfrac{1}{2}\mathbf{h}_{mn}q^{mn}q_{ab} \quad \text{and} \quad \gamma'_{ab} \hat{=} \mathbf{h}'_{ab} - \tfrac{1}{2}\mathbf{h}'_{mn}q^{mn}q_{ab}.$$

Thus,

$$\hat{\omega}(\{h\},\{h'\}) = \Omega(\hat{\Phi}\circ\{h\},\hat{\Phi}\circ\{h'\}). \quad \square$$

Remarks. (i) In Sect. 3.1, the restriction to linearized fields h_{ab} with compact spatial support was for convenience only: As noted in the remarks following Lemma 1.2 and Theorem 1, we could have enlarged $\bar{V}$ to $\bar{V}'$ without affecting the results and their proofs in any essential way. The situation is different in Sect. 3.2 because the Geroch–Xanthopoulos analysis [20] itself has to be extended to incorporate the linearized fields which fail to have compact spatial support. Because of the curvature singularity at i^0, such an extension would not be straightforward: αh_{ab} may not be smooth on $\mathscr{I}$ [23]. However, a preliminary investigation into these problems has shown that the possible loss of differentiability of αh_{ab} would not be severe enough to upset the relation between the symplectic structures $\hat{\omega}$ and Ω. (ii) The condition on the behavior at i^+ is required in Theorem 2 to ensure that the integral over a Cauchy surface Σ can be replaced by the one over $\mathscr{I}^+$, i.e., that the leakage through $i^\pm$ can be ignored. Although there do exist examples of source-free test Maxwell fields on asymptotically simple background space-times in which the corresponding leakages for Maxwell fields cannot be ignored [24], one expects from the investigations of gravitational geons that for linearized Einstein fields off vacuum backgrounds, "bound states" would not develop, and hence, that the assumption is not too restrictive. In particular, one hopes that a large family of vacuum space-times may exist in which the entire ADM mass is radiated away by some finite retarded time so that the space-time is *flat* in a neighborhood of i^+. Linearized fields off such backgrounds would automatically satisfy the condition in question. (iii) The results of this subsection shed some light on two issues in particular. The first concerns the Geroch–Xanthopoulos analysis where the result that $\alpha^2 h_{ab}$ vanishes on $\mathscr{I}$ came as a surprise: since $\alpha^2 h_{ab}$ is the perturbation of the rescaled metric $\hat{g}_{ab}$, the intuitive ideas on stability of $\mathscr{I}$ lead one to expect only that $\alpha^2 h_{ab}$ would admit smooth limits to $\mathscr{I}$. From the symplectic viewpoint, on the other hand, the vanishing at $\mathscr{I}$ of $\alpha^2 h_{ab}$ is essential: For fields which vanish in a neighborhood of $i^\pm$, for example, one can argue, via Theorem 2, that, if $\alpha^2 h_{ab}$ admits a smooth limit to $\mathscr{I}$, this limit must be zero. That is, in the light of Theorem 2, the stronger than expected fall-off of h_{ab} is no more a surprise. The second issue concerns the choice of $\{D\}$ as the basic variable in the construction of Γ: Intuitive considerations lead one to think of *metrics* as the basic variables in Einstein's theory, rather than connections. The analysis preceding Lemma 2.1 resolves this apparent paradox: since $\delta D \hat{=} \mathbf{h}_{ab}$, it is the connections on $\mathscr{I}$ which are the appropriate analogues of the metrics in space-time.

To conclude this discussion, let us summarize the main results of this section. Fix a non-compact 3-manifold Σ and consider thereon a pair $(q_{ab}, p^{ab}{}_{mnr})$ which is asymptotically flat at spatial infinity in a suitable sense and which satisfies the constraint equations (2.a) and (2.b). Let us assume that the vacuum solution g_{ab} of Einstein's equation obtained by evolving $(q_{ab}, p^{ab}{}_{mnr})$ is asymptotically flat at null infinity and free of horizons. Then, for all elements $\{h\}$ and $\{h'\}$ of $\hat{V}$ (which are well-behaved at i^+/i^-), one has:

$$\hat{\Omega}|_{(q,p)}(\hat{\Psi}\circ\{h\}, \hat{\Psi}\circ\{h'\}) = \Omega|_{\{D\}}(\hat{\Phi}\circ\{h\}, \hat{\Phi}\circ\{h'\}),$$

where (q,p) in $\hat{\Gamma}$ is the equivalence class of elements in $\bar{\Gamma}$ to which $(q_{ab}, p^{ab}{}_{mnr})$

belongs and $\{D\}$ is the element of Γ singled out by g_{ab}.

The key question now is whether or not the assumptions on the solution g_{ab} can be satisfied by a large class of initial data sets $(q_{ab}, p^{ab}{}_{mnr})$. If such a class were not to exist, most results from the gravitational radiation theory itself would have to be regarded as uninteresting. In particular, the phase space (Γ, Ω) would have very little physical significance at least in classical general relativity and one would have to make a fresh start all over again. However, if the recent results of Christodoulou et al [25–27] on the "boost problem" as well as those of Friedrich [28] on the characteristic initial value problem on $\mathscr{I}$ are any indications, such a prospect seems rather unlikely.

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