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OF  $GL_3$**

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# ON THE TENSOR PRODUCT OF THETA REPRESENTATIONS OF $GL_3$

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Let  $V$  be the theta representation of  $\widetilde{GL}_3$ —the two fold central extension of  $GL_3$ . Let  $W$  be a spherical representation of  $GL_3$ . We show that there is a nonzero  $GL_3$  invariant trilinear form on  $V \otimes V^* \otimes W$  if and only if  $W$  is a lift from  $SL_2$ . In this case the form is unique up to a scalar.

**Introduction.** Let  $k$  be a global field and  $A$  its ring of adeles. Let  $\sigma$  be an irreducible 3 dimensional representation of the Galois group  $\Gamma$  of  $k$ . Assume, for simplicity, that  $\sigma(\Gamma) \subset SL_3(\mathbb{C})$ . Then, according to Langlands there exists an automorphic representation  $\pi \subset L^2(PGL_3(k) \backslash PGL_3(A))$  such that the corresponding  $L$ -functions are equal. Consider the symmetric square of the representation  $\sigma$ . Then, conjecturally, the corresponding  $L$  function will have a pole only if the symmetric square representation contains a copy of trivial representation. But this means that there is a quadratic form invariant under  $\sigma$  and therefore  $\sigma(\Gamma) \subset SO_3(\mathbb{C})$ . Since  $SO_3(\mathbb{C}) = {}^L SL_2$ , the automorphic representation  $\pi$  should be a lift of an automorphic representation of  $SL_2$ . Let  $\pi_v$  be a local component of  $\pi$ . If it is spherical,  $\pi_v$  is the local lift of a representation of  $SL_2$  if  $\pi_v = \text{ind}_B^{\text{PGL}_3} \chi$  where  $\chi$  is a character of the diagonal subgroup of  $PGL_3$  given by

$$\chi \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = \mu \left( \frac{a}{c} \right)$$

for some unramified character  $\mu: k_v^* \rightarrow \mathbb{C}^*$ .

On the other hand, Patterson and Piatetski-Shapiro [PP] have constructed the symmetric square  $L$ -function corresponding to a cuspidal automorphic representation  $\pi$  of  $PGL_3$ . Moreover, they showed that the residue at  $s = 1$  of this  $L$ -function is

$$\int_{PGL_3(k) \backslash PGL_3(A)} \varphi(g) \theta(g) \theta'(g) dg$$

where  $\varphi \in \pi$  and  $\theta, \theta'$  are “theta functions” of Kazhdan and Patterson [KP]. They are certain automorphic forms on  $\widetilde{GL}_3$ —the two

fold central extension of  $\mathrm{GL}_3$ . Let  $F$  be a local field. In [FKS] we have constructed a smooth model  $(\theta_3, V)$  of the local component of “theta functions”. Let  $(\pi, W)$  be an irreducible representation of  $\mathrm{PGL}_3(F)$ . From what was explained above, it is natural to ask whether there is a  $\mathrm{GL}_3$  invariant trilinear form on  $V \otimes V^* \otimes W$ . We have the following result:

**THEOREM.** *Let  $F$  be a local field of the characteristic  $\neq 2$ . Let  $(\pi, W)$  be a spherical representation of  $\mathrm{PGL}_3$ . Then there exists a  $\mathrm{GL}_3$  invariant trilinear form on  $V \otimes V^* \otimes W$  if and only if  $\pi$  is the lift of a representation of  $\mathrm{SL}_2$ . Moreover, the form is unique up to a scalar.*

We remark that the article of Prasad [P] was perhaps the first result indicating relationship between special values of  $L$ -functions and invariant functionals.

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**Preliminaries and notation.** Let  $P_1$  (resp.  $P_2$ ) be the standard  $(2, 1)$  (resp.  $(1, 2)$ ) parabolic subgroup of  $\mathrm{GL}_3$ . Let  $P_1 = M_1 U_1$  and  $P_2 = M_2 U_2$  be standard Levi decompositions. We shall use the letter  $N$  to denote the unipotent group of uppertriangular matrices of  $\mathrm{GL}_2$  and  $\mathrm{GL}_3$  and the letter  $T$  to denote the group of diagonal matrices of  $\mathrm{GL}_2$  and  $\mathrm{GL}_3$ . It will be clear from the context which is meant. Finally put  $N_1 = N \cap M_1$ ,  $N_2 = N \cap M_2$  and  $B = TN$ .

Let  $P = MU$  be a parabolic subgroup and  $(\pi, V)$  a smooth module. Define  $V(U) = \mathrm{span}\{v - \pi(u)v \mid v \in V, u \in U\}$ . Then  $V_U = V/V(U)$  is the module of coinvariants.

Let  $X$  be an algebraic variety over the field  $F$ . Then  $S(X)$  will denote the space of locally constant, compactly supported functions on  $X$ . Obviously,  $S(X_1 \times X_2) = S(X_1) \otimes S(X_2)$ . Let  $q$  be an algebraic function on  $X$ . Then one can define a representation  $\pi$  of  $N$  on  $S(X)$  by

$$\pi \left( \begin{pmatrix} 1 & n \\ & n \end{pmatrix} \right) f(x) = \phi(nq(x))f(x)$$

where  $\phi$  is an additive character of  $F$ . It is easy to check that  $S(X)_N = S(Y)$  where  $Y$  is the subvariety of  $X$  defined by  $q = 0$ .

We need to recall some facts about the principal series representations of  $\mathrm{GL}_3(F)$ . Let  $\mathrm{St}$  denote the Steinberg representation of  $\mathrm{GL}_2$ .

Let  $\lambda$  be a multiplicative character of  $F^*$  such that  $\lambda^2 = 1$ . Put  $\text{St}_\lambda = \text{St} \otimes \lambda(\det)$ .

Let  $\mu$  be a character of  $F^*$ . It defines a character  $\chi_\mu: T \rightarrow \mathbb{C}^*$  by the following formula:

$$\chi_\mu \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = \mu \left( \frac{a}{c} \right).$$

Let  $\pi(\mu) = \text{ind}_B^{\text{GL}_3} \chi_\mu$  (normalized induction). To describe the composition series of  $\pi(\mu)$  we need to introduce  $\sigma_1, \sigma_2$  representations of  $\text{GL}_3$  defined as follows:

$$\begin{aligned} 0 \rightarrow 1 \rightarrow \text{ind}_{P_1}^{\text{GL}_3} 1 \rightarrow \sigma_1 \rightarrow 0, \\ 0 \rightarrow 1 \rightarrow \text{ind}_{P_2}^{\text{GL}_3} 1 \rightarrow \sigma_2 \rightarrow 0. \end{aligned}$$

Here the induction is not normalized! We need the following result about the principal series representations. A reader can find details in Cartier's article [C, §III].

**PROPOSITION 1.** *The representations  $\pi(\mu)$  are irreducible and  $\pi(\mu) \cong \pi(\mu^{-1})$  unless  $\mu$  is of the two following types:*

(a)  $\mu = \lambda | \cdot |^{\pm 1/2}$ ,  $\lambda^2 = 1$ . *The composition series consists of  $\text{ind}_P^{\text{GL}_3} \lambda(\det)$  and  $\text{ind}_P^{\text{GL}_3} \text{St}_\lambda$ .*

(b)  $\mu = | \cdot |^{\pm 1}$ . *The composition series consists of the trivial representation, the Steinberg representation,  $\sigma_1$  and  $\sigma_2$ .*

**The central extension and theta representation.** Let  $(\cdot, \cdot): F^* \times F^* \rightarrow \{\pm 1\}$  be the Hilbert symbol. Let  $\widetilde{\text{GL}}_n(F)$  be the 2-fold central extension of  $\text{GL}_n(F)$  and  $\mathbf{s}: \text{GL}_n \rightarrow \widetilde{\text{GL}}_n$  the section as in [FKS]. The extension can be characterized in the following way:

$$\mathbf{s}(\text{diag}(a_i))\mathbf{s}(\text{diag}(b_i)) = \mathbf{s}(\text{diag}(a_i b_i)) \prod_{i < j} (a_i, b_j),$$

where  $\text{diag}(a_i)$  denotes the diagonal matrix with entries  $a_i$ . Moreover the section  $\mathbf{s}$  is an isomorphism on  $N$  and we will identify  $N$  and  $\mathbf{s}(N)$ . Fix a nontrivial additive character  $\phi: F \rightarrow \mathbb{C}^*$ . Define a function  $\gamma = \gamma_\phi: F^* \rightarrow \mathbb{C}^*$  by

$$\gamma(a) = \frac{|a|^{1/2} \int \phi(ax^2) dx}{\int \phi(x^2) dx}.$$

LEMMA 1 (Weil [W1]). *The function  $\gamma$  has the following properties:*

(a)  $\gamma(ab) = \gamma(a)\gamma(b)(a, b),$

(b)  $\overline{\gamma}_\phi = \gamma_{\overline{\phi}}.$  □

DEFINITION 1. Let  $C_2(F)$  be the space of locally constant functions on  $F^*$  such that

(a)  $f(x) = 0$  if  $|x| > c,$

(b)  $f(y^2x) = f(x)$  if  $|x|, |y^2x| < 1/c,$

where  $c$  is a constant depending on  $f$ .

The theta representation  $\theta_2$  of  $\widetilde{\text{GL}}_2$  can be realized on the space of functions  $f$  on  $F^*$  such that  $|x|^{1/4}f(x) \in C_2(F)$ . The action of  $\widetilde{\text{GL}}_2$  is given by the following formulae [F]:

$$\theta_2 \left( \begin{pmatrix} a & \\ & 1 \end{pmatrix} \right) f(x) = |a|^{1/2} f(ax),$$

$$\theta_2 \left( \begin{pmatrix} z & \\ & z \end{pmatrix} \right) f(x) = (x, z) \gamma(z) f(x),$$

$$\theta_2 \left( \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \right) f(x) = \phi(nx) f(x),$$

$$\theta_2 \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) f(x) = c \gamma(x) |x|^{1/2} \int_F |y|^{1/2} f(xy^2) \phi(xy) dy$$

for some constant  $c$ .

PROPOSITION 2. *Let  $\lambda$  be a multiplicative character of  $F^*$  such that  $\lambda^2 = 1$ . Then  $\theta_2 \cong \theta_2 \otimes \lambda(\det)$ .*

*Proof.* Let  $\theta$  be the even Weil representation of  $\widetilde{\text{SL}}_2$ . Let  $G$  be the subgroup of  $\widetilde{\text{GL}}_2$  consisting of the elements whose determinant is a square in  $F^*$ . It is easy to see that  $\theta$  extends to  $G$  and that  $\theta_2 = \text{ind}_G^{\widetilde{\text{GL}}_2} \theta$ . Since  $\widetilde{\text{GL}}_2/G \cong F^*/(F^*)^2$  the proposition follows.

DEFINITION 2. Let  $H$  be a group and  $C$  its center. We say that  $H$  is a Heisenberg group if  $H/C$  is abelian.

To give a characterization of  $\theta_3$  we need a simple result about Heisenberg groups (see [KP, §0.3]):

LEMMA 2. *Let  $H$  be a Heisenberg group and  $C$  its center. Let  $\delta$  be a character of  $C$ . Assume that  $\delta$  is faithful on  $[H, H] \subset C$ . Then there is unique irreducible representation  $\pi_\delta$  of  $H$  such that  $C$  acts by multiplication by  $\delta$ . Moreover,  $\pi_\delta \otimes \pi_{\bar{\delta}}$  is just the regular representation of  $H/C$ .  $\square$*

Let  $\tilde{T}$  be the inverse image of  $T$  in  $\widetilde{\text{GL}}_3$ . Let  $Z$  be the center of  $\text{GL}_3$  and  $\tilde{Z}$  the inverse image in  $\widetilde{\text{GL}}$ . It is easy to check that  $\tilde{Z}$  is the center of  $\widetilde{\text{GL}}_3$ . The group  $\tilde{T}$  is a Heisenberg group with center  $C = \tilde{Z} \cdot \mathbf{s}(T^2)$  where  $T^2$  is the group of diagonal matrices whose entries are squares. Define a character  $\delta$  of  $C$  by

$$\delta(\mathbf{s}(z)\mathbf{s}(t^2)\zeta) = \gamma(z)\zeta, \quad \zeta \in \{\pm 1\}.$$

Let  $\pi_\delta$  be the corresponding representation of  $\tilde{T}$ . Define  $\rho$  to be, as usual,

$$\rho \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = \left| \frac{a}{c} \right|.$$

In [FKS] we have the following theorem.

THEOREM 1. *There is a unique representation  $(\theta_3, V)$  of  $\widetilde{\text{GL}}_3$  such that  $V_N \cong \rho^{1/2} \otimes \pi_\delta$ . The properties of  $\theta_3$  are:*

- (1)  $\theta_3(\mathbf{s}(z)) = \gamma(z) \text{Id}$ ,  $z \in Z$ .
- (2)  $V_{U_1} \cong \theta_2 \otimes |\det|^{1/4}$ ,  $V_{U_2} \cong \theta_2 \otimes |\det|^{-1/4}$ .
- (3) Let  $V_0 = V(U_1) \cap V(U_2)$ ; then  $V_0 \cong S(F^* \times F)$  with the action of  $\tilde{B}$  given by

$$\begin{aligned} \theta_3 \left( \mathbf{s} \begin{pmatrix} a & b & \\ & d & \\ & & 1 \end{pmatrix} \right) f(x, y) &= (x, d) |ad|^{1/2} f(ax, bx + dy), \\ \theta_3 \left( \begin{pmatrix} 1 & & u \\ & 1 & v \\ & & 1 \end{pmatrix} \right) f(x, y) &= \phi(ux + vy) f(x, y). \end{aligned}$$

- (4)  $V(U_1) + V(U_2) = V(N)$ .

*In particular, it follows that we have a filtration of  $V$  as a  $\tilde{B}$  module such that the quotients are  $V_0$ ,  $V_{U_1}(N_1)$ ,  $V_{U_2}(N_2)$  and  $V_N$ .*

REMARK. Note that the dual representation  $\theta_3^*$  is obtained by replacing  $\phi$  by  $\bar{\phi}$ .

*Proof of the Theorem.* Let  $(\pi, W)$  be a representation of  $\mathrm{GL}_3$ . Then the existence of a nontrivial trilinear  $\mathrm{GL}_3$  invariant form is equivalent to the existence of a nontrivial  $\mathrm{GL}_3$  intertwining map from  $V \otimes V^*$  to  $W^*$ . Hence we have to compute  $\dim \mathrm{Hom}_G(V \otimes V^*, W)$ . Assume that  $\pi = \mathrm{ind}_B^G \chi$ . Then by the Frobenius reciprocity we have  $\dim \mathrm{Hom}_G(V \otimes V^*, W) = \dim \mathrm{Hom}_T((V \otimes V^*)_N, \rho\chi)$ . In other words we have restricted the problem to computing the  $T$ -equivariant functionals on  $(V \otimes V^*)_N$ . From  $\tilde{B}$  filtration of  $V$  it follows that  $(V \otimes V^*)_N$  has a filtration whose quotients are  $(V_0 \otimes V_0^*)_N$ ,  $(V_{U_1}(N_1) \otimes V_{U_1}^*(N_1))_{N_1}$ ,  $(V_{U_2}(N_2) \otimes V_{U_2}^*(N_2))_{N_2}$  and  $V_N \otimes V_N^*$ .

Let  $\Gamma_\mu$  be the functional on  $S(F^*)$  given by

$$\Gamma_\mu(f) = \int_{F^*} f(x) \mu(x^{-1}) \frac{dx}{|x|}.$$

Obviously, we have the following simple proposition:

**PROPOSITION 3.** *The functional  $\Gamma_\mu$  is unique up to a nonzero constant  $\mu$ -equivariant functional on  $S(F^*)$  with respect to the standard action of  $F^*$ .*  $\square$

Next we need to describe  $F^*$  equivariant functionals on  $C_2(F)$ .

**PROPOSITION 4** (see [W2]). *Let  $\mu$  be a character of  $F^*$ . The functional  $\Gamma_\mu$  extends to  $C_2(F)$  if  $\mu^2 \neq 1$ .*

*Proof.* Let  $\mathcal{O}$  be the ring of integers of  $F$  and  $\varpi$  a uniformizing element. Put  $q = |\varpi|^{-1}$ . Assume that  $\mu^2(x) = |x|^{-s}$ . Let  $f \in C_2(F)$ . Consider the integral

$$\Lambda_s(f) = \int_{F^*} (f(x) - f(\varpi^2 x)) |x|^s \frac{dx}{|x|}.$$

Obviously  $\Lambda_s(f)$  is defined for every  $s$  and if  $\mathrm{Re}(s) > 0$  then

$$\Lambda_s(f) = (1 - q^s) \Gamma_\mu(f).$$

This formula extends the functional  $\Gamma_\mu$  to  $C_2(F)$  if  $\mu^2(x) = |x|^{-s}$  and  $s \neq 0$ . If  $\mu^2$  is ramified then  $\Gamma_\mu$  extends by taking the Principal Value integral. The proposition is proved.

**PROPOSITION 5.** *Let  $\chi$  be a character of  $T$ . The space of  $\chi$ -equivariant linear functionals on  $V_N \otimes V_N^*$  is at most 1-dimensional. It has the*

dimension one if and only if  $\chi = \rho\lambda$

$$\lambda \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = \mu_1(a)\mu_2(b)\mu_3(c),$$

$$\mu_i^2 = 1 \text{ and } \mu_1 \cdot \mu_2 \cdot \mu_3 = 1.$$

*Proof.* It follows from Lemma 2.

Let  $\mu$  be a multiplicative character of  $F^*$ . Let  $\delta_{1,2}(\mu)$ ,  $\delta_{1,3}(\mu)$  and  $\delta_{1,3}(\mu)$  be the characters of  $T$  defined by

$$\begin{aligned} \delta_{12}(\mu)(t) &= \mu(a)\mu(b)^{-1}, \\ \delta_{23}(\mu)(t) &= \mu(b)\mu(c)^{-1}, \\ \delta_{13}(\mu)(t) &= \mu(a)\mu(c)^{-1} \end{aligned}$$

where  $t = \text{diag}(a, b, c)$ . Let  $W^{T,\chi}$  denote the space of  $\chi$ -equivariant functionals on a smooth  $T$  module  $W$ .

**PROPOSITION 6.**

$$\begin{aligned} \text{(a)} \quad \dim(V_{U_1}(N_1) \otimes V_{U_1}^*(N_1))_{N_1}^{T,\chi} &= \begin{cases} 1 & \text{if } \chi = \rho\delta_{12}(\mu), \\ 0 & \text{otherwise.} \end{cases} \\ \text{(b)} \quad \dim(V_{U_2}(N_2) \otimes V_{U_2}^*(N_2))_{N_2}^{T,\chi} &= \begin{cases} 1 & \text{if } \chi = \rho\delta_{23}(\mu), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

*Proof.* Using property (2) of  $\theta_3$  and the description of  $\theta_2$  it is easy to check that  $V_{U_1}(N_1) \cong S(F^*)$  and therefore  $(V_{U_1}(N_1) \otimes V_{U_1}^*(N_1))_{N_1} \cong S(F^*)$  with the action of  $T$  given by

$$\theta_3 \otimes \theta_3^* \left( \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \right) f(x) = \left| \frac{a}{c} \right| \left| \frac{a}{b} \right|^{1/2} f\left(\frac{a}{b}x\right).$$

Part (a) now follows from Proposition 3. Part (b) is proved analogously.

**PROPOSITION 7.**

$$(V_0 \otimes V_0^*)_N^{T,\chi} = \begin{cases} 1 & \text{if } \chi = \rho\delta_{13}(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Using property (3) of  $\theta_3$  it follows that

$$(V_0 \otimes V_0^*)_{U_1} \cong S(F^* \times F)$$



with the action of  $TN_1$  given by

$$\begin{aligned}\theta_3 \otimes \theta_3^* \left( \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \right) f(x, y) &= \left| \frac{ab}{c^2} \right| f\left(\frac{a}{c}x, \frac{b}{c}y\right) \quad \text{and} \\ \theta_3 \otimes \theta_3^* \left( \begin{pmatrix} 1 & n & \\ & 1 & \\ & & 1 \end{pmatrix} \right) f(x, y) &= f(x, nx + y).\end{aligned}$$

After taking the Fourier transform in the second variable the action becomes

$$\begin{aligned}\theta_3 \otimes \theta_3^* \left( \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \right) f(x, y) &= \left| \frac{a}{c} \right| f\left(\frac{a}{c}x, \frac{c}{b}y\right) \quad \text{and} \\ \theta_3 \otimes \theta_3^* \left( \begin{pmatrix} 1 & n & \\ & 1 & \\ & & 1 \end{pmatrix} \right) f(x, y) &= f(x, y)\phi(nxy).\end{aligned}$$

Therefore  $(V_0 \otimes V_0^*)_N \cong S(F^*)$  with the action of  $T$  given by

$$\theta_3 \otimes \theta_3^* \left( \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \right) f(x) = \left| \frac{a}{c} \right| f\left(\frac{a}{c}x\right).$$

The proposition follows from Proposition 3.

Let us call  $T$  equivariant functionals appearing in Proposition 5 (resp. Propositions 6 and 7) of type I (resp. II and III). Since  $V_N \otimes V_N^*$  is a quotient of  $(V \otimes V^*)_N$ , functionals of type I extend to  $(V \otimes V^*)_N$ . In the next several propositions we are studying extension of the functionals of type II and III to  $(V \otimes V^*)_N$ .

**PROPOSITION 8.** *The functionals of type II extend to  $(V \otimes V^*)_N$  if and only if  $\mu^2 \neq 1$ .*

*Proof.* Since  $V_{U_1}$  is a quotient of  $V$  it follows that  $(V_{U_1} \otimes V_{U_1}^*)_{N_1}$  is a quotient of  $(V \otimes V^*)_N$ . Recall that  $V_{U_1} \cong \theta_2 \otimes |\det|^{1/4}$ . The value of a  $\rho\delta_{12}(\mu)$  equivariant functional on  $(V_{U_1}(N_1) \otimes V_{U_1}^*(N_1))_{N_1}$  is given by the following integral.

$$I_\mu(f \otimes f^*) = \int_{F^*} |x|^{1/2} f(x) f^*(x) \mu(x^{-1}) \frac{dx}{|x|}.$$

If  $f \in \theta_2 \otimes |\det|^{1/4}$  and  $f^* \in \theta_2^* \otimes |\det|^{1/4}$  it follows from the description of  $\theta_2$  that  $|x|^{1/4} f(x)$  and  $|x|^{1/4} f^*(x) \in C_2(F)$ . Therefore

$I_\mu$  defines a  $\rho\delta_{12}(\mu)$  equivariant functional on  $(V \otimes V^*)_N$  if  $\mu^2 \neq 1$  by Proposition 4. It remains to deal with  $\mu$ ,  $\mu^2 = 1$ . Let  $\varphi \in V_{U_1}$  be a function given by

$$\varphi(x) = \begin{cases} |x|^{-1/4} & \text{if } |x| \leq 1 \text{ and } x \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$v = |\varpi|^2 \varphi \otimes \varphi^* - |\varpi| \theta_2 \otimes \theta_2^* \left( \begin{pmatrix} \varpi^2 & \\ & 1 \end{pmatrix} \right) \varphi \otimes \varphi^*.$$

The projection of  $v$  on  $(V_{U_1} \otimes V_{U_1}^*)_{N_1}$  lies in  $(V_{U_1}(N_1) \otimes V_{U_1}^*(N_1))_{N_1} \cong S(F^*)$  and is given by

$$\begin{aligned} & |\varpi|^2 \varphi(x) \varphi^*(x) - |\varpi|^3 \varphi(\varpi^2 x) \varphi^*(\varpi^2 x) \\ &= \begin{cases} -|\varpi| & \text{if } |x| = |\varpi|^{-2} \text{ and } x \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that  $I_\mu(v) < 0$ . On the other hand, if the functional  $I_\mu$  extends to  $(V \otimes V^*)_N$  then the equivariance implies  $I_\mu(v) = 0$ . Contradiction. Similar conclusions can be obtained for the characters  $\rho\delta_{23}(\mu)$ . The proposition is proved.

Let  $m_{ij}(\mu)$  be the multiplicity of  $\rho\delta_{ij}(\mu)$  equivariant functionals on  $(V \otimes V^*)_N$ . If the principal series representation  $\pi(\mu)$  is irreducible then  $\pi(\mu) = \text{ind}_B^G \delta_{ij}(\mu)$  for all  $1 \leq i < j \leq 3$  [C]. In particular,  $m_{ij}(\mu)$  is independent of  $i, j$ . Therefore, we have obtained the following corollary.

**COROLLARY 1.** *The functional  $\rho\delta_{13}(\mu)$  of type III extends to  $(V \otimes V^*)_N$  if  $\mu \neq |\cdot|^{\pm 1}$ ,  $\mu^2 \neq |\cdot|^{\pm 1}$  and  $\mu^2 \neq 1$ . If  $\mu^2 = 1$  and  $\mu \neq 1$  then it does not extend.*  $\square$

It remains to deal with  $\rho\delta_{13}(1)$ .

**PROPOSITION 9.** *The functional  $\rho\delta_{13}(1)$  of type III does not extend to  $(V \otimes V^*)_N$ .*

*Proof.* The value of a  $\rho\delta_{13}(\mu)$  equivariant functional on  $(V_0 \otimes V_0^*)_N$  is given by the following integral:

$$I_\mu(f \otimes f^*) = \int_{F^*} \int_F f(x, y) f^*(x, y) \mu(x^{-1}) \frac{dx}{|x|} dy.$$

Let  $\varphi$  be a function on  $F^* \times F$  given by

$$\varphi(x, y) = \begin{cases} 1 & \text{if } |x| \leq 1 \text{ and } |y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\varphi \in V(U_1)$  and let

$$v = |\varpi| \varphi \otimes \varphi^* - \theta_3 \otimes \theta_3^* \left( \begin{pmatrix} \varpi & \\ & 1 \\ & & 1 \end{pmatrix} \right) \varphi \otimes \varphi^*.$$

The projection of  $v$  on  $(V(U_1) \otimes V^*(U_1))_{U_1}$  lies in  $(V_0 \otimes V_0^*)_{U_1} \cong S(F^* \times F)$  and is given by

$$\begin{aligned} & |\varpi| \varphi(x, y) \varphi^*(x, y) - |\varpi| \varphi(\varpi x, y) \varphi^*(\varpi x, y) \\ &= \begin{cases} -|\varpi| & \text{if } |x| = |\varpi|^{-1} \text{ and } |y| = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that  $I_1(v) < 0$ . On the other hand, if the functional  $I_1$  extends to  $(V \otimes V^*)_N$  then the equivariance implies  $I_1(v) = 0$ . Contradiction. The proposition is proved.

**COROLLARY 2.** *Let  $\chi$  be a character of  $T$ . Then  $\dim(V \otimes V^*)_N^{T, \chi} \leq 1$ .  $\square$*

Let  $\mu_1, \mu_2, \mu_3$  be three characters of  $F^*$  such that  $\mu_i^2 = 1$  and  $\mu_1 \cdot \mu_2 \cdot \mu_3 = 1$ . Let  $\chi$  be a character of  $T$  defined by  $\chi(\text{diag}(a, b, c)) = \mu_1(a) \mu_2(b) \mu_3(c)$ . Define  $\pi(\mu_1, \mu_2, \mu_3) = \text{ind}_B^G \chi$  (normalized induction). It is unitary irreducible representation. We are now ready to formulate our main result:

**THEOREM.** *Let  $(\pi, W)$  be a quotient of a principal series representation of  $\text{GL}_3$ . Then the space of  $\text{GL}_3$  invariant trilinear forms on  $V \otimes V^* \otimes W$  is 0 or 1 dimensional. The dimension is 0 unless  $\pi$  is one of the following:*

- (a)  $\pi(\mu_1, \mu_2, \mu_3)$ ,  $\mu_i^2 = 1$  and  $\mu_1 \mu_2 \mu_3 = 1$ ,
- (b)  $\pi(\mu)$ ,  $\mu^2 \neq |\cdot|^{\pm 1}$  and  $\mu \neq |\cdot|^{\pm 1}$ ,
- (c) trivial representation,
- (d)  $\text{ind}_P^{\text{GL}_3} \mu$ ,  $\mu^2 = 1$ ,
- (e)  $\text{ind}_P^{\text{GL}_3} \text{St}_\mu$ ,  $\mu^2 = 1$ ,
- (f)  $\sigma_1, \sigma_2$  and  $\text{St}$ .

*In cases (a)–(d) the dimension is 1. In cases (e) and (f) the dimension is  $\leq 1$ .*

*Proof.* Clearly the dimension is 0 unless  $\pi$  is one of the representations in (a)–(f). Since representations in (a) and (b) are irreducible these two cases follow from Corollary 2. The trace  $\text{tr}: V \times V^* \rightarrow \mathbb{C}$

is a  $\mathrm{GL}_3$  invariant trilinear form for  $\pi = 1$ . We can similarly deal with the representations in (d). Indeed,  $V_U \otimes V_U^*$  is a quotient of  $(V \otimes V^*)_U$ . Since  $V_U \cong \theta_2 \otimes |\det|^{1/4}$  and  $\theta_2 \cong \theta_2 \otimes \mu(\det)$  by Proposition 2 we can define an appropriate  $P$ -equivariant functional on  $(V \otimes V^*)_U$  defining a map from  $V \otimes V^*$  into  $\mathrm{ind}_P^G \mu$ . The theorem is proved.

**COROLLARY.** *Let  $(\pi, W)$  be a spherical representation of  $\mathrm{GL}_3$ . Then there exists a  $\mathrm{GL}_3$  invariant trilinear form on  $V \otimes V^* \otimes W$  if and only if  $\pi$  is the lift of a representation of  $\mathrm{SL}_2$ . In this case the form is unique up to a scalar.*

*Proof.* Note that there is only one nontrivial unramified character  $\mu$  of  $F^*$  such that  $\mu^2 = 1$ . Therefore if  $\pi(\mu_1, \mu_2, \mu_3)$  is spherical then  $\pi(\mu_1, \mu_2, \mu_3) \cong \pi(\mu, 1, \mu^{-1})$  for some unramified character  $\mu$ ,  $\mu^2 = 1$ .

**A final remark.** Recently Bump and Ginsburg [BG] have generalized the work of Patterson and Piatetski-Shapiro to construct an integral representation of the symmetric square  $L$ -function corresponding to a cuspidal automorphic representation  $\pi$  of  $\mathrm{PGL}_n$ . As in the case  $n = 3$ , the residue at  $s = 1$  of the  $L$ -function is

$$\int_{\mathrm{PGL}_n(k) \backslash \mathrm{PGL}_n(\mathbf{A})} \varphi(g) \theta(g) \theta'(g) dg$$

where  $\varphi \in \pi$  and  $\theta, \theta'$  are “theta functions” of  $\widetilde{\mathrm{GL}}_n$ —the two fold central extension of  $\mathrm{GL}_n$ . The result of Bump and Ginzburg suggests the following generalization of our result:

**CONJECTURE.** *Let  $F$  be a local field of the characteristic  $\neq 2$  and let  $(\theta, V)$  be the theta representation of  $\widetilde{\mathrm{GL}}_n$ . Let  $(\pi, W)$  be a spherical representation of  $\mathrm{PGL}_n$ . Then there exists a  $\mathrm{GL}_n$  invariant trilinear form on  $V \otimes V^* \otimes W$  if and only if  $\pi$  is the lift of a representation of  $\mathrm{Sp}(2m)$  if  $n = 2m + 1$  or  $\pi$  is the lift of a representation of  $\mathrm{SO}(2m)$  if  $n = 2m$ .*

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