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ON THE TANGENT SPHERE BUNDLE OF A RIEMANNIAN 2-MANIFOLD

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1. Introduction. Let V be an oriented Riemannian 2-manifold. The bundle $T_1(V)$ of the tangent unit vectors of V can be equipped with a family of natural Riemannian metrics given by the following line element:

$$d\sigma^2 = g_{ik}dx^i dx^k + \rho g_{ik}\delta_y{}^i \delta_y{}^k ,$$

where g_{ik} is the metric tensor of the basic manifold V, ρ is an arbitrary non-zero real constant and we have put

$$(1) \qquad g_{ik}y^i y^k = 1 , \qquad \delta y^i = dy^i + \left\{ {i \atop j \; k} \right\} y^j dx^k .$$

This metric in the case $\rho = 1$ was introduced and studied by S. SASAKI [2]. In a recent paper [1] W. KLINGEBERG and S. SASAKI investigated the tangent sphere bundle of a 2-sphere. The geometry of the tangent sphere bundle of a Euclidean 3-space was investigated by A. M. VASIL'EV in another approach [3].

In this paper we consider the tangent sphere bundle of an arbitrary Riemannian 2-manifold equipped with the generalized Sasaki-metric (1). We carry out our discussions using a special orthogonal frame: the first vector of the frame is the horizontal lift of the supporting element (i.e., of the regarded point of $T_1(V)$), the second and the third ones are the horizontal and vertical lifts of the normalized vector which is orthogonal to the supporting element.

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The author is indebted to A. Szücs (Budapest) for the verbal observation that if $(x(t), y(t))$ forms a geodesic in $T_1(M)$ then $y(t)$ moves on a simple helix relative to the parallel displacement necessarily (cf. Theorem 1).

3. The structure equations. The Riemannian connection of the manifold V defines a direct sum decomposition of the tangent spaces of TV. Let $x \in V$ and y be a tangent unit vector at x. We denote by e_1 the horizontal lift of $y \in T_x V$ to $T_{(x,y)}TV$. Let $z \in T_x V$ be the tangent unit vector at x which is orthogonal to y and such that the 2-frame (y, z) at x has a positive orientation. We denote by e_2 and e_3 the horizontal and vertical lifts of $z \in T_x V$ to $T_{(x,y)}TV$ respectively. It is easy to see that the vectors e_1, e_2, e_3 are tangent to the tangent sphere bundle $T_1(V)$ of the manifold V. Let ω^1, ω^2, ω^3 be the linear forms on $T_1(V)$ forming a dual basis to the frame e_1, e_2, e_3.

PROPOSITION 1. *The linear forms ω^1, ω^2, ω^3 on $T_1(V)$ satisfy the following structure equations*

$$
\begin{aligned}
d\omega^1 &= -\omega^2 \wedge \omega^3 , \\
d\omega^2 &= \omega^1 \wedge \omega^3 , \\
d\omega^3 &= -K\omega^1 \wedge \omega^2 ,
\end{aligned} \tag{2}
$$

where K is the Gaussian curvature of the Riemannian manifold V.

For the proof it is sufficient to note that the bundle of unit vectors of an oriented Riemannian 2-manifold can be identified with its bundle of positively oriented orthonormal 2-frames in a natural manner. Then the equations (2) correspond to the structure equations on the frame-bundle.

PROPOSITION 2. *The components θ^i_k of the Riemannian connection form Θ corresponding to the generalized Sasaki metric can be expressed in the form*

$$
\begin{array}{lll}
\theta^1_1 = 0 , & \theta^2_1 = -\left(\frac{\rho}{2}K - 1\right)\omega^3 , & \theta^3_1 = -\frac{1}{2}K\omega^2 , \\
\theta^1_2 = \left(\frac{\rho}{2}K - 1\right)\omega^3 , & \theta^2_2 = 0 , & \theta^3_2 = +\frac{1}{2}K\omega^1 , \\
\theta^1_3 = \frac{\rho}{2}K\omega^2 , & \theta^2_3 = -\frac{\rho}{2}K\omega^1 , & \theta^3_3 = 0 .
\end{array}
$$

PROOF. The forms θ^i_k satisfy the structure equations

$$d\omega^i = -\theta^i_k \wedge \omega^k .$$

On the other hand we have

$$dg_{ik} + g_{im}\theta^m_k + g_{mk}\theta^m_i = 0 ,$$

where the components g_{ik} of the metric tensor are

$$g_{11} = g_{22} = 1 \,, \quad g_{33} = \rho \,, \quad g_{12} = g_{13} = g_{23} = 0 \,.$$

If follows that

$$\theta_1^1 = \theta_2^2 = \theta_3^3 = 0 \,, \quad \theta_2^1 + \theta_1^2 = 0 \,, \quad \theta_3^1 + \rho\theta_1^3 = 0 \,, \quad \theta_3^2 + \rho\theta_2^3 = 0 \,.$$

It is easy to see using Cartan's lemma that the theorem holds.

4. **Geodesics on $T_1(V)$.** We shall prove the following theorem.

THEOREM 1. *The curve $(x(t), y(t))$ is a geodesic in $T_1(V)$ with respect to the Riemannian metric* (1) *if and only if*

a) *the geodesic curvature κ of $x(t)$ is proportional to the Gaussian curvature K of V along $x(t)$, that is $\kappa = aK$ (a = constant);*

b) *the endpoint of the vector $y(t)$ moves on a simple helix with respect to the parallel displacement along the curve $x(t)$ and has the constant angular velocity a/ρ concerning the arc-length parameter of $x(t)$.*

REMARK. $x(t)$ may reduce to a point x_0. In this case ($a = \infty$) the vector $y(t)$ moves on the unit circle in the tangent plane as x_0.

PROOF. The coordinates ω^i of the tangent vectors of a geodesic with respect to the frame (e_1, e_2, e_3) satisfy the differential equations

$$\dot\omega^i + \theta_k^i \omega^k = 0 \,,$$

where the point denotes the derivation by the affine parameter t. This equations can be written on account of Proposition 2 as

$$\begin{aligned} &\dot\omega^1 - \omega^2\omega^3 + \rho K\omega^2\omega^3 = 0 \,, \\ &\dot\omega^2 + \omega^1\omega^3 - \rho K\omega^1\omega^3 = 0 \,, \\ &\dot\omega^3 = 0 \,. \end{aligned}$$

If $z(t)$ is the vectorfield along $x(t)$ such that $(y(t), z(t))$ forms an oriented orthonormal frame at $x(t)$ on V, than we can write the above equations in the form

$$\text{(3)} \qquad \begin{aligned} &\dot x = \omega^1 y + \omega^2 z \,, \qquad \nabla_{\dot x} y = \omega^3 z \,, \\ &\nabla_{\dot x}\dot x = -\rho K\omega^2\omega^3 y + \rho K\omega^1\omega^3 z \,, \qquad \nabla_{\dot x}\nabla_{\dot x} y = -(\omega^3)^2 y \end{aligned}$$

(c.f. [1], p. 51, equations (2.1)).

If we put $c = \|\dot x\| = \sqrt{(w^1)^2 + (\omega^2)^2}$, we see from $d/dt\langle \dot x, \dot x\rangle = 2\langle \dot x, \nabla_{\dot x}\dot x\rangle = 0$ and

$$\frac{d}{dt}\langle \nabla_{\dot x} y, \nabla_{\dot x} y\rangle = 2\langle \nabla_{\dot x} y, \nabla_{\dot x}\nabla_{\dot x} y\rangle = 0$$

that c and ω^3 are constants.

If $c = 0$ we have the case mentioned in the Remark.

Let s be the arc-length of $x(t)$ and dash denotes the derivation by it. We can write the equations (3) as follows

$$x^1 = \frac{\omega^1}{c}y + \frac{\omega^2}{c}z\,, \qquad \nabla_{x'}y = \frac{\omega^3}{c}z\,,$$

$$\nabla_{x'}x' = -\rho K\frac{\omega^2}{c^2}\omega^3 y + \rho K\frac{\omega^1}{c^2}\omega^3 z\,, \qquad \nabla_{x'}\nabla_{x'}y = -\left(\frac{\omega^3}{c}\right)^2 y\,.$$

The geodesic curvature κ of $x(s)$ satisfies

$$|\kappa| = \|\nabla_{x'}x'\| = \left|\frac{\rho K\omega^3}{c}\right|\,,$$

$$\operatorname{sign}\kappa = \operatorname{sign}\det\begin{vmatrix}\omega^1; & \omega^2 \\ -\rho K\omega^2\omega^3; & \rho K\omega^1\omega^3\end{vmatrix} = \operatorname{sign}(\rho K\omega^3)\,.$$

So we have $\kappa = \rho K\omega^3/c$ i.e. $a = \rho\omega^3/c$.

The equation $\nabla_{x'}\nabla_{x'}y = -(\omega^3/c)^2 y$ means that the endpoint of the vector y moves on a simple helix along the curve $x(t)$ with respect to the parallel displacement.

On the other hand let $x(s)$ be a curve in V (s is its arc-length parameter) such that the geodesic curvature κ of $x(s)$ is proportional to the Gaussian curvature K of V along $x(s)$: $\kappa = aK$. Let $y(s)$ be a vectorfield along $x(s)$, the endpoint of which moves on a simple helix along $x(s)$ and has the constant angular velocity a/ρ. Now we state that $(x(s), y(s))$ is a geodesic in $T_1(V)$ with respect to the metic (1).

Let $z(s)$ be the vectorfield along $x(s)$ such that $(y(s), z(s))$ forms an oriented orthonormal frame. We can write the differential equations of $x(s)$, $y(s)$, $z(s)$ as

$$x' = \gamma^1 y + \gamma^2 z\,,$$

$$\nabla_{x'}x' = -\kappa\gamma^2 y + \kappa\gamma^1 z\,,$$

$$\nabla_{x'}y = \frac{a}{\rho}z$$

$$\nabla_{x'}\nabla_{x'}y = -\left(\frac{a}{\rho}\right)^2 y\,.$$

If we take

$$\gamma^1 = \frac{\omega^1}{c}\,, \quad \gamma^2 = \frac{\omega^2}{c}\,, \quad \kappa = \frac{\rho K\omega^3}{c}\,, \quad a = \frac{\rho\omega^3}{c}\,,$$

we get the equations of geodesics (3).

COROLLARY 1. *If V is an elliptic or hyperbolic plane then the geodesics on $T_1(V)$ are helices around circles on V.*

In fact, in this case $K \neq 0$ is constant on V, and the geodesic curvature κ of $x(t)$ is constant.

COROLLARY 2. *If V is a Euclidean space, then the geodesics on $T_1(V)$ are helices around straight lines ($\kappa = a \circ 0 = 0$).*

5. The curvature of $T_1(V)$.

PROPOSITION 3. *The components of the curvature forms $\Omega^i_k = d\theta^i_k + \theta^i_l \wedge \theta^l_k$ in our frame can be expressed as follows:*

$$\Omega^1_2 = K\left(1 - \frac{3\rho}{4}K\right)\omega^1 \wedge \omega^2 + \frac{\rho}{2}K_1\omega^1 \wedge \omega^3 + \frac{\rho}{2}K_2\omega^2 \wedge \omega^3 ,$$

$$\Omega^1_3 = \frac{\rho}{2}K_1\omega^1 \wedge \omega^2 + \frac{\rho^2}{4}K^2\omega^1 \wedge \omega^3 ,$$

$$\Omega^2_3 = \frac{\rho}{2}K_2\omega^1 \wedge \omega^2 + \frac{\rho^2}{4}K^2\omega^2 \wedge \omega^3 ,$$

where $dK = K_1\omega^1 + K_2\omega^2$.

The proof is obtained by a simple calculation using the results of Proposition 2.

Now we can find in which case will be the 3-manifold $T_1(V)$ of recurrent curvature or locally symmetric or of constant curvature with respect to the metric (1).

THEOREM 2. *The Riemannian manifold $T_1(V)$ equipped with the metric (1) is of recurrent curvature if and only if $K = 0$ or $\rho K = 1$. In the case $K = 0$ it is flat, in the case $\rho K = 1$ $T_1(V)$ has the constant curvature $K/4$.*

PROOF. We get from Proposition 3 the components of the curvature tensor:

$$R_{1212} = K\left(1 - \frac{3\rho}{4}K\right), \quad R_{1213} = \frac{\rho}{2}K_1 , \quad R_{1223} = \frac{\rho}{2}K_2 ,$$

$$R_{1312} = \frac{\rho}{2}K_1 , \quad R_{1313} = \frac{\rho^2}{4}K^2 , \quad R_{1323} = 0 ,$$

$$R_{2312} = \frac{\rho}{2}K_2 , \quad R_{2313} = 0 , \quad R_{2323} = \frac{\rho^2}{4}K^2 .$$

Using these expressions we can calculate the components of the covariant derivative of the curvature tensor. For example we have

$$R_{1323;1} = -\frac{\rho^2}{4}KK_2 \quad \text{and} \quad R_{1323;2} = -\frac{\rho^2}{4}KK_1 .$$

From the recurrence of the curvature tensor it follows $K_1 = K_2 = 0$, that is $K =$ constant. But in this case we have

$$R_{1212} = K\left(1 - \frac{3\rho}{4}K\right), \quad R_{1313} = \frac{\rho^2}{4}K^2, \quad R_{2323} = \frac{\rho^2}{4}K^2,$$

and the other components are 0. Now we calculate

$$R_{1213;1} = \frac{1}{2}\frac{\rho^2}{4}K^3 - \frac{\rho}{2}K^2\left(1 - \frac{3\rho}{4}K\right) = \frac{\rho}{2}K^2(\rho K - 1) .$$

We obtained that if $T_1(V)$ is of recurrent curvature than $K = 0$ or $\rho K = 1$. In the first case we have evidently that $T_1(V)$ is flat.

Now we suppose that $\rho K = 1$. We get

$$R_{1212} = \frac{K}{4} = \frac{K}{4}(g_{11}g_{22} - g_{12}^2), \quad R_{1313} = \frac{K}{4}\rho = \frac{K}{4}(g_{11}g_{33} - g_{13}^2), \quad R_{2323} = \frac{K}{4}\rho = \frac{K}{4}(g_{22}g_{33} - g_{13}^2),$$

and $R_{ijkl} = 0$ in the other combination of the indices. This completes the proof.

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