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On the symplectic structures on moduli space of stable sheaves over a K3 or abelian surface and on Hilbert scheme of points.

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ON THE SYMPLECTIC STRUCTURES ON MODULI SPACE OF STABLE SHEAVES OVER A K3 OR ABELIAN SURFACE AND ON HILBERT SCHEME OF POINTS

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Abstract. Fix a smooth very ample curve C on a $K3$ or abelian surface X. Let $\mathcal{M}$ denote the moduli space of pairs of the form (F, s), where F is a stable sheaf over X whose Hilbert polynomial coincides with that of the direct image, by the inclusion map of C in X, of a line bundle of degree d over C, and s is a nonzero section of F. Assume d to be sufficiently large such that F has a nonzero section. The pullback of the Mukai symplectic form on moduli spaces of stable sheaves over X is a holomorphic 2–form on $\mathcal{M}$. On the other hand, $\mathcal{M}$ has a map to a Hilbert scheme parametrizing 0-dimensional subschemes of X that sends (F, s) to the divisor, defined by s, on the curve defined by the support of F. We prove that the above 2–form on $\mathcal{M}$ coincides with the pullback of the symplectic form on Hilbert scheme.

1. Introduction

Let X be a connected smooth projective surface over $\mathbb{C}$ with trivial canonical bundle. In other words, X is either a $K3$ surface or an abelian surface.

Let C be a smooth very ample curve on X. We will use C for defining the *degree* of a coherent sheaf on X. Let $\iota : C \longrightarrow X$ be the inclusion map.

Fix an integer $d \geq 1$. Let $\mathcal{M}_C(d)$ denote the moduli space of stable sheaves F over X such that the Hilbert polynomial of F coincides with the Hilbert polynomial of $\iota_* L$, where L is a holomorphic line bundle of degree d over the curve C. The construction of $\mathcal{M}_C(d)$ can be found in [6, Ch. 4], [11]. The condition on the Hilbert polynomial for F implies that F is a torsion sheaf supported over some curve D on X and, furthermore, F is isomorphic to $j_* V$, where j denotes the inclusion map of D in X and V is a stable sheaf over D. The condition further implies that the support D is homologically equivalent to C.

Since a stable sheaf is simple, from [10, Theorem 0.1] we know that $\mathcal{M}_C(d)$ is a smooth quasiprojective variety of dimension

$$2H^0(X,\, \mathcal{O}_X(C)) + 2 - 2\chi(\mathcal{O}_X) = 2 \cdot \text{genus}(C)\,.$$

Note that $\chi(\mathcal{O}_X) = 0$ if X is an abelian surface and $\chi(\mathcal{O}_X) = 2$ if X is a $K3$ surface.

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Fix a trivialization of K_X. By a well-known construction of Mukai [10, Corollary 0.2], the variety $\mathcal{M}_C(d)$ has a natural holomorphic symplectic structure. This symplectic form on $\mathcal{M}_C(d)$ will be denoted by Θ.

Henceforth, we will assume that $d \geq 2 \cdot \text{genus}(C) - 1$. Therefore, $H^1(X, F) = 0$ for $F \in \mathcal{M}_C(d)$. Note that since F is supported on a curve, we have $H^2(X, F) = 0$. So, $\dim H^0(X, F) = d - \text{genus}(C) + 1$ for any $F \in \mathcal{M}_C(d)$. Let

$$\phi : \mathcal{M} \longrightarrow \mathcal{M}_C(d)$$

be the projectivized Picard bundle over $\mathcal{M}_C(d)$. In other words, $\mathcal{M}$ is a projective bundle and for any $F \in \mathcal{M}_C(d)$ the fiber $\phi^{-1}(F)$ is $PH^0(X, F)$, the space of lines in $H^0(X, F)$.

The variety $\mathcal{M}$ is the moduli space pairs of the form (F, s), where $F \in \mathcal{M}_C(d)$ and $s \in H^0(X, F) \setminus \{0\}$. Note that such a pair is a very special case of more general objects introduced by Le Potier which are known as *coherent systems* (see [8], [9]).

Let

$$\Omega := \phi^*\Theta$$

be the holomorphic two form which is the pullback of Mukai form.

Let $\text{Hilb}^d(X)$ denote the *Hilbert scheme*, which is the moduli space parametrizing 0-dimensional subschemes of X of length d.

A well-known result of Beauville, [1], says that $\text{Hilb}^d(X)$ has a natural holomorphic symplectic structure. Let ω denote the symplectic form on $\text{Hilb}^d(X)$.

Clearly there is a morphism

$$\pi : \mathcal{M} \longrightarrow \text{Hilb}^d(X)$$

that sends a pair (F, s), where F is a stable sheaf supported on a curve D in X and $s \in PH^0(X, F)$, to the divisor on the curve D defined by s. Note that using the inclusion map D in X, a divisor on D is identified with a 0-dimensional subschemes of X.

Although we do not consider the case $d = \text{genus}(C)$, it may be pointed out that if $d = \text{genus}(C)$, then both the maps ϕ and π are birational.

Let

$$\Omega' := \pi^*\omega$$

be the holomorphic two form on $\mathcal{M}$ which is the pullback of Beauville form.

The aim here is to prove

Theorem 1.1. *The two holomorphic 2–forms on $\mathcal{M}$, namely Ω and Ω', coincide.*

Theorem 1.1 will be proved in Section 3.

See [4] and [13] for relationship between Hilbert scheme of points and semistable sheaves on a $K3$ surface. A result relating the symplectic structure on a moduli space of Higgs bundles on a compact Riemann surface Y with that on a Hilbert scheme of points on K_Y can be found in [2].

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2. Properties of the forms

In this section we briefly recall the constructions of Θ and ω and will note some of their properties that will be useful for our purpose.

We once and for all fix a trivialization of the canonical bundle K_X. The section of K_X that defines this trivialization, which we will denote by τ, is a symplectic form on X.

Let $x = \{x_1, x_2, \cdots, x_d\} \in \mathrm{Hilb}^d(X)$ be a point with all x_i distinct. Clearly we have

$$T_x\mathrm{Hilb}^d(X) = \bigoplus_{i=1}^{d} T_{x_i}X .$$

If $v^j = \{v_1^j, v_2^j, \cdots, v_d^j\} \in T_x\mathrm{Hilb}^d(X)$, where $j = 1, 2$ and $v_i^j \in T_{x_i}X$, then

$$\omega(v^1, v^2) = \sum_{i=1}^{d} \tau(v_i^1, v_i^2) , \tag{2.1}$$

where τ is the above defined symplectic form on X. Clearly this defines a holomorphic symplectic form on the Zariski open subset of $\mathrm{Hilb}^d(X)$ defined by reduced subschemes, that is, distinct d points of X. It was proved by Beauville that this form extends to a holomorphic symplectic form on $\mathrm{Hilb}^d(X)$ [1, p. 766–767], which has been denoted by ω.

For any $F \in \mathcal{M}_C(d)$, the tangent space $T_F\mathcal{M}_C(d)$ coincides with $\mathrm{Ext}^1_{\mathcal{O}_X}(F, F)$ [6, Corollary 4.5.2], where Ext is the global ext. Now consider the composition

$$\mathrm{Ext}^1_{\mathcal{O}_X}(F,F) \otimes \mathrm{Ext}^1_{\mathcal{O}_X}(F,F) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(F,F) \longrightarrow H^2(X, \mathcal{O}_X) \xrightarrow{\cup\tau} H^2(X, K_X) \cong \mathbb{C} , \tag{2.2}$$

where the second homomorphism is defined using the trace map [10, p. 114]. This bilinear pairing on $\mathrm{Ext}^1_{\mathcal{O}_X}(F,F)$ is clearly skew-symmetric. The 2–form on $\mathcal{M}_C(d)$ defined by (2.2) is the Mukai symplectic form, which has been denoted by Θ.

Let F be supported on a divisor D in X. The restriction of F to D will also be denoted by F. We recall that the spectral sequence for base change, [12], gives an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_D}(F,F) \xrightarrow{\sigma} \mathrm{Ext}^1_{\mathcal{O}_X}(F,F) \xrightarrow{q} H^0(D, \mathcal{O}_X(D)|_D) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_D}(F,F) . \tag{2.3}$$

Since $\dim D = 1$, we have $\mathrm{Ext}^2_{\mathcal{O}_D}(F, F) = 0$. So (2.3) is a short exact sequence.

If F is a line bundle over D, then $\mathrm{Ext}^1_{\mathcal{O}_D}(F, F) \cong H^1(D, \mathcal{O}_D)$. We note that if D is a smooth divisor, then the line bundle $\mathcal{O}_X(D)|_D$ over D is identified by the Poincaré adjunction formula with the normal bundle $N_D := j^*TX/TD$, where j is the inclusion map, of D.

We noted earlier that $\mathrm{Ext}^1_{\mathcal{O}_X}(F, F)$ parametrizes infinitesimal deformation of F. The projection q in (2.3) corresponds to the infinitesimal deformations of the support of the sheaf, and the inclusion σ corresponds to deforming the sheaf keeping its support fixed.

Suppose that there is a line bundle ξ over X such that $F \cong j_*j^*\xi$, where j, as before, denotes the inclusion map of D to X. In that case F has the projective resolution

$$0 \longrightarrow \xi \otimes \mathcal{O}_X(-D) \longrightarrow \xi \longrightarrow F \longrightarrow 0$$

which is obtained by tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow j_*\mathcal{O}_D \longrightarrow 0\,.$$

Therefore, from the definition of Ext [7, Ch. V, §3], it follows that

$$\mathrm{Ext}^1_{\mathcal{O}_X}(F, F) \cong H^1(D, \mathcal{O}_D) \oplus H^0(D, N_D)\,.$$

In other words, in this case the exact sequence (2.3) has a canonical splitting if F is of the above form $j_*j^*\xi$. As D moves over a family, the restrictions of ξ, which is defined over X, give the splitting. Note that ξ is uniquely determined the condition $F \cong j_*j^*\xi$. To explain this, let ξ_0 be a line bundle over X such that $c_1(\xi_0) = 0$. Now, since the divisor D is ample and $K_X \cong \mathcal{O}_X$, from Kodaira vanishing theorem [5, p. 154] and Serre duality it follows immediately that $H^1(X, \xi_0 \otimes \mathcal{O}_X(-D)) = 0$. Consequently, for the exact sequence

$$0 \longrightarrow \xi_0 \otimes \mathcal{O}_X(-D) \longrightarrow \xi_0 \longrightarrow j_*j^*\xi_0 \longrightarrow 0\,,$$

the homomorphism $H^0(X, \xi_0) \longrightarrow H^0(D, j^*\xi_0)$ is surjective. Therefore, given two line bundles ξ_1 and ξ_2 on X with $c_1(\xi_1) = c_1(\xi_2)$, setting $\xi_0 = \xi_1^* \otimes \xi_2$ we conclude that if $j^*\xi_1 \cong j^*\xi_2$ then $\xi_1 \cong \xi_2$. In other words, the restriction of line bundles to D is injective. Of course, the injectivity of the differential of this restriction homomorphism is the restatement of the fact that $H^1(X, \mathcal{O}_X) \longrightarrow H^1(D, \mathcal{O}_D)$ is injective which in turn follows from the above observation that $H^1(X, \xi_0 \otimes \mathcal{O}_X(-D)) = 0$ by setting $\xi_0 \cong \mathcal{O}_X$.

Note that the symplectic form τ on X identifies N_D with the canonical bundle K_D. The form τ defines a symplectic structure on j^*TX. Since the restriction of this symplectic form to the subbundle $TD \subset j^*TX$, vanishes, it identifies the quotient N_D of j^*TX with $K_D := (TD)^*$. Using this identification and Serre duality, we have

$$H^1(D, \mathcal{O}_D) \cong H^0(D, N_D)^*\,. \tag{2.4}$$

From the definition of Θ in (2.2) it follows immediately that for any $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_D}(F,F)$ and $\beta \in \mathrm{Ext}^1_{\mathcal{O}_X}(F,F)$, the identity

$$\Theta(\sigma(\alpha), \beta) = \alpha(q(\beta)) \tag{2.5}$$

is valid, where σ, q are as in (2.3), and α is identified, using (2.4), with the corresponding element in $H^0(D,\, N_D)^*$.

3. Equality of forms

We start with a simple proposition.

Proposition 3.1. *There is a 2–form Ω_0 on $\mathcal{M}_C(d)$ such that its pullback $\phi^*\Omega_0$ to $\mathcal{M}$ coincides with Ω'.*

Proof. Recall that the map ϕ, defined in the introduction, is a projective bundle. Since there is no nonzero holomorphic 2–form on a projective space, the restriction of the form Ω' to a fiber of ϕ vanishes.

Take any point $y \in \mathcal{M}_C(d)$ and a tangent vector $v \in T_y\mathcal{M}_C(d)$. Take a point $z \in \phi^{-1}(y)$ and $w \in T_z\mathcal{M}$ which is in the kernel of the differential map

$$d\phi(z) \,:\, T_z\mathcal{M} \longrightarrow T_y\mathcal{M}_C(d)$$

for ϕ at z. Take $\overline{v} \in T_z\mathcal{M}$ such that $d\phi(z)(\overline{v}) = v$. Since the restriction of Ω' to $\phi^{-1}(y)$ vanishes, $\Omega'(w, \overline{v})$ is independent of the choice of lift $\overline{v}$ of v. Therefore, v defines a holomorphic 1–form on $\phi^{-1}(y)$. Since a projective space does not have a nonzero holomorphic 1-form, this form must vanish identically.

Therefore, given another tangent vector $v' \in T_y\mathcal{M}_C(d)$ and a lift $\overline{v}' \in T_z\mathcal{M}$ with

$$d\phi(z)(\overline{v}') = v' \,,$$

the pairing $\Omega'(\overline{v}, \overline{v}')$ is independent of the choice of the lifts. Therefore, the pair (v, v') defines a holomorphic function on $\phi^{-1}(y)$ which sends any z to $\Omega'(\overline{v}, \overline{v}')$. Since $\phi^{-1}(y)$ is complete and connected, this must be a constant function.

Let Ω_0 be the holomorphic 2–form on $\mathcal{M}_C(d)$ that sends any pair of tangent vectors (v, v') to the constant $\Omega'(\overline{v}, \overline{v}')$. It is clear from the construction that $\phi^*\Omega_0 = \Omega'$. This completes the proof of the proposition. □

Note that since ψ is surjective, the condition $\phi^*\Omega_0 = \Omega'$ uniquely determines Ω_0.

Note that if we know that $\dim H^0(\mathcal{M}_C(d),\, \Omega^2_{\mathcal{M}_C(d)}) = 1$, then from Proposition 3.1 it follows immediately that Ω' must be a constant scalar multiple of Ω.

We have a natural projection from $\mathcal{M}_C(d)$ to the space of divisors on X homologically equivalent to C – we will denote this space by $\mathcal{S}$ – that sends any sheaf F to its support.

Let

$$\psi : \mathcal{M}_C(d) \longrightarrow \mathcal{S} \tag{3.2}$$

be this projection map. If $D \in \mathcal{S}$ is a reduced smooth curve, then the fiber $\psi^{-1}(D)$ is the component $\mathrm{Pic}^d(D)$ of the Picard group of D. Note that since C is very ample, the points in $\mathcal{S}$ corresponding to reduced smooth divisors in X constitute a Zariski open dense subset.

Proposition 3.3. *For any divisor $D \in \mathcal{S}$, the restrictions of both Θ and Ω_0 to the subvariety $\psi^{-1}(D) \subset \mathcal{M}_C(d)$ vanish identically.*

Proof. Since the support of any sheaf in $\psi^{-1}(D)$ is the fixed divisor D, and when restricted to D, the symplectic form τ vanishes identically, from the definition of ω in (2.1) it follows immediately that the restriction of Ω_0 to $\psi^{-1}(D)$ vanishes identically.

From the identity (2.5) for Θ it follows immediately that the restriction of Θ to $\psi^{-1}(D)$ vanishes identically. This is also proved in Lemma 1.3 of [3]. This completes the proof of the proposition. $\square$

Now we will show that $\Theta - \Omega_0$ is a pullback from $\mathcal{S}$.

Lemma 3.4. *There is a holomorphic 2–form Ω_1 on $\mathcal{S}$ such that*

$$\psi^*\Omega_1 = \Theta - \Omega_0 ,$$

where ψ is defined in (3.2).

Proof. Let $U \subset \mathcal{S}$ be the subvariety defined by all reduced smooth curves in X. It was already noted that U is a Zariski open dense subvariety.

Take any $D \in U$ and a tangent vector $v \in T_D\mathcal{S}$. Take any $L \in \psi^{-1}(D)$. Let

$$d\psi(L) : T_L\mathcal{M}_C(d) \longrightarrow T_D\mathcal{S}$$

be the differential of the map ψ at the point L.

Take any $w \in T_L\mathcal{M}_C(d)$ such that $d\psi(L)(w) = 0$. From Proposition 3.3 it follows that for any $\overline{v} \in T_L\mathcal{M}_C(d)$, with $d\psi(L)(\overline{v}) = v$, both $\Theta(\overline{v}, w)$ and $\Omega_0(\overline{v}, w)$ are independent of the choice of the lift $\overline{v}$ of v.

The identity (2.5) describes $\Theta(\overline{v}, w)$. We need to compare the expression in (2.5) for $\Theta(\overline{v}, w)$ with $\Omega_0(\overline{v}, w)$, which is described using (2.1). For this we will recall a general fact.

Let Y be a compact connected Riemann surface, and let

$$\underline{y} := (y_1, y_2, \cdots, y_l) \in Y^l$$

be a point of the l-fold Cartesian product. Let

$$\gamma : \Delta \longrightarrow Y^l$$

be a holomorphic map of the disk $\Delta := \{z \in \mathbb{C} \big| |z| < 1\}$ such that $\gamma(0) = \underline{y}$. The composition of γ with the projection of Y^l to the i-th factor will be denoted by γ_i. So, $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_l)$. Sending any $t \in \Delta$ to the line bundle over Y defined by the divisor γ we get a map

$$\overline{\gamma} : \Delta \longrightarrow \text{Pic}^l(Y) .$$

The differential of $\overline{\gamma}$ at $0 \in \Delta$ is a homomorphism

$$\zeta : \mathbb{C} \longrightarrow H^1(Y, \mathcal{O}_Y) .$$

Finally, take a holomorphic 1–form

$$\theta \in H^0(Y, K_Y) .$$

Note that $H^0(Y, K_Y) \cong H^1(Y, \mathcal{O}_Y)^*$. For this duality pairing, the identity

$$\theta(\zeta(1)) = \sum_{i=1}^{l} \theta(y_i)(d\gamma_i(1)) \tag{3.5}$$

is valid, where $d\gamma_i$ denotes the differential of the map γ_i at 0.

Using the identity (3.5) it is straight-forward to see that

$$\Theta(\overline{v}, w) = \Omega_0(\overline{v}, w) \tag{3.6}$$

for all w with $d\psi(L)(w) = 0$. Indeed, since $T_D\mathcal{S} \cong H^0(D, N_D) \cong H^0(D, K_D)$, where N_D and K_D are respectively the normal bundle and the canonical bundle of D, the vector $v \in T_D\mathcal{S}$ corresponds to a holomorphic 1–form on D. (We assume D to be smooth; since C is very ample, it suffices to consider smooth divisors.) Let $\theta' \in H^0(D, K_D)$ be the 1–form given by v. If we set the above map γ such that $\zeta(1) = w$, then setting θ in (3.5) to be θ' we conclude that $\Theta(\overline{v}, w)$ coincides with $\sum \theta'(y_i)(d\gamma_i(1))$. On the other hand, from the definition of the symplectic form ω on Hilbert scheme it follows immediately that $\Omega_0(\overline{v}, w) = \sum \theta'(y_i)(d\gamma_i(1))$. This establishes the equality in (3.6).

As we noted earlier, the fiber $\psi^{-1}(D)$ is a component of the Picard group of D. Therefore, there is no nonconstant holomorphic function on $\psi^{-1}(D)$. Now imitating the final part of the proof of Proposition 3.1 we conclude that there is a holomorphic 2–form Ω_1 on U such that $\psi^*\Omega_1 = \Theta - \Omega_0$ over $\psi^{-1}(U)$. But U is Zariski open and dense in $\mathcal{S}$, and $\Theta - \Omega_0$ is defined on $\mathcal{M}_C(d)$. Therefore, Ω_1 also extends to $\mathcal{S}$. This completes the proof of the lemma. □

If X is a $K3$ surface, then Theorem 1.1 follows easily from Lemma 3.4. Indeed, for a $K3$ surface X we have $H^1(X, \mathcal{O}_X) = 0$. Therefore, any deformation of the divisor C is linearly (rationally) equivalent to C. Consequently, $\mathcal{S}$ is the space of all divisors rationally equivalent to C. In other words, it is the projective space $PH^0(X, \mathcal{O}_X(C))$.

But $PH^0(X,\, \mathcal{O}_X(C))$ does not have any nonzero holomorphic 2–form. Consequently, the form Ω_1 in Lemma 3.4 must vanish identically. Therefore, we have

$$\Theta \,=\, \Omega_0$$

if X is a $K3$ surface.

To complete the proof of Theorem 1.1 we assume that X is an abelian surface.

In that case, $\mathcal{S}$ is a projective bundle over a component $\text{Pic}^c(X)$ of the Picard group of X, where $c \,\in\, H^2(X,\, \mathbb{Z})$ is the homology class of C. The fiber of the projection

$$f \,:\, \mathcal{S} \,\longrightarrow\, \text{Pic}^c(X) \tag{3.7}$$

over $L \,\in\, \text{Pic}^c(X)$ is $PH^0(X,\, L)$. Since f is a projective bundle, from the proof of Proposition 3.1 it follows that

$$\Omega_1 \,=\, f^*\Omega_2\, ,$$

where Ω_2 is a holomorphic 2–form on $\text{Pic}^c(X)$.

Using the additive structure of X, and the divisor C, the variety $\text{Pic}^c(X)$ gets identified with X. More precisely, for any $x \in X$, the corresponding divisor is the translation

$$C_x \,:=\, C + x$$

of C by x. Fix once and for all a reduced divisor

$$B \,:=\, \{x_1, x_2, \cdots, x_d\} \,\subset\, C$$

of degree d the Riemann surface C. Note that $B_x \,:=\, B + x$ is a divisor of degree d on C_x. In other word, we have map

$$\lambda \,:\, X \,\longrightarrow\, \mathcal{M}_C(d)$$

that sends x to the pair $(j_*\mathcal{O}_{C_x}(B_x), 1_x)$, where j is the inclusion map of C_x in X and 1_x is the section of the line bundle $\mathcal{O}_{C_x}(B_x)$ defined by the constant function 1. So $\pi(\lambda(x))$ coincides with $B_x \,\in\, \text{Hilb}^d(X)$ for every $x \,\in\, X$. It is clear that using the identification of X with $\text{Pic}^c(X)$, the composition $f \circ \psi \circ \lambda$ coincides with the identity map of X, where ψ is defined in (3.2) and f is defined in (3.7). Therefore, to complete the proof of Theorem 1.1 it suffices to show that $\lambda^*\Theta \,=\, \lambda^*\Omega_0$.

The symplectic form τ on X is invariant under the translations of the abelian variety. Using this it is easy to deduce that $\lambda^*\Omega_0$ coincides with $d \cdot \tau$. Similarly, it is straightforward to deduce that $\lambda^*\Theta \,=\, d\cdot\tau$. This completes the proof of Theorem 1.1 assuming that X is an abelian surface. We already proved it when X is a $K3$ surface and hence the proof of Theorem 1.1 is complete.

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