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Published in:
Modern Physics Letters A

DOI:
10.1142/S0217732396001107

Modern Physics Letters A, Vol. 11, No. 13 (1996) 1069–1080

ON THE TENSOR FORMULATION OF EFFECTIVE VECTOR LAGRANGIANS AND DUALITY TRANSFORMATIONS

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Received 28 February 1996

Using two different methods inspired by duality transformations we present the equivalence between effective Lagrangians for massive vector mesons using a vector field and an antisymmetric tensor field. This completes the list of explicit field transformations between the various effective Lagrangian methods to describe massive vector and axial vector mesons. Our method automatically generates the point-like terms needed for off-shell equivalence.

1. Introduction

Dual transformations have been used to a large extent to prove the equivalence of apparently different Lagrangian formulations with relevant consequences for solid state physics and gauge field theories.[1]

Self-duality has been proven for massive vector theories in odd dimensions[2] and their equivalence with topologically massive Abelian gauge theory in $(2+1)$ dimensions has been shown in Ref. 3. Some physical implications of the dual formulation of various three-dimensional field theories have been studied in Ref. 4 and Ref. 6 cited therein.

Dual formulation of some gauge field theories in four dimensions has also been considered[5,6] (for the construction of massive gauge theories in $d = 4$ see Refs. 7 and 8). This was also used to prove the equivalence of the Thirring model to a gauge theory.[9] The latter reference triggered the present work.

Recently, in the framework of chiral effective theories describing low energy strong interactions, a tensor formalism to describe an ordinary vector field has been developed in Ref. 10 and an attempt to prove the equivalence of the vector and tensor formulation was done in Ref. 11 for the nonanomalous sector of the low energy effective action and in Ref. 12 for the anomalous one.

Various relations between parameters of the two formulations were found as a phenomenological consequence of QCD dispersion relations. The equivalence of all the possible representations for massive vector fields in chiral Lagrangians was also conjectured in Ref. 11. For those transforming as a vector gauge field the equivalence was shown in Ref. 13 and the relation to the vector matter field used here can be found in Ref. 11.

In this letter we prove that a duality-type relationship connects the two different Lagrangian descriptions of the same physics at the classical level. This implies that the tensor and vector formulations give rise to the same partition function and the equivalence between them holds in the sense of the path integral. Nevertheless, we do not consider the quantum level since in order to describe massive vector fields in a renormalizable fashion we need to use the Higgs mechanism.

Our transformation provides a new systematic way to obtain the form of terms in the tensor formalism that are equivalent to those in the more standard formulations. In previous work these terms were obtained by looking at specific physical processes and including the extra terms not involving vectors needed for off-shell equivalence. Our method automatically generates these extra point-like terms and it is valid to any order in the derivative expansion at the classical level.

During the calculation it will also become obvious that there is no simple power-counting possible for the massive fields. In our method we explicitly show how the number of derivatives in interaction terms can be changed. The general approach has some similarity with the so-called *first-order formulation* in which the field strength ($F_\mu = \partial_\mu \Phi$ for spin-0 and $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ for spin-1) is treated as an independent variable.

We first describe in detail the method which is most easily generalized to terms with powers of quark masses or more derivatives and then shortly describe the other method that leads to identical results. We also present a few short comments on the previously derived phenomenological consequences.[11]

2. The Equivalence

The theory we are going to use describes an ordinary (not gauge) massive vector field interacting with pseudoscalar mesons whose Lagrangian is explicitly *local* chiral invariant due to the addition of external sources.

We refer for the nomenclature to the particular case which is the effective field theory of low energy QCD with the inclusion of vector mesons,[11] although our derivation can be easily generalized.

The Lagrangian for the interacting vector field V_μ is written as follows:

$$\begin{aligned}\mathcal{L}_V &= -\frac{1}{4}\langle V_{\mu\nu}V^{\mu\nu}\rangle + \frac{1}{2}m^2\langle V_\mu V^\mu\rangle + \langle V_{\mu\nu}J^{\mu\nu}\rangle \\ J^{\mu\nu} &= -\frac{f_V}{2\sqrt{2}}\, f_+^{\mu\nu} - i\frac{g_V}{2\sqrt{2}}[u^\mu, u^\nu]\,,\end{aligned} \tag{1}$$

where $\langle\cdots\rangle$ stands for the trace over flavor indices. The formalism used here is that of Ref. 11. This allows us to directly compare our results to those in Ref. 11. The current $J^{\mu\nu}$ contains two terms with couplings f_V and g_V. In principle there are more interaction terms with external sources which can appear at the leading order (i.e. $O(p^3)$) and higher orders of the chiral expansion. It will be clear at the end how our analysis can be easily extended to a more general form of the interaction

Lagrangian. The fields $f_+^{\mu\nu}$ and u_μ are defined as

$$\begin{aligned} f_+^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger + u^\dagger F_R^{\mu\nu}u\,, \\ u_\mu &= iu^\dagger D_\mu U u^\dagger = u_\mu^\dagger\,, \end{aligned} \tag{2}$$

where $F_{L,(R)}^{\mu\nu}$ is the field strength tensor associated with the non-Abelian external source $v_\mu - a_\mu$, $(v_\mu + a_\mu)$ and $u = \sqrt{U} = \exp\{i\Phi/f\}$ is the square root of the usual exponential representation of the pseudoscalar Goldstone boson field with flavor matrix Φ. $V_{\mu\nu} = D_\mu V_\nu - D_\nu V_\mu$ is the field strength tensor of the vector field where the covariant derivative $D_\mu = \partial_\mu + [\Gamma_\mu\cdot]$ with $\Gamma_\mu = 1/2\{u^\dagger[\partial_\mu - i(v_\mu + a_\mu)]u + u[\partial_\mu - i(v_\mu - a_\mu)]u^\dagger\}$ guarantees the *local* chiral invariance of the kinetic term. The fields V_μ, $V_{\mu\nu}$, $f_+^{\mu\nu}$ and u_μ transform homogeneously and nonlinearly under a chiral transformation $g_L \times g_R \in G = \mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$ as

$$\mathcal{O} \xrightarrow{G} h(\Phi)\mathcal{O}h^\dagger(\Phi)\,, \tag{3}$$

where $h(\Phi)$ is the nonlinear realization of G which defines the action of the group on a coset element $u(\Phi)$ via

$$u(\Phi) \xrightarrow{G} g_R u(\Phi)h^\dagger(\Phi) = h(\Phi)u(\Phi)g_L^\dagger\,. \tag{4}$$

This guarantees that the full vector Lagrangian (1) is *local* chiral invariant with the inclusion of the mass term for the vector field.

In the case of a *global* chiral invariant formulation the path integral for the vector Lagrangian (1), where the replacement $D_\mu \to \partial_\mu$ has been done, would be

$$Z[L_\mu, R_\mu, u_\mu] = \int \mathcal{D}V_\mu \delta(\partial_\mu V^\mu) e^{i\int d^4x \mathcal{L}_V}\,, \tag{5}$$

where the transversality constraint $\partial_\mu V^\mu = 0$ reduces the number of independent degrees of freedom in four dimensions to three. The transversality condition on the vector field in (3+1) dimensions guarantees that it admits a representation in terms of its dual antisymmetric tensor field as $V_\mu = \partial^\lambda H_{\lambda\mu}$, which automatically satisfies the constraint $\partial_\mu V^\mu = 0$.

The extension to *local* chiral invariance is more delicate. In this case the correct dual transformation is the one which does not break the homogeneous transformation properties (3) of the vector field. A choice which reduces to the above one in the absence of other fields and sources is for example $V_\mu \simeq D^\lambda H_{\lambda\mu}$, where the tensor field transforms homogeneously like in (3).

The transversality constraint $\partial_\mu V^\mu = 0$ is no longer automatically satisfied. But at leading order in fields it is still $\partial_\mu V^\mu = \mathcal{O}(\phi^2)$ with ϕ any field or source. The condition $V_\mu = D^\lambda H_{\lambda\mu}$ thus still removes one degree of freedom from the V_μ field. We write the partition function of the *local* chiral invariant vector Lagrangian (1) in terms of a generalized transversality constraint as

$$Z[L_\mu, R_\mu, u_\mu] = \int \mathcal{D}V_\mu \delta(\mathcal{F}[V^\mu]) e^{i\int d^4x \mathcal{L}_V} , \tag{6}$$

where $\mathcal{F}[V^\mu] = 0$ is consistent with the dual transformation $V_\mu \simeq D^\lambda H_{\lambda\mu}$.

At the end of this section we briefly formulate an alternative method to prove the equivalence. The constraint there will again be consistent with the dual transformation of the type $V_\mu \simeq D^\lambda H_{\lambda\mu}$.

For the dual transformation of the vector field there are in fact two possibilities:

$$\begin{aligned} &(\mathrm{I}) \qquad V_\mu = \frac{1}{m} D^\lambda H_{\lambda\mu} , \\ &(\mathrm{II}) \qquad V_\mu = \frac{1}{2m} \varepsilon_{\mu\nu\alpha\beta} D^\nu \tilde{H}^{\alpha\beta} . \end{aligned} \tag{7}$$

We also notice that the present dual transformation is strictly valid only for massive vector fields where the mass plays the role of an ir cutoff of the theory. For an alternative method in $(2+1)$ dimensions that also works in the massless case see Ref. 9.

The two choices in (7) correspond to two different assignments of parity transformation property of the dual tensor field. The vector field V_μ is a $J^{\mathrm{PC}} = 1^{--}$ state, i.e. $V_\mu^P = \varepsilon(\mu) V_\mu$ and $V_\mu^C = -V_\mu^T$. This implies that in choice (I) the tensor field is a vector-like field for a 1^{--} state, with $H_{\mu\nu}^P = \varepsilon(\mu)\varepsilon(\nu) H_{\mu\nu}$ and $H_{\mu\nu}^C = -H_{\mu\nu}^T$. While in choice (II) the tensor field is an axial-like field for a state 1^{--}, with $\tilde{H}_{\mu\nu}^P = -\varepsilon(\mu)\varepsilon(\nu)\tilde{H}_{\mu\nu}$ and $\tilde{H}_{\mu\nu}^C = -\tilde{H}_{\mu\nu}^T$. In the case of axial vectors the choice is of course the opposite.

We present the full derivation of the equivalence for choice (I), while for choice (II) we shall point out the differences and the final result.

For any of the two choices, we refer to choice (I) from now on, the path integral (6) on the vector field can be rewritten as a path integral on the dual tensor field due to the following identity:

$$\int \mathcal{D}V_\mu \delta(\mathcal{F}[V^\mu]) \cdots = \int \mathcal{D}V_\nu \mathcal{D}H_{\mu\nu} \delta\left(V_\mu - \frac{1}{m} D^\lambda H_{\lambda\mu}\right) \cdots . \tag{8}$$

The integration over the vector field V_μ then becomes trivial due to the δ-function and one gets the path integral for the Lagrangian of the dual tensor field $H_{\mu\nu}$

$$Z[L_\mu, R_\mu, u_\mu] = \int \mathcal{D}H_{\mu\nu}\, e^{i\int d^4x \mathcal{L}_H} , \tag{9}$$

where $\mathcal{L}_H$, for choice (I), is given by

$$\begin{aligned} \mathcal{L}_H = &-\frac{1}{4m^2} \langle (D_\mu D^\lambda H_{\lambda\nu} - D_\nu D^\lambda H_{\lambda\mu})^2 \rangle + \frac{1}{2} \langle (D^\lambda H_{\lambda\mu})^2 \rangle \\ &- \frac{f_V}{2\sqrt{2}m} \langle (D_\mu D^\lambda H_{\lambda\nu} - D_\nu D^\lambda H_{\lambda\mu}) f_+^{\mu\nu} \rangle \\ &- i\frac{g_V}{2\sqrt{2}m} \langle (D_\mu D^\lambda H_{\lambda\nu} - D_\nu D^\lambda H_{\lambda\mu}) [u^\mu, u^\nu] \rangle . \end{aligned} \tag{10}$$

At this level we have the problem that there is no explicit mass term for the $H_{\mu\nu}$ field but there exist a two-derivative and a four-derivative kinetic like term. The latter implies the naive existence of a second pole. This one is at zero mass, see below. The underlying reason for the appearance of the extra pole is the presence of a derivative in the field redefinition of (7). A constant field $H_{\mu\nu}$ does not contribute to V_μ. We therefore would like to lower the number of derivatives in the kinetic terms.

We can remove the first term in (10) by adding a new auxiliary tensor field in a way that leaves the original path integral invariant. This is similar to the first-order formalism for gauge theories. We can always write

$$Z[L_\mu, R_\mu, u_\mu] = \int \mathcal{D}I'_{\mu\nu}\, e^{i\int d^4x I'^2_{\mu\nu}} \int \mathcal{D}H_{\mu\nu}\, e^{i\int d^4x \mathcal{L}_H}\,. \tag{11}$$

The path integral in (11) is equivalent to that in (9). They differ by an overall normalization constant given by the Gaussian integral over the auxiliary tensor field $I'_{\mu\nu}$. Redefining $I'_{\mu\nu}$ with a linear transformation with unit Jacobian the original path integral (9) is equivalent to the one where we add to $\mathcal{L}_H$ the quadratic term

$$+\frac{1}{4m^2}\left[D_\mu D^\lambda H_{\lambda\nu} - D_\nu D^\lambda H_{\lambda\mu} - \alpha I_{\mu\nu} - \beta f^+_{\mu\nu} - \delta[u_\mu, u_\nu]\right]^2 \tag{12}$$

and integrate over the original tensor field $H_{\mu\nu}$ and the new auxiliary field $I_{\mu\nu}$.

The full tensor Lagrangian contains now two tensor fields:

$$\begin{aligned}
\mathcal{L}_{HI} = {} & \frac{1}{2}\langle (D^\lambda H_{\lambda\mu})^2\rangle + \frac{\alpha^2}{4m^2}\langle I_{\mu\nu}I^{\mu\nu}\rangle - \frac{\alpha}{2m^2}\langle (D_\mu D^\lambda H_{\lambda\nu} - D_\nu D^\lambda H_{\lambda\mu})I^{\mu\nu}\rangle \\
& - \left(\frac{f_V}{2\sqrt{2}m} + \frac{\beta}{2m^2}\right)\langle (D_\mu D^\lambda H_{\lambda\nu} - D_\nu D^\lambda H_{\lambda\mu}) f_+^{\mu\nu}\rangle \\
& - \left(i\frac{g_V}{2\sqrt{2}m} + \frac{\delta}{2m^2}\right)\langle (D_\mu D^\lambda H_{\lambda\nu} - D_\nu D^\lambda H_{\lambda\mu})[u^\mu, u^\nu]\rangle \\
& + \frac{\alpha\beta}{2m^2}\langle I_{\mu\nu} f_+^{\mu\nu}\rangle + \frac{\alpha\delta}{2m^2}\langle I_{\mu\nu}[u^\mu, u^\nu]\rangle \\
& + \frac{\beta^2}{4m^2}\langle f_+^{\mu\nu} f^+_{\mu\nu}\rangle + \frac{\delta^2}{4m^2}\langle [u^\mu, u^\nu][u_\mu, u_\nu]\rangle + \frac{\beta\delta}{2m^2}\langle f^+_{\mu\nu}[u^\mu, u^\nu]\rangle\,.
\end{aligned} \tag{13}$$

There is no kinetic term for the auxiliary field $I_{\mu\nu}$ while it is coupled to the tensor field $H_{\mu\nu}$ via the last term in the first line of (13). At this stage both the fields H and I interact with external sources. Parameters β, δ can be chosen in order to eliminate unwanted interaction terms with derivative couplings on the tensor field H. This implies the choice

$$\beta = -\frac{m f_V}{\sqrt{2}}\,, \qquad \delta = -i\frac{m g_V}{\sqrt{2}}\,. \tag{14}$$

As can be seen here we have a choice of whether to add interaction terms or not to (12). The number of derivatives in the interaction terms can thus be easily changed. This shows again that the usual chiral power counting is not possible for massive fields.

At this point we show that a two-step orthogonal transformation of the tensor fields permits one to rewrite the two-tensors Lagrangian in terms of rotated tensor fields which simultaneously are eigenstates of the kinetic operator and diagonalize the mass term. Since the Jacobian of the transformation is trivial, the final path integral will be equivalent to the original one.

The first orthogonal transformation ensures the diagonalization of the kinetic term. Defining the rotated fields as

$$\begin{aligned} H_{\mu\nu} &= s_\theta G_{\mu\nu} + c_\theta G'_{\mu\nu}\,, \\ I_{\mu\nu} &= c_\theta G_{\mu\nu} - s_\theta G'_{\mu\nu}\,, \end{aligned} \tag{15}$$

the Lagrangian for the fields G and G' becomes

$$\begin{aligned} \mathcal{L}_{GG'} = {} & \left(\frac{s_\theta^2}{2} + \frac{\alpha}{m^2} s_\theta c_\theta\right) \langle (D^\lambda G_{\lambda\mu})^2\rangle + \left(\frac{c_\theta^2}{2} - \frac{\alpha}{m^2} s_\theta c_\theta\right) \langle (D^\lambda G'_{\lambda\mu})^2\rangle \\ & + \frac{\alpha^2}{4m^2}\left[c_\theta^2 \langle G_{\mu\nu}G^{\mu\nu}\rangle + s_\theta^2\langle G'_{\mu\nu}G'^{\mu\nu}\rangle - 2s_\theta c_\theta \langle G^{\mu\nu}G'_{\mu\nu}\rangle\right] \\ & + \frac{\alpha}{2m^2} c_\theta \left[\beta\langle G_{\mu\nu} f_+^{\mu\nu}\rangle + \delta\langle G_{\mu\nu}[u^\mu, u^\nu]\rangle\right] \\ & - \frac{\alpha}{2m^2} s_\theta \left[\beta\langle G'_{\mu\nu} f_+^{\mu\nu}\rangle + \delta\langle G'_{\mu\nu}[u^\mu, u^\nu]\rangle\right] \\ & + \left[s_\theta c_\theta + \frac{\alpha}{m^2}(c_\theta^2 - s_\theta^2)\right] \langle D^\lambda G_{\lambda\mu} D_{\lambda'} G'^{\lambda'\mu}\rangle \\ & + \frac{\beta^2}{4m^2}\langle f_+^{\mu\nu} f^+_{\mu\nu}\rangle + \frac{\delta^2}{4m^2}\langle [u^\mu, u^\nu][u_\mu, u_\nu]\rangle + \frac{\beta\delta}{2m^2}\langle f^+_{\mu\nu}[u^\mu, u^\nu]\rangle\,. \end{aligned} \tag{16}$$

In (16) five types of terms appear in order: kinetic terms, mass terms, interaction terms for G and G' individually, G, G' mixed terms and *local* or contact terms with only external fields or the other degrees of freedom. These latter terms are precisely the ones that were required by the high energy constraints in Ref. 11. In this approach they appear automatically.

The condition that the mixed derivative term $\langle D^\lambda G_{\lambda\mu} D_{\lambda'} G'^{\lambda'\mu}\rangle$ vanishes implies one constraint on the parameter α

$$\alpha = -\frac{m^2}{2}\tan 2\theta\,. \tag{17}$$

With this constraint the kinetic terms of G and the G' fields become

$$\mathcal{L}_{\text{kin}} = -\frac{s_\theta^2}{2\cos 2\theta}\langle (D^\lambda G_{\lambda\mu})^2\rangle + \frac{c_\theta^2}{2\cos 2\theta}\langle (D^\lambda G'_{\lambda\mu})^2\rangle\,. \tag{18}$$

For a given choice of the rotation angle θ the kinetic terms of the two fields have opposite signs. The choice of the correct relative sign of kinetic and mass terms is determined in the Minkowski case by the requirement that there be no tachyons in the final theory. Hence, the physical solution has to be the one where the tensor field with the unphysical ("wrong") sign in the kinetic term "decouples" in the sense that it acquires zero mass and it does not interact with any other field.

Choosing $\cos 2\theta > 0$, this is always possible from (17), the rescaled G and G' fields are defined via the wave function renormalization constant as:

$$K_{\mu\nu} = \sqrt{\frac{s_\theta^2}{\cos 2\theta}}\, G_{\mu\nu}\,, \qquad K'_{\mu\nu} = \sqrt{\frac{c_\theta^2}{\cos 2\theta}}\, G'_{\mu\nu}\,. \tag{19}$$

The rescaled fields $K_{\mu\nu}$ and $K'_{\mu\nu}$ are not mass eigenstates since the mixed term $\langle G^{\mu\nu} G'_{\mu\nu}\rangle$ is present in (16).

The second step of the orthogonal transformation is the one which leaves invariant the kinetic piece and diagonalizes the mass term:

$$\begin{aligned} K_{\mu\nu} &= ch_\phi I_{\mu\nu} + sh_\phi I'_{\mu\nu}\,, \\ K'_{\mu\nu} &= sh_\phi I_{\mu\nu} + ch_\phi I'_{\mu\nu}\,. \end{aligned} \tag{20}$$

With this substitution and defining

$$c_1 \equiv \frac{\alpha^2}{4m^2}\frac{c_\theta^2}{s_\theta^2}\cos 2\theta\,, \qquad c_2 \equiv \frac{\alpha^2}{4m^2}\frac{s_\theta^2}{c_\theta^2}\cos 2\theta\,, \tag{21}$$

with $\sin 2\theta > 0$, the Lagrangian for the I, I' fields becomes

$$\begin{aligned} \mathcal{L}_{I,I'} = &-\frac{1}{2}\langle (D^\lambda I_{\lambda\mu})^2\rangle + \frac{1}{2}\langle (D^\lambda I'_{\lambda\mu})^2\rangle + (\sqrt{c_1}\, ch_\phi - \sqrt{c_2}\, sh_\phi)^2 \langle I_{\mu\nu} I^{\mu\nu}\rangle \\ &+ (\sqrt{c_1}\, sh_\phi - \sqrt{c_2}\, ch_\phi)^2 \langle I'_{\mu\nu} I'^{\mu\nu}\rangle \\ &+ 2\left[(c_1 + c_2) sh_\phi ch_\phi - \sqrt{c_1 c_2}(sh_\phi^2 + ch_\phi^2)\right] \langle I_{\mu\nu} I'^{\mu\nu}\rangle \\ &+ \frac{1}{m}(\sqrt{c_1}\, ch_\phi - \sqrt{c_2}\, sh_\phi)\left(\beta\langle I_{\mu\nu} f_+^{\mu\nu}\rangle + \delta\langle I_{\mu\nu}[u^\mu, u^\nu]\rangle\right) \\ &+ \frac{1}{m}(\sqrt{c_1}\, sh_\phi - \sqrt{c_2}\, ch_\phi)\left(\beta\langle I'_{\mu\nu} f_+^{\mu\nu}\rangle + \delta\langle I'_{\mu\nu}[u^\mu, u^\nu]\rangle\right) \\ &+ \frac{\beta^2}{4m^2}\langle f_+^{\mu\nu} f^+_{\mu\nu}\rangle + \frac{\delta^2}{4m^2}\langle [u^\mu, u^\nu][u_\mu, u_\nu]\rangle + \frac{\beta\delta}{2m^2}\langle f^+_{\mu\nu}[u^\mu, u^\nu]\rangle\,. \end{aligned} \tag{22}$$

From (22) one deduces that the constraint equation which diagonalizes the mass term is given by

$$(c_1 + c_2) sh_\phi ch_\phi - \sqrt{c_1 c_2}(sh_\phi^2 + ch_\phi^2) = 0\,. \tag{23}$$

The solution in terms of $ch2\phi = ch_\phi^2 + sh_\phi^2$ is $ch^2 2\phi = (c_1+c_2)^2/(c_1-c_2)^2$. It is then easy to find by direct substitution that the mass terms for $I_{\mu\nu}$ and $I'_{\mu\nu}$ fields are

$$\mathcal{L}_{\text{mass}} = (c_1 - c_2)\langle I_{\mu\nu} I^{\mu\nu}\rangle + 0\cdot\langle I'_{\mu\nu} I'^{\mu\nu}\rangle\,. \tag{24}$$

Using Eqs. (21) and (17) we find $c_1 - c_2 = m^2/4$ so that the free Lagrangian is

$$\mathcal{L}^0_{II'} = -\frac{1}{2}\langle (D^\lambda I_{\lambda\mu})^2\rangle + \frac{1}{2}\langle (D^\lambda I'_{\lambda\mu})^2\rangle + \frac{1}{4}m^2\langle I_{\mu\nu} I^{\mu\nu}\rangle\,. \tag{25}$$

As expected, the tensor field which is massive is the one with the correct relative sign for the kinetic and mass terms (i.e. it has causal propagation), while the tensor field with the "wrong" sign assignment (i.e. it has tachyonic propagation) remains massless and is the artefact expected from the transformation (7). At the same time all the interaction terms of the unphysical field $I'_{\mu\nu}$ with external currents vanish as a consequence of Eq. (23) and the final Lagrangian for the physical tensor field $I_{\mu\nu}$ becomes

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_T + \frac{\beta^2}{4m^2}\langle f_+^{\mu\nu} f^+_{\mu\nu}\rangle + \frac{\delta^2}{4m^2}\langle [u^\mu, u^\nu][u_\mu, u_\nu]\rangle + \frac{\beta\delta}{2m^2}\langle f^+_{\mu\nu}[u^\mu, u^\nu]\rangle\,,\\ \mathcal{L}_T &= -\frac{1}{2}\langle (D^\lambda I_{\lambda\mu})^2\rangle + \frac{m^2}{4}\langle I_{\mu\nu} I^{\mu\nu}\rangle + \frac{\beta}{2}\langle I_{\mu\nu} f_+^{\mu\nu}\rangle + \frac{\delta}{2}\langle I_{\mu\nu}[u^\mu, u^\nu]\rangle\,.\end{aligned} \tag{26}$$

This is our main result. We have shown that the vector Lagrangian (1), with the constraint $\mathcal{F}[V^\mu] = 0$ consistent with the dual transformation (I) of (7), is *equivalent* in the sense of the path integral to the tensor Lagrangian (26) for a tensor vector-like field describing a 1^{--} state, where additional *local* terms (i.e. terms with external sources only) are present. These terms are precisely the ones whose presence was required by the constraints in Ref. 11. Using the values of β and δ given by Eq. (14) the following equivalence relation holds:

$$\mathcal{L}_T \equiv \mathcal{L}_V - \frac{f_V^2}{8}\langle f_+^{\mu\nu} f^+_{\mu\nu}\rangle + \frac{g_V^2}{8}\langle [u^\mu, u^\nu][u_\mu, u_\nu]\rangle - i\frac{f_V g_V}{4}\langle f^+_{\mu\nu}[u^\mu, u^\nu]\rangle\,. \tag{27}$$

For choice (II) of (7), where the dual tensor field $\tilde{H}_{\mu\nu}$ is an axial-like tensor field, we are also able to produce the equivalence of the vector Lagrangian (1) under the constraint $\mathcal{F}[V^\mu] = 0$ with a Lagrangian for an axial-like tensor field describing a 1^{--} state. Exactly the same procedure as before can be followed but using instead of $I, I', G, \ldots$ the fields $\tilde{I}, \tilde{I}', \tilde{G}, \ldots$ with

$$\tilde{X}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} X^{\alpha\beta}\,. \tag{28}$$

The two-step diagonalization proceeds as for choice (I). Elimination of unwanted interaction terms with derivative couplings leads again to the constraints (14) for β and δ and the elimination of nondiagonal terms induces again constraint (17) on the parameter α. Of the two final mass eigenstates only $\tilde{I}_{\mu\nu}$ (the one with the correct sign of the kinetic term) gets massive as before and the final Lagrangian for the tensor field $\tilde{I}_{\mu\nu}$ follows

$$\begin{aligned}\mathcal{L}_T = {} & \frac{1}{4}\langle D_\lambda \tilde{I}_{\mu\nu} D^\lambda \tilde{I}^{\mu\nu} - 2D^\lambda \tilde{I}_{\lambda\mu} D_{\lambda'} \tilde{I}^{\lambda'\mu}\rangle - \frac{m^2}{4}\langle \tilde{I}_{\mu\nu}\tilde{I}^{\mu\nu}\rangle \\ & + \frac{\beta}{4}\langle \varepsilon_{\mu\nu\alpha\beta}\tilde{I}^{\mu\nu} f_+^{\alpha\beta}\rangle + \frac{\delta}{4}\langle \varepsilon_{\mu\nu\alpha\beta}\tilde{I}^{\mu\nu}[u^\alpha, u^\beta]\rangle \\ & + \frac{\beta^2}{4m^2}\langle f_+^{\mu\nu} f_{\mu\nu}^+\rangle + \frac{\delta^2}{4m^2}\langle [u^\mu, u^\nu][u_\mu, u_\nu]\rangle + \frac{\beta\delta}{2m^2}\langle f_{\mu\nu}^+[u^\mu, u^\nu]\rangle . \end{aligned} \tag{29}$$

Note that the structure of the kinetic term corresponds to the case $a + 2b = 0$ in Appendix A of Ref. 11. The choice (I) led to the case $b = 0$. Our derivation shows that both choices are possible and leads to a good description for a vector meson. Note that because of the opposite intrinsic parity required for case (b) the interaction terms also contain an extra Levi–Civita tensor. The signs of the interaction terms can also be changed by multiplying the dual transformations of (7) by -1.

In the end we have four possibilities. Case (a), case (b) and both with an extra minus sign in (7). Case (a) corresponds to the case where the components I^{0i}, $i = 1, 2, 3$, propagate in the rest frame. Obtaining the correct parity for these requires $I_{\mu\nu}$ to have positive intrinsic parity as already remarked above. In case (b) $\tilde{I}^{ij}$, with $i, j = 1, 2, 3$, are the components that propagate in the rest frame. This in turn requires $\tilde{I}_{\mu\nu}$ to have negative intrinsic parity so that the $\tilde{I}^{ij}$ can describe the propagating components of a vector.

In all cases we proved the equivalence to the original vector Lagrangian (1) with the constraint $\mathcal{F}[V^\mu] = 0$ in the sense of the path integral and with the addition of the SAME set of *local* terms.

The alternative approach we mentioned before is more similar to the well-known *first-order formalism.* In order to treat $V_{\mu\nu}$ and V_μ as independent fields let us rewrite the partition function (6) as

$$Z[J] = \int \mathcal{D}V_{\mu\nu}\delta(V_{\mu\nu} - (D_\mu V_\nu - D_\nu V_\mu)) \int \mathcal{D}V_\mu \delta(\mathcal{F}[V^\mu]) e^{i\int d^4x \mathcal{L}_V} . \tag{30}$$

The first δ-function can be rewritten as a Gaussian integral over an auxiliary tensor field in two possible ways:

$$\begin{aligned} & \int \mathcal{D}V_{\mu\nu}\delta(V_{\mu\nu} - (D_\mu V_\nu - D_\nu V_\mu)) \ldots \\ & \quad (\mathrm{I}) = \int \mathcal{D}V_{\mu\nu}\mathcal{D}H_{\mu\nu}\, e^{i\int d^4x \alpha H^{\mu\nu}[V_{\mu\nu} - (D_\mu V_\nu - D_\nu V_\mu)]} \ldots \\ & \quad (\mathrm{II}) = \int \mathcal{D}V_{\mu\nu}\mathcal{D}\tilde{H}_{\mu\nu}\, e^{i\int d^4x \alpha \varepsilon_{\mu\nu\alpha\beta}\tilde{H}^{\alpha\beta}[V^{\mu\nu} - (D^\mu V^\nu - D^\nu V^\mu)]} \ldots . \end{aligned} \tag{31}$$

Integrating out the field $V_{\mu\nu}$ one gets for choice (I)

$$\begin{aligned} Z[J] &= \int \mathcal{D}V_\mu \mathcal{D}H_{\mu\nu}\delta(\mathcal{F}[V^\mu])e^{i\int d^4x \mathcal{L}_{V,H}}\,, \\ \mathcal{L}_{V,H} &= \frac{1}{2}m^2\langle V_\mu V^\mu\rangle + \langle (J_{\mu\nu}+\alpha H_{\mu\nu})^2\rangle - \alpha\langle H_{\mu\nu}(D^\mu V^\nu - D^\nu V^\mu)\rangle\,. \end{aligned} \tag{32}$$

The integration over V_μ can be done simply if we integrate by parts in the last term. If the boundary condition $\int d^4x\langle D^\mu(H_{\mu\nu}V^\nu)\rangle = 0$ is satisfied, which is obviously the case, this can be done. Then the integral over V_μ reduces to a Gaussian integral and the final partition function is the one for a tensor Lagrangian

$$\mathcal{L}_T = -\left(\frac{\alpha\sqrt{2}}{m}\right)^2 \langle D^\lambda H_{\lambda\mu} D_{\lambda'} H^{\lambda'\mu}\rangle + \alpha^2\langle H^2_{\mu\nu}\rangle + 2\alpha\langle H_{\mu\nu}\, J^{\mu\nu}\rangle + \langle J^2_{\mu\nu}\rangle\,. \tag{33}$$

It is immediate to verify that the choices $\alpha = \pm m/2$ reproduce choice (I) of the previous approach with both possible signs for the interaction terms. The analogous procedure for choice (II) of (31) leads to the tensor Lagrangian of case (b) of the first approach.

Note that in both methods the presence of the mass term in the original Lagrangian was crucial. In the first method it directly produced the final kinetic term and in the second method it produced the quadratic part of the Gaussian integral. We could of course have expected this since in the massless case there is a singularity of the type $1/q^2$ possible while in the tensor formalism this singularity is at most $q_\mu q_\nu/q^2$ in interactions with other fields. In the approach of Ref. 9 the presence at intermediate stages of inverse derivatives in the Lagrangian shows the same problem.

3. Some Implications of the Equivalence

In Ref. 11 relations between the parameters of the two formulations were obtained at the lowest order in the derivative expansion. These we reproduce trivially in (35) below. In addition within the tensor model in the low energy limit values for the low-energy constants L_i were obtained there (see below for their definition). These were zero in the vector model. Our dual transformation generates this difference in a systematic fashion and is easily extendable to higher derivative terms.

We notice first that the two tensor Lagrangians obtained with choice (I) or (II) in (7) correspond to the two possible choices $a + 2b = 0$ and $b = 0$ in the Appendix of Ref. 10. These two choices of the parameters in the most general tensor Lagrangian are all the possible ones which reduce from six to three the propagating components of the tensor field. In the case $b = 0$, which corresponds to choice (I) in our formalism, the usual tensor Lagrangian for vector meson fields is written in terms of two couplings F_V and G_V of the tensor field to the external currents as[10]

$$L_T = -\frac{1}{2}\langle (D^\lambda I_{\lambda\mu})^2\rangle + \frac{1}{4}m^2\langle I_{\mu\nu}I^{\mu\nu}\rangle + \frac{F_V}{2\sqrt{2}}\langle I_{\mu\nu}f_+^{\mu\nu}\rangle + i\frac{G_V}{2\sqrt{2}}\langle I_{\mu\nu}[u^\mu, u^\nu]\rangle\,. \quad (34)$$

Comparing with Eq. (26) and using the constraints (14) we get

$$F_V = -mf_V\,, \qquad G_V = -mg_V\,, \quad (35)$$

where only the relative sign between F_V and G_V is fixed due to the arbitrariness in (7).

The last three terms on the right-hand side of (27) are the additional *local* terms which guarantee the equivalence of the vector and tensor Lagrangians in Ref. 11. Writing $f_{\mu\nu}^+$ and u_μ in terms of the external left- and right-handed currents and the pseudo-Goldstone boson field as given in (2) we get some of the $O(p^4)$ terms of the CHPT Lagrangian[14]:

$$\langle f_+^{\mu\nu}f_{\mu\nu}^+\rangle = \langle F_{L\mu\nu}^2 + F_{R\mu\nu}^2 + 2F_{L\mu\nu}U^\dagger F_R^{\mu\nu}U\rangle = P_{H_1} + 2P_{10}\,, \quad (36)$$

$$\begin{aligned}\langle [u^\mu, u^\nu]^2\rangle &= 2\langle D_\mu U D_\nu U^\dagger D^\mu U D^\nu U^\dagger - D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger\rangle\\ &= -6P_3 + P_1 + 2P_2\,, \end{aligned}\quad (37)$$

$$-i\langle f_{\mu\nu}^+[u^\mu, u^\nu]\rangle = -2i\langle F_L^{\mu\nu}D_\mu U^\dagger D_\nu U + F_R^{\mu\nu}D_\mu U D_\nu U^\dagger\rangle = 2P_9\,. \quad (38)$$

The P_i's are the usual terms of the $O(p^4)$ chiral Lagrangian.[14]

Referring to the conventional definition of the coefficients of the $O(p^4)$ CHPT Lagrangian $L_1, L_2, \ldots, L_{10}$, H_1, H_2 we find that the path integral equivalence of vector and tensor models (a) fixes the contribution of vector mesons to some of the low energy coefficients and (b) implies relations amongst them. Both (a) and (b) classes of identities have been derived in other ways, but never proven at the formal level as it is shown here. The structure of the local term in Eq. (36) implies

$$H_1^V = -\frac{f_V^2}{8}\,, \qquad L_{10}^V(\gamma_{10}^{II}) = -\frac{f_V^2}{4} \qquad \text{and} \qquad L_{10}^V = 2H_1^V\,. \quad (39)$$

The coefficient L_{10}^V is also the coefficient γ_{10}^{II} of Ref. 11 of the same local term added to the vector Lagrangian in order to satisfy the off-shell equivalence with the tensor one.

The local term in Eq. (37) can be reduced to a more familiar form via the use of SU(3) relations for flavour traces.[14] Its structure implies

$$L_1^V, \gamma_1^{II} = \frac{g_V^2}{8}\,, \qquad L_2^V, \gamma_2^{II} = \frac{g_V^2}{4}\,, \qquad L_3^V, \gamma_3^{II} = -\frac{3}{4}g_V^2\,, \quad (40)$$

which give the identities $L_2^V = 2L_1^V$ and $L_3^V = -3L_2^V$.

The local term (38) fixes the vector contribution to the low energy parameter L_9 (which also corresponds to the coefficient γ_9^{II} of the same local term in Ref. 11) to be:

$$L_9^V = \frac{f_V g_V}{2}\,. \quad (41)$$

We thus derive the same relations as those previously obtained. In order to get the final values for the vector resonance contribution to the L_i we need additional arguments. Our duality argument of course does not tell us which vector representation is the one with no point-like pseudoscalar couplings present. The VMD argument of Ref. 11 shows that this is the tensor version.

4. Conclusions

In this letter we have explicitly shown the relation between the vector field and the antisymmetric tensor field descriptions of massive spin-1 particles. The equivalence is proven at the classical level where the vector field obeys a transversality condition compatible with the dual transformation of (7). The relation of the vector representation used here for the Hidden gauge model and others can be found in Refs. 11 and 13.

This work has added to the list of known field redefinitions and also those that end up with the tensor representation. The method used here can be systematically extended to terms that contain powers of quark masses and derivatives beyond those explicitly considered here, as well as to the "anomalous" or abnormal intrinsic parity sector of vector meson Lagrangians. The extension to axial vector mesons is also trivial.

Acknowledgments

This work was partially supported by NorFA grant 93.15.078/00. J.B. would like to thank IPN-Orsay, where part of this work was done, for hospitality. The work of EP was supported by the EU Contract No. ERBCHBGCT 930442.

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