This case includes all the systems usually met with in the Quantum Theory, the vanishing of the C's for certain values of the suffixes manifesting itself by selection principles which allow only a finite number of changes for each of the quantum numbers except at most two of them. The quantum integrals of such systems are invariant under any adiabatic change except at a finite set of points where

$$\omega_r/\omega_s = \frac{\partial \omega_r}{\partial a} \bigg/ \frac{\partial \omega_s}{\partial a},$$

ω and ω_s being the frequencies corresponding to the quantum numbers whose changes are unrestricted, and at another finite set (or possibly infinite enumerable set tending to points of the previous set) where relations of the type $\Sigma m_r \omega_r = 0$ hold.

The writer is much obliged to Mr. R. H. Fowler for suggesting this investigation, and for his help during its progress.

On the Theory of Elastic Stability.

By W. R. Dean, B.A., Fellow of Trinity College, Cambridge.

(Communicated by Prof. G. I. Taylor, F.R.S.—Received December 20, 1924.)

The object of the present paper is to derive equations that are adequate to decide questions of the stability under stress of thin shells of isotropic elastic material. Equations for the same purpose have been given by R. V. Southwell,* who used a method that is closely followed in a part of this paper.

Such equations must contain terms that may be, and are, neglected in applications of the theory of elasticity to problems in which the stability of configurations is not considered. The truth of Kirchhoff's uniqueness theorem,† which has reference to the ordinary equations of elasticity, in which powers of the displacement co-ordinates above the first are neglected, is sufficient proof of this statement. In practice it is generally sufficient to retain only the first and second order terms,‡ and no terms of higher order are considered

* "On the General Theory of Elastic Stability," 'Phil. Trans.,' A, vol. 213, p. 187.

† A. E. H. Love, 'Mathematical Theory of Elasticity' (3rd Edition), § 118.

‡ There are exceptions to this. *Cf.* a paper by J. Prescott, 'Phil. Mag.,' vol. 43, p. 97 (1922), which, though not immediately concerned with elastic stability, obtains equations which can be applied to this theory. See also § 9 below.

here. To obtain such equations an extended form of Hooke's Law is necessary; the extension made by Southwell* is used in this paper. There are then two methods available for the derivation of the equations. Either we may obtain the three conditions for the equilibrium of an elementary volume of the substance by considering the forces acting upon it, or we may calculate the energy of strain correct to the *third* order of displacement co-ordinates, and deduce the equations by variation of this function. The first method has been used in one place here, as it would appear to be the simpler in the particular case of a plane plate, in which only one of the equations, and that the simplest, is required. However, the stability equations for a cylindrical shell are also obtained, and then all three equations are necessary. The derivation by the first method of each one of these is a laborious matter, while using the second method there is only one calculation, that of the strain energy function, to be made. Consequently, for this purpose, as in general, the second method seems to be preferable.

The equations that are obtained by either of the methods outlined above refer in the first instance to the co-ordinates of displacement of any point of the shell. Yet it is clearly desirable to have instead equations which connect the displacements of points only of the middle surface, for equations of this type will be simpler in so far as these displacements are functions of two variables only, while a knowledge of the behaviour of the middle surface is evidently sufficient to decide a question of stability. What is wanted, in fact, is a method of reduction of equations involving the displacements of all points of the shell, to equations involving only the displacements of points of the middle surface,† precisely similar to that used in the Theory of Thin Shells.‡ The assumptions used in the reduction by this theory, however, are such that it is not clear how they can be used, or extended, to effect the reduction of second-order general equations that is required here.

Consequently, no use of them has been made. It is merely supposed that the displacement co-ordinates of any point of the shell can be expanded in power series of the normal distance of the point from the middle surface. Second-order shell equations can then be deduced from the general equations by using the boundary conditions at the two faces of the shell. The method will in the same way reduce general equations of the first order to the corre-

* *Loc. cit.*, p. 192.

† It will be convenient in what follows to call equations of these two types "general" equations and "shell" equations respectively.

‡ Love, *op. cit.*, chap. 24.

sponding shell equations. The assumption, therefore fundamental, as to the expansions of displacements may exclude some problems from the range of the method, but it does not appear likely that in elastic stability, where attention is of necessity confined to a consideration of the simplest types of stress, there will be any difficulty on this score. Moreover, the equations in their final form contain no explicit reference to the assumption, so that they may be, and certainly in first order problems often are, valid beyond the limits that might appear to be imposed. The equations thus obtained are not here applied to any new problem, but as it appeared desirable to check the results of a new method by a comparison with known formulæ, the stability of a tubular strut has been briefly considered. The condition for instability in a symmetrical mode of distortion has been deduced by Southwell from general theory, and in other modes from the Theory of Thin Shells.* Equivalent results are obtained in each case by the methods of this paper.

The Strain Energy Function.

2. We proceed to develop a method of finding in a suitable form the strain energy function of a thin cylindrical shell. The energy of strain is a quadratic function of the components of strain, which ordinarily need only be evaluated correctly to the second order of displacement co-ordinates. This accuracy is not sufficient if problems of stability are to be considered, and a more exact determination demands a complete revision of the usual processes of evaluation. In the first place an extended statement of Hooke's Law is necessary, as in its usual form it is not framed precisely enough if squares and products of displacements are to be included. We take that statement given by Southwell.†

In a distortion of any magnitude of an elastic body there are associated with each point of the body three linear elements orthogonal both before and after strain,‡ and these elements undergo stationary extension, as defined below. If a parallelopiped is constructed with these elements as coterminous edges, only normal stresses will act on its faces after strain, and if relations are assumed between these stresses and the corresponding strains, called respectively principal stresses and principal strains, they are sufficient to determine stress in terms of displacement completely. The extended form of Hooke's Law is that principal stresses and strains in a distortion of any magnitude are

* *Loc. cit.*, pp. 227–236.

† *Loc. cit.*, p. 193.

‡ Love, *op. cit.*, Appendix to chap. 1.

connected by the usual equations; that is to say, if the principal strains are extensions e_1, e_2, e_3, and the corresponding principal stresses are S_1, S_2, S_3, then

$$S_1 = 2\mu e_1 + \lambda(e_1 + e_2 + e_3), \text{ etc.}, \quad . \quad . \quad . \quad . \quad . \quad . \tag{1}$$

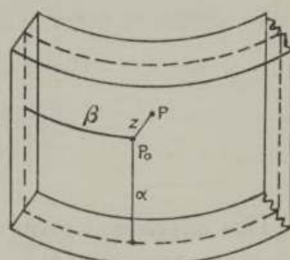
with Lamé's notation for the elastic constants.

Extension is defined as the ratio of the increase in length of an element to its length *before* strain, and stress as the total action over an element of surface divided by the area of the element *before* strain. The energy of strain per unit volume of *unstrained* material, W, is then given by the equation

$$W = \frac{\lambda + 2\mu}{2}(e_1 + e_2 + e_3)^2 - 2\mu(e_2e_3 + e_3e_1 + e_1e_2). \quad . \quad . \tag{2}$$

With these assumptions and definitions the strain energy function of any elastic body can be calculated to any degree of accuracy that is required.

We confine attention to a thin shell of uniform thickness of which the middle surface is generated by parallel straight lines. The position of any point P_0 of the middle surface is specified by α, the distance of P_0, measured along a generator, from an arbitrary line of curvature; and by β, the distance of P_0, measured along a line of curvature, from an arbitrary generator. ρ, the radius of curvature of the normal section of this surface perpendicular to the generator at any point, is a function of β only.* The position of any point P of the shell is specified by drawing PP_0 normal to the middle surface; then if the length P_0P is z, the position of P is given by the orthogonal curvilinear co-ordinates α, β, z, as in the accompanying figure.



3. We have first to find an expression for the extension of a linear element under strain. Let the displacements of P, (α, β, z), be u, v, w with regard to α, β, z axes at P; these axes being the normal to the middle surface through P, (z), a line through P parallel to the generators, (α), and a third perpendicular, (β); u, v, w are then functions of α, β, z. Taking a neighbouring point P′, ($\alpha + \delta\alpha$, $\beta + \delta\beta$, $z + \delta z$), let the length PP′ be r, and let its direction cosines with regard to α, β, z axes at P be l, m, n. The displacements of P′ are

$$u + \frac{\partial u}{\partial \alpha}\delta\alpha + \frac{\partial u}{\partial \beta}\delta\beta + \frac{\partial u}{\partial z}\delta z, \text{ etc.},$$

* Although the only application in this paper is to a problem wherein is constant, other applications are meditated in which is a function of β. The simplification in the work if ρ is supposed constant is very slight.

along α, β, z axes at P′, while to the first order of r

$$\delta\alpha = lr, \qquad \delta\beta = \frac{m\rho r}{\rho - z},^* \qquad \delta z = nr.$$

The angle between the z axes of P and P′ is to the same order $mr/(\rho - z)$. Thus the co-ordinates of P′ after strain with regard to α, β, z axes at P are

$$u + r\left[l\left(1 + \frac{\partial u}{\partial\alpha}\right) + \frac{m\rho}{\rho - z}\frac{\partial u}{\partial\beta} + n\frac{\partial u}{\partial z}\right],$$

$$v + r\left[l\frac{\partial v}{\partial\alpha} + m\left(1 + \frac{\rho}{\rho - z}\frac{\partial v}{\partial\beta} - \frac{w}{\rho - z}\right) + n\frac{\partial v}{\partial z}\right],$$

and

$$w + r\left[l\frac{\partial w}{\partial\alpha} + m\left(\frac{\rho}{\rho - z}\frac{\partial w}{\partial\beta} + \frac{v}{\rho - z}\right) + n\left(1 + \frac{\partial w}{\partial z}\right)\right].$$

The co-ordinates of P after strain are u, v, w, so that denoting by $r\,(1 + e)$ the length of PP′ after strain, there results

$$\begin{aligned}(1 + e)^2 = &\left[l\left(1 + \frac{\partial u}{\partial\alpha}\right) + \frac{m\rho}{\rho - z}\frac{\partial u}{\partial\beta} + n\frac{\partial u}{\partial z}\right]^2 \\ &+ \left[l\frac{\partial v}{\partial\alpha} + m\left(1 + \frac{\rho}{\rho - z}\frac{\partial v}{\partial\beta} - \frac{w}{\rho - z}\right) + n\frac{\partial v}{\partial z}\right]^2 \\ &+ \left[l\frac{\partial w}{\partial\alpha} + m\left(\frac{\rho}{\rho - z}\frac{\partial w}{\partial\beta} + \frac{v}{\rho - z}\right) + n\left(1 + \frac{\partial w}{\partial z}\right)\right]^2. \qquad (3)\end{aligned}$$

This expression gives e, the extension of a linear element, and shows that $(1 + e)$, the ratio of the length of the element after strain to that before, is inversely proportional to the central radius vector of a quadric in the direction of the element before strain. The equation to this quadric can be written

$$\mathrm{F}\,(\xi, \eta, \zeta) = \text{const.}, \qquad (4)$$

if $\mathrm{F}\,(l, m, n)$ denotes the right-hand side of (3), and the ξ, η, ζ axes coincide with the α, β, z axes at P. But if e_1, e_2, e_3 are the principal extensions at P we can also write the equation

$$(1 + e_1)^2\,\mathrm{X}^2 + (1 + e_2)^2\,\mathrm{Y}^2 + (1 + e_3)^2\,\mathrm{Z}^2 = \text{const.}, \qquad (5)$$

the X, Y, Z axes being through P and in the directions before strain of the linear elements that undergo principal extension. A comparison of equations (4) and (5) gives relations starting from which the expression (2) can be put in terms of u, v, w correctly to the third order of displacement co-ordinates. In the first place $\Sigma\,(1 + e_1)^2$ is equal to the sum of the coefficients of ξ^2, η^2

* With a proper choice of the sign of ρ.

and ζ^2 in F (ξ, η, ζ). Subtracting 3 from each of these expressions and squaring,

$$4(\Sigma e_1)^2+4(\Sigma e_1)(\Sigma e_1{}^2)$$

$$=4\left[\frac{\partial u}{\partial\alpha}+\frac{\rho}{\rho-z}\frac{\partial v}{\partial\beta}-\frac{w}{\rho-z}+\frac{\partial w}{\partial z}\right]^2$$

$$+4\left[\frac{\partial u}{\partial\alpha}+\frac{\rho}{\rho-z}\frac{\partial v}{\partial\beta}-\frac{w}{\rho-z}+\frac{\partial w}{\partial z}\right]\left[\left(\frac{\partial u}{\partial\alpha}\right)^2+\left(\frac{\rho}{\rho-z}\frac{\partial u}{\partial\beta}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial\alpha}\right)^2\right.$$

$$+\left(\frac{\rho}{\rho-z}\frac{\partial v}{\partial\beta}-\frac{w}{\rho-z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial w}{\partial\alpha}\right)^2$$

$$\left.+\left(\frac{\rho}{\rho-z}\frac{\partial w}{\partial\beta}+\frac{v}{\rho-z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right];$$

on each side terms of order higher than the third in e_1, e_2, e_3, or in u, v, w have been ignored. The equation puts $(\Sigma e_1)^2$ in terms of u, v, w, but for some terms of the third order in e_1, e_2, e_3. To evaluate the latter to our approximation it is clear that only relations of the first order in u, v, w are required. Thus we may use

$$\Sigma e_1=\frac{\partial u}{\partial\alpha}+\frac{\rho}{\rho-z}\frac{\partial v}{\partial\beta}-\frac{w}{\rho-z}+\frac{\partial w}{\partial z}, \tag{6}$$

and

$$\Sigma e_2e_3=\left(\frac{\rho}{\rho-z}\frac{\partial v}{\partial\beta}-\frac{w}{\rho-z}\right)\frac{\partial w}{\partial z}+\frac{\partial w}{\partial z}\frac{\partial u}{\partial\alpha}+\frac{\partial u}{\partial\alpha}\left(\frac{\rho}{\rho-z}\frac{\partial v}{\partial\beta}-\frac{w}{\rho-z}\right)$$

$$-\tfrac{1}{4}\left[\left(\frac{\partial v}{\partial z}+\frac{\rho}{\rho-z}\frac{\partial w}{\partial\beta}+\frac{v}{\rho-z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial\alpha}\right)^2+\left(\frac{\partial v}{\partial\alpha}+\frac{\rho}{\rho-z}\frac{\partial u}{\partial\beta}\right)^2\right], \tag{7}$$

two of the invariants of the ordinary theory of elasticity. Hence finally

$$(\Sigma e_1)^2=\left[\frac{\partial u}{\partial\alpha}+\frac{\rho}{\rho-z}\frac{\partial v}{\partial\beta}-\frac{w}{\rho-z}+\frac{\partial w}{\partial z}\right]^2$$

$$+\left[\frac{\partial u}{\partial\alpha}+\frac{\rho}{\rho-z}\frac{\partial v}{\partial\beta}-\frac{w}{\rho-z}+\frac{\partial w}{\partial z}\right]\left[\left(\frac{\rho}{\rho-z}\frac{\partial u}{\partial\beta}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial\alpha}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right.$$

$$+\left(\frac{\partial w}{\partial\alpha}\right)^2+\left(\frac{\rho}{\rho-z}\frac{\partial w}{\partial\beta}+\frac{v}{\rho-z}\right)^2$$

$$-\tfrac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\rho}{\rho-z}\frac{\partial w}{\partial\beta}+\frac{v}{\rho-z}\right)^2$$

$$\left.-\tfrac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial\alpha}\right)^2-\tfrac{1}{2}\left(\frac{\partial v}{\partial\alpha}+\frac{\rho}{\rho-z}\frac{\partial u}{\partial\beta}\right)^2\right]. \tag{8}$$

$\Sigma\, e_2\, e_3$ is similarly expressed in terms of u, v, w. Evidently $\Sigma\,[(1+e_2)^2 (1+e_3)^2]$, the second invariant of quadrics (4) and (5), can be expressed in terms of u, v, w, while to our approximation

$$4\Sigma e_2 e_3 = 3 + \Sigma\,[(1+e_2)^2\,(1+e_3)^2] - 2\Sigma\,(1+e_1)^2 - 2\,(\Sigma e_1)\,(\Sigma e_2 e_3) + 6 e_1 e_2 e_3.$$

Thus we have $\Sigma e_2 e_3$ in terms of u, v, w, but for some terms of the third order in the principal extensions. As above these may be evaluated by (6) and (7) together with the third invariant of the ordinary theory,

$$\begin{aligned} e_1 e_2 e_3 = {} & \frac{\partial u}{\partial \alpha}\left(\frac{\rho}{\rho - z}\frac{\partial v}{\partial \beta} - \frac{w}{\rho - z}\right)\frac{\partial w}{\partial z} \\ & + \tfrac{1}{4}\left(\frac{\partial v}{\partial z} + \frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta} + \frac{v}{\rho - z}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial \alpha}\right)\left(\frac{\partial v}{\partial \alpha} + \frac{\rho}{\rho - z}\frac{\partial u}{\partial \beta}\right) \\ & - \tfrac{1}{4}\frac{\partial u}{\partial \alpha}\left(\frac{\partial v}{\partial z} + \frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta} + \frac{v}{\rho - z}\right)^2 - \tfrac{1}{4}\left(\frac{\rho}{\rho - z}\frac{\partial v}{\partial \beta} - \frac{w}{\rho - z}\right)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial \alpha}\right)^2 \\ & - \tfrac{1}{4}\frac{\partial w}{\partial z}\left(\frac{\partial v}{\partial \alpha} + \frac{\rho}{\rho - z}\frac{\partial u}{\partial \beta}\right)^2. \end{aligned}$$

There results

$$\begin{aligned} 4\Sigma\, e_2 e_3 = {} & 4\left[\left(\frac{\rho}{\rho - z}\frac{\partial v}{\partial \beta} - \frac{w}{\rho - z}\right)\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \alpha}\left(\frac{\rho}{\rho - z}\frac{\partial v}{\partial \beta} - \frac{w}{\rho - z}\right) - \tfrac{1}{4}\left(\frac{\partial v}{\partial \alpha} + \frac{\rho}{\rho - z}\frac{\partial u}{\partial \beta}\right)^2\right] \\ & + 2\frac{\partial u}{\partial \alpha}\left[\left(\frac{\rho}{\rho - z}\frac{\partial u}{\partial \beta}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta} + \frac{v}{\rho - z}\right)^2\right] \\ & + 2\left(\frac{\rho}{\rho - z}\frac{\partial v}{\partial \beta} - \frac{w}{\rho - z}\right)\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial \alpha}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial \alpha}\right)^2\right] \\ & + 2\frac{\partial w}{\partial z}\left[\left(\frac{\rho}{\rho - z}\frac{\partial u}{\partial \beta}\right)^2 + \left(\frac{\partial v}{\partial \alpha}\right)^2 + \left(\frac{\partial w}{\partial \alpha}\right)^2 + \left(\frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta} + \frac{v}{\rho - z}\right)^2\right] \\ & - \left[\frac{\partial v}{\partial z} + \frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta} + \frac{v}{\rho - z} + \frac{\rho}{\rho - z}\frac{\partial u}{\partial \beta}\frac{\partial u}{\partial z} + \left(\frac{\rho}{\rho - z}\frac{\partial v}{\partial \beta} - \frac{w}{\rho - z}\right)\frac{\partial v}{\partial z} + \left(\frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta} + \frac{v}{\rho - z}\right)\frac{\partial w}{\partial z}\right]^2 \\ & - \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial \alpha} + \frac{\partial u}{\partial \alpha}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial \alpha}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial \alpha}\frac{\partial w}{\partial z}\right]^2 \\ & - 2\left[\frac{\partial v}{\partial \alpha} + \frac{\rho}{\rho - z}\frac{\partial u}{\partial \beta}\right]\left[\frac{\partial u}{\partial \alpha}\cdot\frac{\rho}{\rho - z}\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\left(\frac{\rho}{\rho - z}\frac{\partial v}{\partial \beta} - \frac{w}{\rho - z}\right) + \frac{\partial w}{\partial \alpha}\left(\frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta} + \frac{v}{\rho - z}\right)\right] \end{aligned}$$

$$+\tfrac{1}{2}\left[\frac{\partial u}{\partial \alpha}+\frac{\rho}{\rho-z}\frac{\partial v}{\partial \beta}-\frac{w}{\rho-z}+\frac{\partial w}{\partial z}\right]\left[\left(\frac{\partial v}{\partial z}+\frac{\rho}{\rho-z}\frac{\partial w}{\partial \beta}+\frac{v}{\rho-z}\right)^2\right.$$
$$\left.+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial \alpha}\right)^2+\left(\frac{\partial v}{\partial \alpha}+\frac{\rho}{\rho-z}\frac{\partial u}{\partial \beta}\right)^2\right]$$
$$+\tfrac{3}{2}\left[\left(\frac{\partial v}{\partial z}+\frac{\rho}{\rho-z}\frac{\partial w}{\partial \beta}+\frac{v}{\rho-z}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial \alpha}\right)\left(\frac{\partial v}{\partial \alpha}+\frac{\rho}{\rho-z}\frac{\partial u}{\partial \beta}\right)\right.$$
$$-\frac{\partial u}{\partial \alpha}\left(\frac{\partial v}{\partial z}+\frac{\rho}{\rho-z}\frac{\partial w}{\partial \beta}+\frac{v}{\rho-z}\right)^2$$
$$\left.-\left(\frac{\rho}{\rho-z}\frac{\partial v}{\partial \beta}-\frac{w}{\rho-z}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial \alpha}\right)^2-\frac{\partial w}{\partial z}\left(\frac{\partial v}{\partial \alpha}+\frac{\rho}{\rho-z}\frac{\partial u}{\partial \beta}\right)^2\right]. \quad (9)$$

Using equations (8) and (9) we have from (2) an expression for the strain energy function correct to the third order of u, v, w. By variation three conditions of equilibrium correct to the second-order of the co-ordinates of displacement can be derived. These are, of course, general equations; they are not needed in the deduction of shell equations, so that as they are complicated they are not set down here.

The Boundary Conditions.

4. The reduction of general equations to shell equations is effected by means of the boundary conditions at the faces, $z = \pm h$, of the shell.

It happens that it is not necessary to calculate these conditions to the second-order in full: with the strain energy method first-order boundary conditions can be used, the extra terms of the more accurate conditions disappearing on substitution, while with the other method it is only necessary to know the form of the second-order conditions.

It is supposed in what follows that the faces are free from all external surface forces.* We write $\overline{\alpha\alpha}$, $\overline{\beta\beta}$, $\overline{zz}$, $\overline{\beta z}$, $\overline{z\alpha}$, and $\overline{\alpha\beta}$ for stresses referred to the actual (strained) elements of area upon which they act, and referred also to (α, β, z) axes at the point $(\bar{\alpha}, \bar{\beta}, \bar{z})$, to which point (α, β, z) is displaced by the strain. Thus $\overline{\bar{\alpha}\bar{\alpha}}$, for instance, is the stress acting in the direction of the α axis of $(\bar{\alpha}, \bar{\beta}, \bar{z})$ upon an element of area normal to this direction, where by stress is meant action divided by strained element of area. Second order expressions for these stresses are not needed; to the first order they are of course known.

Suppose now that any point P of either face of the shell is displaced by the

* This is done as no problem is considered here, or is contemplated, in which the contrary is the case. It is pointed out below, §6, that no loss of generality is involved.

strain to P_1, which will be in the corresponding surface of the strained shell. The direction cosines of three orthogonal elements through P_1, two of which are in the strained surface, are required to the first order. They must be referred to α, β, z axes at P_1. The linear elements through P, whose direction cosines are (1, 0, 0) and (0, 1, 0) before strain, go through P_1 and lie in the surface after strain. A common perpendicular to these elements after strain gives the normal to the strained surface at P_1. Hence we find that the direction cosines of an orthogonal set are given by the scheme

	$\bar{\alpha}$	$\bar{\beta}$	$\bar{z}$
1	1	$\frac{\partial v}{\partial \alpha}$	$\frac{\partial w}{\partial \alpha}$
2	$-\frac{\partial v}{\partial \alpha}$	1	$\frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta}$
3	$-\frac{\partial w}{\partial \alpha}$	$-\frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta}$	1

,

the element denoted by 3 being the normal to the strained surface at P_1.

The conditions that there should be no action upon either face of the shell may now be written down at once. These are

$$\left.\begin{aligned} &\overline{zz} - \frac{2\rho}{\rho - z}\frac{\partial w}{\partial \beta}\overline{\beta z} - 2\frac{\partial w}{\partial \alpha}\overline{z\alpha} = 0, \\ &\overline{\beta z} - \frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta}(\overline{\beta\beta} - \overline{zz}) - \frac{\partial v}{\partial \alpha}\overline{z\alpha} - \frac{\partial w}{\partial \alpha}\overline{\alpha\beta} = 0, \\ \text{and}\quad &\overline{z\alpha} - \frac{\partial w}{\partial \alpha}(\overline{\alpha\alpha} - \overline{zz}) + \frac{\partial v}{\partial \alpha}\overline{\beta z} - \frac{\rho}{\rho - z}\frac{\partial w}{\partial \beta}\overline{\alpha\beta} = 0, \end{aligned}\right\} z = \pm h, \qquad (10)$$

which are correct to the second order of u, v, w.

The Stability of a Plane Plate.

5. Sufficient information has now been obtained to reduce the strain energy function to terms of the co-ordinates of displacement of points of the middle

surface of the shell, and hence to determine the shell equations. But it is also possible to write down the conditions for the equilibrium of an element of volume, and reduce them to shell equations. This is the first of the methods mentioned above as available. The work with the cylindrical shell considered hitherto is long, but in the case of a plane plate it is greatly simplified. The equation for a plane plate is, therefore, deduced in this way, in part as an example of the method, but more particularly because the process of approximation is exactly the same as that in the general case of the cylindrical shell. Consequently, the work in the latter instance can be set down more concisely after an easier problem has been handled by a similar procedure.

We take the middle surface of the plane plate in its unstressed configuration to be the $x\,y$ plane, and any normal as the z axis. All co-ordinates are referred to these fixed axes. Suppose that any point (x, y, z) of the shell is displaced by the strain to $(\bar{x}, \bar{y}, \bar{z})$. Then the condition that the force in the direction of the z axis on an element of volume should vanish* is

$$\frac{\partial \overline{zx}}{\partial \bar{x}} + \frac{\partial \overline{yz}}{\partial \bar{y}} + \frac{\partial \overline{zz}}{\partial \bar{z}} = 0. \tag{11}$$

The stresses that occur in this equation are defined exactly as are those above. It is easy to see that to the first order of u, v, w,

$$\frac{\partial}{\partial \bar{x}} = \left(1 - \frac{\partial u}{\partial x}\right)\frac{\partial}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial}{\partial y} - \frac{\partial w}{\partial x}\frac{\partial}{\partial z}, \text{ etc.} \tag{12}$$

After calculation of the stresses to the second order, the general equation can be written down, but it need not be given here as the approximate shell equation can be derived directly.

Suppose that u, v, w can be expanded in power series of z, so that

$$u = u_0 + u_1 z + u_2 z^2 + \ldots,$$

$$v = v_0 + v_1 z + v_2 z^2 + \ldots,$$

and

$$w = w_0 + w_1 z + w_2 z^2 + \ldots,$$

where u_0, v_0, w_0, and the various coefficients of powers of z are all functions of x and y. u_0, v_0 and w_0 are evidently the co-ordinates of displacement of a point of the middle surface of the plate. With these values of u, v, w, we may reduce the left-hand side of (11) to a power series in z; the equation must be

* *Cf.* Southwell, *loc. cit.*, p. 196; also for equation (12) below.

satisfied for all values of z, so that in particular the term independent of z in this expression must be zero. The resulting equation is

$$\left(1-\frac{\partial u_0}{\partial x}\right)\frac{\partial}{\partial x}(\overline{zx})_0-\frac{\partial v_0}{\partial x}\frac{\partial}{\partial y}(\overline{zx})_0-\frac{\partial w_0}{\partial x}(\overline{zx})_1$$
$$-\frac{\partial u_0}{\partial y}\frac{\partial}{\partial x}(\overline{yz})_0+\left(1-\frac{\partial v_0}{\partial y}\right)\frac{\partial}{\partial y}(\overline{yz})_0-\frac{\partial w_0}{\partial y}(\overline{yz})_1$$
$$-u_1\frac{\partial}{\partial x}(\overline{zz})_0-v_1\frac{\partial}{\partial y}(\overline{zz})_0+(1-w_1)(\overline{zz})_1=0, \quad (13)$$

where, for instance, $(\overline{zx})_0$ stands for the term independent of z, and $(\overline{zx})_1$ for the coefficient of z, in the expansion of $\overline{zx}$. Equation (13) is not yet a shell equation, owing to the appearance of such terms as u_1, v_1, w_1; to eliminate these the boundary conditions are used. For a plane plate equations (10) reduce to

$$\overline{zz}-2\frac{\partial w}{\partial y}\overline{yz}-2\frac{\partial w}{\partial x}\overline{zx}=0, \quad (14)$$

$$\overline{yz}-\frac{\partial w}{\partial y}(\overline{yy}-\overline{zz})-\frac{\partial v}{\partial x}\overline{zx}-\frac{\partial w}{\partial x}\overline{xy}=0, \quad (15)$$

and

$$\overline{zx}-\frac{\partial w}{\partial x}(\overline{xx}-\overline{zz})+\frac{\partial v}{\partial x}\overline{yz}-\frac{\partial w}{\partial y}\overline{xy}=0, \quad (16)$$

each condition holding for $z=\pm h$.

Therefore there are six conditions of which

$$(\overline{yz})_0-\frac{\partial w_0}{\partial y}(\overline{yy}-\overline{zz})_0-\frac{\partial v_0}{\partial x}(\overline{zx})_0-\frac{\partial w_0}{\partial x}(\overline{xy})_0$$
$$+h^2\Big[(\overline{yz})_2-\frac{\partial w_2}{\partial y}(\overline{yy}-\overline{zz})_0-\frac{\partial w_1}{\partial y}(\overline{yy}-\overline{zz})_1-\frac{\partial w_0}{\partial y}(\overline{yy}-\overline{zz})_2$$
$$-\frac{\partial v_2}{\partial x}(\overline{zx})_0-\frac{\partial v_1}{\partial x}(\overline{zx})_1-\frac{\partial v_0}{\partial x}(\overline{zx})_2$$
$$-\frac{\partial w_2}{\partial x}(\overline{xy})_0-\frac{\partial w_1}{\partial x}(\overline{xy})_1-\frac{\partial w_0}{\partial x}(\overline{xy})_2\Big]+\ldots=0, \quad (17)$$

derived from the two equations from (15), is typical. These equations need only be used for the reduction of terms of the first order; for the others, simplified forms of these conditions, obtained by ignoring the second-order terms, are sufficiently accurate. Further, first of all we calculate a first approximation to the equation (13) by ignoring h^2. For the reduction of the second-order terms, then, we have the simple equations

$$(\lambda+2\mu)\,w_1+\lambda\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)=0, \tag{18}$$

$$(\lambda+2\mu)\,2w_2+\lambda\left(\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}\right)=0, \tag{19}$$

$$v_1+\frac{\partial w_0}{\partial y}=0, \tag{20}$$

$$2v_2+\frac{\partial w_1}{\partial y}=0, \tag{21}$$

$$u_1+\frac{\partial w_0}{\partial x}=0, \tag{22}$$

and

$$2u_2+\frac{\partial w_1}{\partial x}=0. \tag{23}$$

These equations can also be used to simplify the more accurate boundary conditions (14), (15) and (16). For example, in (17) the terms $\frac{\partial w_0}{\partial y}(\overline{zz})_0$ and $\frac{\partial v_0}{\partial x}(\overline{zx})_0$ can both be ignored, by reason of equations (18) and (22); the other boundary conditions can be similarly dealt with.

In finding the first approximation, then, even for the reduction of terms of the first order, we have only to use the relatively simple equations

$$(\overline{zz})_0=(\overline{zz})_1=0,$$

$$(\overline{yz})_0-\frac{\partial w_0}{\partial y}(\overline{yy})_0-\frac{\partial w_0}{\partial x}(\overline{xy})_0=0,$$

and

$$(\overline{zx})_0-\frac{\partial w_0}{\partial x}(\overline{xx})_0-\frac{\partial w_0}{\partial y}(\overline{xy})_0=0;$$

the other two conditions are not needed.

Again, what has been said as to the reduction of second-order terms shows that the first approximation to (13) is to be obtained from

$$\frac{\partial}{\partial x}(\overline{zx})_0+\frac{\partial}{\partial y}(\overline{yz})_0+(\overline{zz})_1=0,$$

which by the conditions above is reduced to

$$\frac{\partial}{\partial x}\left[\frac{\partial w_0}{\partial x}(\overline{xx})_0+\frac{\partial w_0}{\partial y}(\overline{xy})_0\right]+\frac{\partial}{\partial y}\left[\frac{\partial w_0}{\partial y}(\overline{yy})_0+\frac{\partial w_0}{\partial x}(\overline{xy})_0\right]=0.$$

All terms in this equation are of the second-order, so that the final reduction to terms of u_0, v_0, w_0 can be effected by equations (18) to (23). The resulting expression is

$$\frac{\partial}{\partial x}\left[\frac{\partial w_0}{\partial x}\left\{\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\frac{\partial u_0}{\partial x}+\frac{2\mu\lambda}{\lambda+2\mu}\frac{\partial v_0}{\partial y}\right\}+\mu\frac{\partial w_0}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[\frac{\partial w_0}{\partial y}\left\{\frac{2\mu\lambda}{\lambda+2\mu}\frac{\partial u_0}{\partial x}+\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\frac{\partial v_0}{\partial x}\right\}+\mu\frac{\partial w_0}{\partial x}\left(\frac{\partial v_0}{\partial y}+\frac{\partial u_0}{\partial y}\right)\right]=0. \quad (24)$$

If a second approximation were required with complete accuracy it would present some difficulty, but it is possible to justify *a posteriori* in practical problems of elastic stability the neglect of terms of the second-order of displacements when multiplied by h^2. Hence, what must be added to the left-hand side of (24) is merely the term in h^2 which appears in the Theory of Thin Shells,* that is to say:

$$-\frac{4\mu(\lambda+\mu)}{3(\lambda+2\mu)}h^2\nabla_1^4 w_0.$$

The shell equation is finally

$$\frac{2(\lambda+\mu)}{3}h^2\nabla_1^4 w_0-\frac{\partial}{\partial x}\left[\frac{\partial w_0}{\partial x}\left\{2(\lambda+\mu)\frac{\partial u_0}{\partial x}+\lambda\frac{\partial v_0}{\partial y}\right\}+\frac{\lambda+2\mu}{2}\frac{\partial w_0}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)\right]$$
$$-\frac{\partial}{\partial y}\left[\frac{\partial w_0}{\partial y}\left\{\lambda\frac{\partial u_0}{\partial x}+2(\lambda+\mu)\frac{\partial v_0}{\partial y}\right\}\right.$$
$$\left.+\frac{\lambda+2\mu}{2}\frac{\partial w_0}{\partial x}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)\right]=0,$$

or

$$\frac{h^2}{3}\nabla_1^4 w_0-\frac{\partial}{\partial x}\left[\frac{\partial w_0}{\partial x}\left(\frac{\partial u_0}{\partial x}+\sigma\frac{\partial v_0}{\partial y}\right)+\frac{1-\sigma}{2}\frac{\partial w_0}{\partial y}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)\right]$$
$$-\frac{\partial}{\partial y}\left[\frac{\partial w_0}{\partial y}\left(\sigma\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)+\frac{1-\sigma}{2}\frac{\partial w_0}{\partial x}\left(\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}\right)\right]=0, \quad (25)$$

σ denoting Poisson's ratio.

The equation of stability follows immediately from (25). Let the plate be stretched by external forces acting in its plane, so that the displacement of any point of the middle surface is $(u_0, v_0, 0)$. If the equilibrium of this configuration, called the "equilibrium configuration," is "neutral," there will be a neighbouring configuration of equilibrium, the "distorted configuration,"

* This expression can be obtained by the general method of this section without trouble, or it comes immediately from equation (32) below.

in which the displacement of a point of the middle surface is (u_0, v_0, w'). Then w' must satisfy the stability equation

$$\frac{h^2}{3}\nabla_1{}^4 w' - \frac{\partial}{\partial x}\left[\frac{\partial w'}{\partial x}\left(\frac{\partial u_0}{\partial x} + \sigma\frac{\partial v_0}{\partial y}\right) + \frac{1-\sigma}{2}\frac{\partial w'}{\partial y}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right)\right]$$
$$- \frac{\partial}{\partial y}\left[\frac{\partial w'}{\partial y}\left(\sigma\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) + \frac{1-\sigma}{2}\frac{\partial w'}{\partial x}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right)\right] = 0, \qquad (26)$$

and certain boundary conditions which need not be considered here. The condition that there should be a non-zero solution w' with these properties determines the "critical" values of the external forces; if these are exceeded the plate will collapse.

The equation (26) can be written*

$$D\nabla_1{}^4 w' - \frac{\partial}{\partial x}\left(T_1\frac{\partial w'}{\partial x} + S\frac{\partial w'}{\partial y}\right) - \frac{\partial}{\partial y}\left(T_2\frac{\partial w'}{\partial y} + S\frac{\partial w'}{\partial x}\right) = 0, \qquad (27)$$

where T_1, T_2, and S are stress resultants† in the equilibrium configuration.

In the general case the stability of an equilibrium configuration (u_0, v_0, w_0) is examined, and three shell equations must be considered. The equilibrium of this configuration is neutral if there is a neighbouring configuration $(u_0 + u'$, $v_0 + v'$, $w_0 + w')$ which will satisfy the equilibrium and boundary conditions.

By subtracting the shell equations for the two configurations we have three equations containing terms of the types u', u_0u', and u'^2. The latter terms can be neglected, for, from the nature of the case, the absolute magnitudes of u', v', and w' can never be obtained; these quantities can therefore be supposed so small that their squares and products are negligible. Consequently, in general the stability equations are linear in u', v' and w', the coefficients being functions of u_0, v_0, and w_0.

The simplification in the case of a plane plate comes from the fact that the question of stability arises only in connection with equilibrium configurations in which the middle surface is plane. It is then found that one stability equation contains only w', if we take as the distorted configuration $(u_0 + u'$, $v_0 + v'$, $w')$, while the other two contain only u' and v'. There are therefore two distinct modes of collapse, and only one of these—that which is determined by the stability equation found here, and which is accompanied by the familiar buckling of the middle surface—is of physical interest.‡

* Special cases of this equation have recently been considered by Southwell and Skan, 'Roy. Soc. Proc.,' A, vol. 105, p. 582, and by the writer, 'Roy. Soc. Proc.,' A, vol. 106, p. 268.

† The notation is that used in Love, *op. cit.*, chap. 22.

‡ See § 8.

The Reduction of the Strain Energy Function.

6. Returning to the general case, we have now to express the strain energy, given by equations (2), (8) and (9), in terms of u_0, v_0 and w_0 by means of the boundary conditions (10). Supposing, as in §5, that u, v and w are expanded in power series in z, an expression for W can be found of the form

$$W = W_0 + W_1 z + W_2 z^2 + \dots ;$$

the energy of the whole plate is then

$$\iint d\alpha\, d\beta \int_{-h}^{h} \left(1 - \frac{z}{\rho}\right)(W_0 + W_1 z + W_2 z^2 + \dots)\, dz.$$

It is proposed to evaluate this, neglecting terms in h^4, terms of the fourth and higher orders of displacement co-ordinates, and terms of the third and higher orders when multiplied by h^2. This accuracy is sufficient for most stability problems.

We have, then, to find W′ given by

$$W' = W_0 + \frac{h^2}{3}(W_2 - W_1/\rho), \tag{28}$$

terms of the third order being neglected in W_1 and W_2. The calculation is divided into two parts. First are obtained the terms not multiplied by h^2; the deduction of the others is a problem in the ordinary theory of elasticity, and does not demand the special accuracy of the results given above. It may be remarked that there is no *a priori* reason to suppose that W_0 does not contain terms in h^2 as it will involve, for instance, terms in u_1, v_1 and w_1, and for the reduction of these to u_0, v_0 and w_0, conditions similar in form to (17) must be used. As pointed out below, however, W_0 does not in fact contain such terms, and this circumstance simplifies the reduction considerably.

W_0 is calculated exactly as is equation (24) above. For the reduction of the third-order terms we have the simple boundary conditions

$$(\overline{zz})_0 = (\overline{zz})_1 = 0,$$

$$(\overline{\beta z})_0 = (\overline{\beta z})_1 = 0,$$

and

$$(\overline{z\alpha})_0 = (\overline{z\alpha})_1 = 0,$$

the stresses being reckoned to the first order only; and these equations can be used to reduce the more accurate boundary conditions (10) to

$$(\overline{zz})_0 = (\overline{zz})_1 = 0,$$

$$(\overline{\beta z})_0 - \frac{\partial w_0}{\partial \beta}(\overline{\beta\beta})_0 - \frac{\partial w_0}{\partial \alpha}(\overline{\alpha\beta})_0 = 0, \text{ etc.}$$

The only one actually required is the first, which can be written

$$(\lambda + 2\mu) w_1 + \lambda\left(\frac{\partial u_0}{\partial \alpha} + \frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho}\right) + \mathrm{F} = 0, \tag{29}$$

where F is an unknown function of the second-order in u_0, u_1, etc.

Simplifying the third-order terms of equations (8) and (9) by the simple boundary conditions, we have

$$\begin{aligned}\left[\Sigma e_1\right]_0^2 = \left[\frac{\partial u_0}{\partial \alpha} + \frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho} + w_1\right]^2 \\ + \left[\frac{\partial u_0}{\partial \alpha} + \frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho} + w_1\right]\left[\left(\frac{\partial u_0}{\partial \beta}\right)^2 + u_1^2 + \left(\frac{\partial v_0}{\partial \alpha}\right)^2 + v_1^2 + \left(\frac{\partial w_0}{\partial \alpha}\right)^2 \right. \\ \left. + \left(\frac{\partial w_0}{\partial \beta} + \frac{v_0}{\rho}\right)^2 - \tfrac{1}{2}\left(\frac{\partial v_0}{\partial \alpha} + \frac{\partial u_0}{\partial \beta}\right)^2\right],\end{aligned}$$

and

$$\begin{aligned}4\left[\Sigma e_2 e_3\right]_0 = 4\left[\left(\frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho}\right) w_1 + w_1 \frac{\partial u_0}{\partial \alpha} + \frac{\partial u_0}{\partial \alpha}\left(\frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho}\right) - \tfrac{1}{4}\left(\frac{\partial v_0}{\partial \alpha} + \frac{\partial u_0}{\partial \beta}\right)^2\right] \\ + 2\frac{\partial u_0}{\partial \alpha}\left[\left(\frac{\partial u_0}{\partial \beta}\right)^2 + u_1^2 + v_1^2 + \left(\frac{\partial w_0}{\partial \beta} + \frac{v_0}{\rho}\right)^2\right] \\ + 2\left(\frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho}\right)\left[u_1^2 + \left(\frac{\partial v_0}{\partial \alpha}\right)^2 + v_1^2 + \left(\frac{\partial w_0}{\partial \alpha}\right)^2\right] \\ + 2w_1\left[\left(\frac{\partial u_0}{\partial \beta}\right)^2 + \left(\frac{\partial v_0}{\partial \alpha}\right)^2 + \left(\frac{\partial w_0}{\partial \alpha}\right)^2 + \left(\frac{\partial w_0}{\partial \beta} + \frac{v_0}{\rho}\right)^2\right] \\ - 2\left(\frac{\partial v_0}{\partial \alpha} + \frac{\partial u_0}{\partial \beta}\right)\left[\frac{\partial u_0}{\partial \alpha}\frac{\partial u_0}{\partial \beta} + \frac{\partial v_0}{\partial \alpha}\left(\frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho}\right) + \frac{\partial w_0}{\partial \alpha}\left(\frac{\partial w_0}{\partial \beta} + \frac{v_0}{\rho}\right)\right] \\ + \tfrac{1}{2}\left(\frac{\partial u_0}{\partial \alpha} + \frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho} - 2w_1\right)\left(\frac{\partial v_0}{\partial \alpha} + \frac{\partial u_0}{\partial \beta}\right)^2.\end{aligned}$$

Using (29) to put w_1 in terms of u_0, v_0, w_0 in the expression for W_0, it is found that the unknown function F disappears; there is therefore no need for the complete second-order boundary condition, a circumstance that reduces considerably the algebra that would otherwise be necessary.

The final result is

$$\begin{aligned}\mathrm{W}_0 = \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\left(\frac{\partial u_0}{\partial \alpha} + \frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho}\right)^2 - 2\mu\left[\frac{\partial u_0}{\partial \alpha}\left(\frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho}\right) - \tfrac{1}{4}\left(\frac{\partial v_0}{\partial \alpha} + \frac{\partial u_0}{\partial \beta}\right)^2\right] \\ + \left(\frac{\partial w_0}{\partial \alpha}\right)^2\left[\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\frac{\partial u_0}{\partial \alpha} + \frac{\lambda\mu}{\lambda+2\mu}\left(\frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho}\right)\right] \\ + \left(\frac{\partial w_0}{\partial \beta} + \frac{v_0}{\rho}\right)^2\left[\frac{\lambda\mu}{\lambda+2\mu}\frac{\partial u_0}{\partial \alpha} + \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\left(\frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho}\right)\right] \\ + \mu\frac{\partial w_0}{\partial \alpha}\left(\frac{\partial w_0}{\partial \beta} + \frac{v_0}{\rho}\right)\left(\frac{\partial v_0}{\partial \alpha} + \frac{\partial u_0}{\partial \beta}\right) + \frac{\mu(3\lambda+2\mu)}{4(\lambda+2\mu)}\left(\frac{\partial u_0}{\partial \alpha} + \frac{\partial v_0}{\partial \beta} - \frac{w_0}{\rho}\right)\left(\frac{\partial v_0}{\partial \alpha} - \frac{\partial u_0}{\partial \beta}\right)^2.\end{aligned}$$

This represents in the first instance a first approximation to W_0 obtained by neglecting terms multiplied by h^2. It remains to find the terms in h^2. The result in this case is known, so that only an indication is given of the method of reaching it by the processes of this paper. The terms of the third-order can be ignored throughout, and we have merely to express in terms of u_0, v_0 and w_0 the coefficient of h^2 in

$$W_0 + \frac{h^2}{3}(W_2 - W_1/\rho),$$

by means of six boundary conditions such as

$$(\lambda + 2\mu)\, w_1 + \lambda\left(\frac{\partial u_0}{\partial\alpha} + \frac{\partial v_0}{\partial\beta} - \frac{w_0}{\rho}\right)$$
$$+ h^2\left[(\lambda + 2\mu)\, 3w_3 + \lambda\left(\frac{\partial u_2}{\partial\alpha} + \frac{\partial v_2}{\partial\beta} + \frac{1}{\rho}\frac{\partial v_1}{\partial\beta} + \frac{1}{\rho^2}\frac{\partial v_0}{\partial\beta} - \frac{w_2}{\rho} - \frac{w_1}{\rho^2} - \frac{w_0}{\rho^3}\right)\right] = 0.$$

This can be done by successive approximation, the process being simple because it is found that upon combining the various terms none of the functions u, v, w appear with suffix greater than 2.

Were this not the case (as it is not if the alternative method of §5 is used for a second approximation), it would be necessary to use the relations that can be derived from general equations by equating to zero the coefficients of the various powers of z, to reduce the suffixes.

The final expression* is

$$\begin{aligned}
W'/\mu = {} & \frac{2(\lambda+\mu)}{\lambda+2\mu}\left(\frac{\partial u}{\partial\alpha} + \frac{\partial v}{\partial\beta} - \frac{w}{\rho}\right)^2 - 2\frac{\partial u}{\partial\alpha}\left(\frac{\partial v}{\partial\beta} - \frac{w}{\rho}\right) + \tfrac{1}{2}\left(\frac{\partial v}{\partial\alpha} + \frac{\partial u}{\partial\beta}\right)^2 \\
& + \left(\frac{\partial w}{\partial\alpha}\right)^2\left[\frac{2(\lambda+\mu)}{\lambda+2\mu}\frac{\partial u}{\partial\alpha} + \frac{\lambda}{\lambda+2\mu}\left(\frac{\partial v}{\partial\beta} - \frac{w}{\rho}\right)\right] \\
& + \left(\frac{\partial w}{\partial\beta} + \frac{v}{\rho}\right)^2\left[\frac{\lambda}{\lambda+2\mu}\frac{\partial u}{\partial\alpha} + \frac{2(\lambda+\mu)}{\lambda+2\mu}\left(\frac{\partial v}{\partial\beta} - \frac{w}{\rho}\right)\right] \\
& + \frac{\partial w}{\partial\alpha}\left(\frac{\partial w}{\partial\beta} + \frac{v}{\rho}\right)\left(\frac{\partial v}{\partial\alpha} + \frac{\partial u}{\partial\beta}\right) + \frac{3\lambda+2\mu}{4(\lambda+2\mu)}\left(\frac{\partial u}{\partial\alpha} + \frac{\partial v}{\partial\beta} - \frac{w}{\rho}\right)\left(\frac{\partial v}{\partial\alpha} - \frac{\partial u}{\partial\beta}\right)^2 \\
& + \frac{2h^2}{3}\left[\frac{\lambda+\mu}{\lambda+2\mu}(\kappa_1+\kappa_2)^2 + \tau^2 - \kappa_1\kappa_2\right. \\
& \qquad + \frac{\lambda}{2(\lambda+2\mu)}\left\{\varepsilon_1 + \frac{\lambda}{\lambda+2\mu}(\varepsilon_1+\varepsilon_2)\right\}\frac{\partial^2}{\partial\alpha^2}(\varepsilon_1+\varepsilon_2) \\
& \qquad + \frac{\lambda}{2(\lambda+2\mu)}\left\{\varepsilon_2 + \frac{\lambda}{\lambda+2\mu}(\varepsilon_1+\varepsilon_2)\right\}\frac{\partial^2}{\partial\beta^2}(\varepsilon_1+\varepsilon_2) \\
& \qquad + \frac{\lambda\tilde{\omega}}{2(\lambda+2\mu)}\frac{\partial^2}{\partial\alpha\,\partial\beta}(\varepsilon_1+\varepsilon_2) + \frac{\tilde{\omega}^2}{4\rho^2}
\end{aligned}$$

* From this point onwards the suffix 0 has been dropped; all co-ordinates of displacement are of points of the middle surface.

$$+\frac{3\lambda+2\mu}{2(\lambda+2\mu)\rho^2}\left\{\varepsilon_2+\frac{\lambda}{\lambda+2\mu}(\varepsilon_1+\varepsilon_2)\right\}^2+\frac{3\lambda+2\mu}{\lambda+2\mu}\cdot\frac{\kappa_1(\varepsilon_1+\varepsilon_2)}{\rho}$$
$$-\frac{(5\lambda+4\mu)(\kappa_1+\kappa_2)}{2(\lambda+2\mu)\rho}\left\{\varepsilon_2+\frac{\lambda}{\lambda+2\mu}(\varepsilon_1+\varepsilon_2)\right\}-\frac{\tilde{\omega}\tau}{2\rho}\Bigg].$$

In the h^2 terms is used the notation of the Theory of Thin Shells,

$$\varepsilon_1=\frac{\partial u}{\partial\alpha},\qquad \varepsilon_2=\frac{\partial v}{\partial\beta}-\frac{w}{\rho},\qquad \tilde{\omega}=\frac{\partial v}{\partial\alpha}+\frac{\partial u}{\partial\beta},$$
$$\kappa_1=\frac{\partial^2 w}{\partial\alpha^2},\quad \kappa_2=\frac{\partial}{\partial\beta}\left(\frac{\partial w}{\partial\beta}+\frac{v}{\rho}\right),\quad \tau=\frac{\partial}{\partial\alpha}\left(\frac{\partial w}{\partial\beta}+\frac{v}{\rho}\right).$$

The result (except for the third-order terms) is equivalent to one given by Basset.*

The discussion has been limited for simplicity to the case in which there is no action on either face of the shell, but were such actions to exist no essential alteration in the procedure would be required. The expressions in equations (10) would then be equal to functions of the surface tractions, and therefore of α and β, and of the displacements of points of the surface.† The successive approximations could be made in exactly the same way, and the only difference would be that in the equations obtained by the method of §5, and in the strain energy of the present method, terms depending on the surface tractions would appear. It is true, as pointed out by the late Lord Rayleigh,‡ that if there are surface tractions no form for the strain energy entirely in terms of the displacements of the middle surface is possible, but for most purposes this does not constitute a difficulty, as the equilibrium or stability of configurations under *given* systems of external force is considered.

The Shell Equations.

7. In a position of equilibrium

$$\iint W'\,d\alpha\,d\beta$$

taken over the middle surface of the whole shell must be a minimum, so that

$$\iint \delta W'\,d\alpha\,d\beta=0,$$

* 'Phil. Trans.,' A, vol. 181.

† *Cf.* Southwell, *loc. cit.*, p. 213.

‡ 'London Math. Soc. Proc.,' vol. 20, p. 372.

where δ denotes an arbitrary variation of u, v and w. This equation can be written in the form

$$\iint (\mathrm{A}\,\delta u + \mathrm{B}\,\delta v + \mathrm{C}\,\delta w)\, d\alpha\, d\beta + \iint \left(\frac{\partial \mathrm{M}}{\partial \alpha} + \frac{\partial \mathrm{N}}{\partial \beta}\right) d\alpha\, d\beta = 0,$$

where A, B, C are functions of u, v, and w, by such relations as*

$$\mathrm{X}\frac{\partial^2 \delta\phi}{\partial\alpha^2} = \frac{\partial^2 \mathrm{X}}{\partial\alpha^2}\delta\phi + \frac{\partial}{\partial\alpha}\left(\mathrm{X}\frac{\partial\delta\phi}{\partial\alpha} - \frac{\partial \mathrm{X}}{\partial\alpha}\delta\phi\right),$$

$$2\mathrm{X}\frac{\partial^2 \delta\phi}{\partial\alpha\,\partial\beta} = 2\,\frac{\partial^2 \mathrm{X}}{\partial\alpha\,\partial\beta}\delta\phi + \frac{\partial}{\partial\alpha}\left(\mathrm{X}\frac{\partial\delta\phi}{\partial\beta} - \frac{\partial \mathrm{X}}{\partial\beta}\delta\phi\right) + \frac{\partial}{\partial\beta}\left(\mathrm{X}\frac{\partial\delta\phi}{\partial\alpha} - \frac{\partial \mathrm{X}}{\partial\alpha}\delta\phi\right),$$

and

$$\mathrm{X}\frac{\partial \delta\phi}{\partial\alpha} = -\frac{\partial \mathrm{X}}{\partial\alpha}\delta\phi + \frac{\partial}{\partial\alpha}(\mathrm{X}\,\delta\phi).$$

The second integral can be transformed into line integrals along the edge of the middle surface, from which the boundary conditions at the edges can be obtained. These are not usually of importance for a theory of elastic stability, and need not be considered here.†

The shell equations are given correctly to the second-order of u, v, w by

$$\mathrm{A} = \mathrm{B} = \mathrm{C} = 0.$$

Written in full these equations are

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\Bigg[\frac{\partial u}{\partial\alpha} + \tfrac{1}{2}\left(\frac{\partial w}{\partial\alpha}\right)^2 + \sigma\left\{\frac{\partial v}{\partial\beta} - \frac{w}{\rho} + \tfrac{1}{2}\left(\frac{\partial w}{\partial\beta} + \frac{v}{\rho}\right)^2\right\} + \frac{1+\sigma}{8}\left(\frac{\partial v}{\partial\alpha} - \frac{\partial u}{\partial\beta}\right)^2 \\
&\qquad + \frac{h^2}{3}\Bigg\{\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\alpha^2}(\varepsilon_1 + \sigma\varepsilon_2) + \frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\beta^2}(\varepsilon_2 + \sigma\varepsilon_1) \\
&\qquad + \frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\alpha^2}(\varepsilon_1 + \varepsilon_2) + \frac{\sigma}{2}\frac{\partial^2\tilde{\omega}}{\partial\alpha\,\partial\beta} + \frac{\sigma(1+\sigma)}{(1-\sigma)^2}\cdot\frac{\varepsilon_2 + \sigma\varepsilon_1}{\rho^2} \\
&\qquad\qquad + (1+\sigma)\frac{\kappa_1}{\rho} - \frac{\sigma(2+\sigma)}{2(1-\sigma)}\frac{\kappa_1 + \kappa_2}{\rho}\Bigg\}\Bigg] \\
&+ \frac{\partial}{\partial\beta}\Bigg[\frac{1-\sigma}{2}\left\{\frac{\partial v}{\partial\alpha} + \frac{\partial u}{\partial\beta} + \frac{\partial w}{\partial\alpha}\left(\frac{\partial w}{\partial\beta} + \frac{v}{\rho}\right)\right\} \\
&\qquad - \frac{1+\sigma}{4}\left(\frac{\partial v}{\partial\alpha} - \frac{\partial u}{\partial\beta}\right)\left(\frac{\partial u}{\partial\alpha} + \frac{\partial v}{\partial\beta} - \frac{w}{\rho}\right) \\
&\qquad\qquad + \frac{h^2}{3}\left\{\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\alpha\,\partial\beta}(\varepsilon_1 + \varepsilon_2) + \frac{1-\sigma}{2}\left(\frac{\tilde{\omega}}{\rho^2} - \frac{\tau}{\rho}\right)\right\}\Bigg] = 0, \qquad (30)
\end{aligned}$$

* *See* Love, 'Phil. Trans.,' A, vol. 179, p. 514.

† For instance, in the problem of the tubular strut examined below, it is impracticable to find the condition for the collapse of a strut of length l under given end conditions. What is done is to find the condition that a distortion of wave-length l may be maintained in an indefinitely long tube.

$$\frac{\partial}{\partial\alpha}\Big[\frac{1-\sigma}{2}\Big\{\frac{\partial v}{\partial\alpha}+\frac{\partial u}{\partial\beta}+\frac{\partial w}{\partial\alpha}\Big(\frac{\partial w}{\partial\beta}+\frac{v}{\rho}\Big)\Big\}+\frac{1+\sigma}{4}\Big(\frac{\partial v}{\partial\alpha}-\frac{\partial u}{\partial\beta}\Big)\Big(\frac{\partial u}{\partial\alpha}+\frac{\partial v}{\partial\beta}-\frac{w}{\rho}\Big)$$
$$+\frac{h^2}{3}\Big\{\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\alpha\,\partial\beta}(\varepsilon_1+\varepsilon_2)+\frac{1-\sigma}{2}\Big(\frac{\tilde{\omega}}{\rho^2}-\frac{\tau}{\rho}\Big)\Big\}\Big]$$
$$+\frac{\partial}{\partial\beta}\Big[\frac{\partial v}{\partial\beta}-\frac{w}{\rho}+\tfrac{1}{2}\Big(\frac{\partial w}{\partial\beta}+\frac{v}{\rho}\Big)^2+\sigma\Big\{\frac{\partial u}{\partial\alpha}+\tfrac{1}{2}\Big(\frac{\partial w}{\partial\alpha}\Big)^2\Big\}+\frac{1+\sigma}{8}\Big(\frac{\partial v}{\partial\alpha}-\frac{\partial u}{\partial\beta}\Big)^2$$
$$+\frac{h^2}{3}\Big\{\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\alpha^2}(\varepsilon_1+\sigma\varepsilon_2)+\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\beta^2}(\varepsilon_2+\sigma\varepsilon_1)$$
$$+\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\beta^2}(\varepsilon_1+\varepsilon_2)+\frac{\sigma}{2}\frac{\partial^2\tilde{\omega}}{\partial\alpha\,\partial\beta}+\frac{1+\sigma}{(1-\sigma)^2}\cdot\frac{\varepsilon_2+\sigma\varepsilon_1}{\rho^2}$$
$$+(1+\sigma)\frac{\kappa_1}{\rho}-\frac{2+\sigma}{2(1-\sigma)}\frac{\kappa_1+\kappa_2}{\rho}\Big\}\Big]$$
$$-\frac{1}{\rho}\Big[\frac{1-\sigma}{2}\frac{\partial w}{\partial\alpha}\Big(\frac{\partial v}{\partial\alpha}+\frac{\partial u}{\partial\beta}\Big)+\Big(\frac{\partial w}{\partial\beta}+\frac{v}{\rho}\Big)\Big(\frac{\partial v}{\partial\beta}-\frac{w}{\rho}+\sigma\frac{\partial u}{\partial\alpha}\Big)$$
$$-\frac{h^2}{3}\Big\{\frac{\partial}{\partial\beta}(\kappa_1+\kappa_2)+(1-\sigma)\Big(2\frac{\partial\tau}{\partial\alpha}-\frac{\partial\kappa_1}{\partial\beta}\Big)$$
$$-\frac{2+\sigma}{2(1-\sigma)}\frac{\partial}{\partial\beta}\Big(\frac{\varepsilon_2+\sigma\varepsilon_1}{\rho}\Big)-\frac{(1-\sigma)}{2\rho}\cdot\frac{\partial\tilde{\omega}}{\partial\alpha}\Big\}\Big]=0, \quad (31)$$

and

$$\frac{h^2}{3}\Big[\Big(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\Big)(\kappa_1+\kappa_2)+(1-\sigma)\Big(2\frac{\partial^2\tau}{\partial\alpha\partial\beta}-\frac{\partial^2\kappa_2}{\partial\alpha^2}-\frac{\partial^2\kappa_1}{\partial\beta^2}\Big)$$
$$+\frac{2-3\sigma^2}{2(1-\sigma)\rho}\frac{\partial^2}{\partial\alpha^2}(\varepsilon_1+\varepsilon_2)-\frac{2+\sigma}{2(1-\sigma)\rho}\frac{\partial^2}{\partial\alpha^2}(\varepsilon_2+\sigma\varepsilon_1)$$
$$-\frac{2+\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\beta^2}\Big(\frac{\varepsilon_2+\sigma\varepsilon_1}{\rho}\Big)-\frac{1-\sigma}{2}\frac{\partial^2}{\partial\alpha\,\partial\beta}\Big(\frac{\tilde{\omega}}{\rho}\Big)\Big]$$
$$-\frac{\partial}{\partial\alpha}\Big[\frac{\partial w}{\partial\alpha}\Big\{\frac{\partial u}{\partial\alpha}+\sigma\Big(\frac{\partial v}{\partial\beta}-\frac{w}{\rho}\Big)\Big\}+\frac{1-\sigma}{2}\Big(\frac{\partial w}{\partial\beta}+\frac{v}{\rho}\Big)\Big(\frac{\partial v}{\partial\alpha}+\frac{\partial u}{\partial\beta}\Big)\Big]$$
$$-\frac{\partial}{\partial\beta}\Big[\Big(\frac{\partial w}{\partial\beta}+\frac{v}{\rho}\Big)\Big(\frac{\partial v}{\partial\beta}-\frac{w}{\rho}+\sigma\frac{\partial u}{\partial\alpha}\Big)+\frac{1-\sigma}{2}\frac{\partial w}{\partial\alpha}\Big(\frac{\partial v}{\partial\alpha}+\frac{\partial u}{\partial\beta}\Big)\Big]$$
$$-\frac{1}{\rho}\Big[\frac{\partial v}{\partial\beta}-\frac{w}{\rho}+\tfrac{1}{2}\Big(\frac{\partial w}{\partial\beta}+\frac{v}{\rho}\Big)^2+\sigma\Big\{\frac{\partial u}{\partial\alpha}+\tfrac{1}{2}\Big(\frac{\partial w}{\partial\alpha}\Big)^2\Big\}+\frac{1+\sigma}{8}\Big(\frac{\partial v}{\partial\alpha}-\frac{\partial u}{\partial\beta}\Big)^2$$
$$+\frac{h^2}{3}\Big\{\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\alpha^2}(\varepsilon_1+\sigma\varepsilon_2)+\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\beta^2}(\varepsilon_2+\sigma\varepsilon_1)$$
$$+\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\beta^2}(\varepsilon_1+\varepsilon_2)+\frac{\sigma}{2}\frac{\partial^2\tilde{\omega}}{\partial\alpha\partial\beta}+\frac{1+\sigma}{(1-\sigma)^2}\cdot\frac{\varepsilon_2+\sigma\varepsilon_1}{\rho^2}$$
$$+(1+\sigma)\frac{\kappa_1}{\rho}-\frac{2+\sigma}{2(1-\sigma)}\frac{\kappa_1+\kappa_2}{\rho}\Big\}\Big]=0. \quad (32)$$

σ as usual denotes Poisson's ratio ; it has been found convenient to use the elastic constants λ, μ in the intermediate stages, though the equations themselves are simpler in terms of σ.

From these equations as in § 5 the equations of stability of any known configuration can be immediately set down.

It will be seen that the complexity of the equations is not due to the second-order terms, which are comparatively few and in the main symmetrical, but to the terms in h^2. The theory of thin shells simplifies the latter considerably by physical assumptions, but it does not appear possible to simplify the equations above on mathematical grounds in the general case. However, in a definite problem it is probable that it would be possible to see without difficulty which of these terms were important, and avoid unnecessary labour as is done in the problem considered below. It is not, therefore, thought worth while to abandon a process of approximation which, though unnecessarily accurate in some cases, is at least definite, in favour of one which it is legitimate to describe in the words of Rayleigh* as of "ill-defined significance."

Examples of General Theory.

8. It has been remarked in § 5 that a plane plate in a configuration of equilibrium in which the middle surface is plane, can collapse in either of two independent modes, of which one only is of physical interest.† The result of this is, as has been seen, considerably to reduce the work necessary in a problem of stability. It is of interest to show that this is the case from equations (30), (31), and (32). From them the corresponding equations for a plane plate can be deduced at once. They are

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}+\tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\sigma\left\{\frac{\partial v}{\partial y}+\tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\}+\frac{1+\sigma}{8}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)^2\right]$$
$$+\frac{\partial}{\partial y}\left[\frac{1-\sigma}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)-\frac{1+\sigma}{4}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]$$
$$+h^2 f_1(u, v)=0, \quad (33)$$

$$\frac{\partial}{\partial x}\left[\frac{1-\sigma}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)+\frac{1+\sigma}{4}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial y}+\tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\sigma\left\{\frac{\partial u}{\partial x}+\tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\}+\frac{1+\sigma}{8}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)^2\right]$$
$$+h^2 f_2(u, v)=0, \quad (34)$$

* *Loc. cit.*, p. 374.

† This was first pointed out by Southwell, *loc. cit.*, p. 202.

and

$$\frac{h^2}{3}\nabla_1{}^4 w - \frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x}\left(\frac{\partial u}{\partial x} + \sigma\frac{\partial v}{\partial y}\right) + \frac{1-\sigma}{2}\frac{\partial w}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]$$
$$- \frac{\partial}{\partial y}\left[\frac{\partial w}{\partial y}\left(\frac{\partial v}{\partial y} + \sigma\frac{\partial u}{\partial x}\right) + \frac{1-\sigma}{2}\frac{\partial w}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] = 0. \quad (35)$$

In equations (33) and (34), $f_1(u, v)$ and $f_2(u, v)$ are functions linear in u and v. (35) is in agreement with equation (25) obtained by the alternative method. It is easy to see that if in these equations we substitute first $(u, v, 0)$ and then $(u + u', v + v', w')$, the displacements of the equilibrium and distorted configurations respectively, and then subtract the two sets of equations ignoring terms above the first-order in u', v' and w', the first two equations contain only u' and v', and the third only w'. Consequently two independent modes of collapse are possible.

There is one point of importance. The condition that there should be a non-zero solution of the third equation will evidently require in general the strains such as $\partial u/\partial x$ in the equilibrium configuration to be of the order of h^2. If instability in this mode is to appear before the breakdown of the material itself, if, that is to say, a theory of stability is to have practical value, the strains in the equilibrium configuration, and consequently h^2, must be very small. But these are precisely the conditions that the approximations used in reaching (35) should be justifiable, for in this equation we have neglected terms similar to those that appear when multiplied either by a displacement or by h^2. Consequently the condition that this instability should be possible in practice, is the condition that equation (35) should be competent to determine when it will take place.

The position in regard to instability of the other type is entirely different. From equations (33) and (34) it is clear that in general a non-zero solution in u' and v' cannot be obtained with small strains; the strains must be of the order of the coefficients of the terms in u' and v' alone. This result regarded as a first approximation shows, however, that there is no ground for neglecting any order of displacement co-ordinates, and there is no reason to suppose that further approximations would alter this conclusion as to the order of magnitude of the strains. This type of instability is then of no practical interest, and it is therefore a matter of indifference that sufficiently accurate equations would, in the general case, be impossible to obtain.* The theory of the stability of plane plates is, therefore, in the

* With the simplest types of stress there is, however, no difficulty.

fortunate position that it is only adequate to deal with problems that are of practical significance.

The Stability of a Tubular Strut.

9. As a more interesting example the stability of a tubular strut is briefly considered. This problem has been solved by Southwell,* using in the case of distortions of axial symmetry his general theory of stability; but in the case of distortions of any type, the theory of thin shells. The general case, then, is here considered for the first time without the latter theory.

The equilibrium configuration of a long circular cylindrical shell in compression due to end thrust is given by

$$u = \theta\alpha, \qquad v = 0, \qquad w = \sigma\theta\rho.$$

θ is a constant and so, in this problem, is ρ. These displacements will evidently satisfy the shell equations. If the equilibrium of this configuration is neutral and a symmetrical distortion of the shell is liable to take place, the shell equations can also be satisfied by the displacements

$$u = \theta\alpha + u', \qquad v = 0, \qquad w = \sigma\theta\rho + w',$$

where u' and v' are independent of β. Subtracting the shell equations for the two configurations, and ignoring terms of order above the first in u' and w', we have the two stability equations†

$$\frac{\partial}{\partial\alpha}\left[\frac{\partial u'}{\partial\alpha} - \frac{\sigma w'}{\rho} + \frac{h^2}{3}\left\{\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\alpha^2}\left(\frac{\partial u'}{\partial\alpha} - \frac{\sigma w'}{\rho}\right)\right.\right.$$
$$+ \frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\alpha^2}\left(\frac{\partial u'}{\partial\alpha} - \frac{w'}{\rho}\right) + \frac{\sigma(1+\sigma)}{(1-\sigma)^2\rho^2}\left(\sigma\frac{\partial u'}{\partial\alpha} - \frac{w'}{\rho}\right)$$
$$\left.\left. + \frac{2-2\sigma-3\sigma^2}{2(1-\sigma)\rho}\frac{\partial^2 w'}{\partial\alpha^2}\right\}\right] = 0, \quad (36)$$

and

$$\frac{h^2}{3}\left[\frac{\partial^4 w'}{\partial\alpha^4} + \frac{2-3\sigma^2}{2(1-\sigma)\rho}\frac{\partial^2}{\partial\alpha^2}\left(\frac{\partial u'}{\partial\alpha} - \frac{w'}{\rho}\right) - \frac{2+\sigma}{2(1-\sigma)\rho}\frac{\partial^2}{\partial\alpha^2}\left(\sigma\frac{\partial u'}{\partial\alpha} - \frac{w'}{\rho}\right)\right]$$
$$- \theta(1-\sigma^2)\frac{\partial^2 w'}{\partial\alpha^2}$$
$$- \frac{1}{\rho}\left[\sigma\frac{\partial u'}{\partial\alpha} - \frac{w'}{\rho} + \frac{h^2}{3}\left\{\frac{\sigma}{2(1-\sigma)}\frac{\partial^2}{\partial\alpha^2}\left(\frac{\partial u'}{\partial\alpha} - \frac{\sigma w'}{\rho}\right)\right.\right.$$
$$\left.\left. + \frac{1+\sigma}{(1-\sigma)^2\rho^2}\left(\sigma\frac{\partial u'}{\partial\alpha} - \frac{w'}{\rho}\right) - \frac{\sigma(1+2\sigma)}{2(1-\sigma)\rho}\frac{\partial^2 w'}{\partial\alpha^2}\right\}\right] = 0. \quad (37)$$

* *Loc. cit.*, pp. 227–236.

† Equation (31) is satisfied identically by both sets of displacements.

Writing

$$u' = \mathrm{U} \sin \frac{q\alpha}{\rho},$$

and

$$w' = \mathrm{W} \cos \frac{q\alpha}{\rho},$$

when U and W are constants, the condition for a non-zero solution of the equations can be written down at once in determinant form. As a first approximation ignore the terms in h^2 in (36) and (37); then

$$\begin{vmatrix} \dfrac{q}{\rho}, & -\dfrac{\sigma}{\rho} \\ -\dfrac{\sigma q}{\rho^2}, & \dfrac{1}{\rho^2} + \theta\,(1-\sigma^2)\,\dfrac{q^2}{\rho^2} \end{vmatrix} = 0,$$

so that

$$\theta = -\frac{1}{q^2}.$$

As a result, q must be large if the stress is to be one of practical interest. The next approximation, in which the terms in h^2 are included, is therefore written in the form

$$\begin{vmatrix} \dfrac{q}{\rho} + \mathrm{A}\dfrac{h^2q^3}{\rho^3} + \mathrm{B}\dfrac{h^2q}{\rho^3}, & -\dfrac{\sigma}{\rho} + \mathrm{C}\dfrac{h^2q^2}{\rho^3} + \mathrm{D}\dfrac{h^2}{\rho^3} \\ -\dfrac{\sigma q}{\rho^2} + \mathrm{E}\dfrac{h^2q^3}{\rho^4} + \mathrm{F}\dfrac{h^2q}{\rho^4}, & \dfrac{1}{\rho^2} + \theta\,(1-\sigma^2)\,\dfrac{q^2}{\rho^2} + \mathrm{H}\dfrac{h^2q^4}{\rho^4} + \mathrm{G}\dfrac{h^2q^2}{\rho^4} + \mathrm{I}\dfrac{h^2}{\rho^4} \end{vmatrix} = 0,$$

where A, B, C, ... are non-dimensional constants independent of q. Further, let

$$\theta = -\frac{1}{q^2} + \theta^1 \frac{h^2}{\rho^2},$$

and evaluate the determinant to order h^2.

Then
$$\theta^1 q^2 = -(\mathrm{L}q^4 + \mathrm{M}q^2 + \mathrm{N}),$$

where L, M, N are further constants independent of q.

The final form for θ is then

$$\theta = -\frac{1}{q^2} - \frac{h^2}{\rho^2}\left(\mathrm{L}q^2 + \mathrm{M} + \frac{\mathrm{N}}{q^2}\right),$$

and the minimum value of θ* for all values of q is given to a first approximation by

$$\theta = -\frac{2h}{\rho}\sqrt{\mathrm{L}},$$

* This corresponds to the minimum thrust that will cause collapse in the mode considered; only this minimum value is of practical importance.

where

$$L = \frac{H}{1-\sigma^2} = \frac{1}{3(1-\sigma^2)}.$$

With this value of θ we arrive at the known formula for the minimum total thrust, S, that in a strut of any radius will cause this type of collapse,

$$S = 8\pi E h^2 \sqrt{\frac{1}{3(1-\sigma^2)}}.$$

To consider any type of distortion, the configuration adjacent to the equilibrium configuration must be taken to be

$$u = \theta\alpha + u', \qquad v = v', \qquad w = \sigma\theta\rho + w';$$

u', v', w' are now functions of α and β.

Forming the stability equations in the usual way, the only terms in θ are found to be $\frac{(1-\sigma^2)}{4}\theta\left(\frac{\partial^2 u'}{\partial\beta^2} - \frac{\partial^2 v'}{\partial\alpha\partial\beta}\right)$, $\frac{(1-\sigma^2)}{4}\theta\left(\frac{\partial^2 v'}{\partial\alpha^2} - \frac{\partial^2 u'}{\partial\alpha\partial\beta}\right)$ and $-(1-\sigma^2)\theta\frac{\partial^2 w'}{\partial\alpha^2}$, in the first, second and third equations respectively.

Setting in these equations

$$u = U \sin\frac{q\alpha}{\rho}\sin\frac{k\beta}{\rho},$$

$$v = V \cos\frac{q\alpha}{\rho}\cos\frac{k\beta}{\rho},$$

and

$$w = W \cos\frac{q\alpha}{\rho}\sin\frac{k\beta}{\rho},$$

a first approximation to θ (ignoring h^2) is found from the determinant

$$\begin{vmatrix} -\frac{1-\sigma}{2}k^2 - q^2 - \frac{1-\sigma^2}{4}\theta k^2, & \frac{1+\sigma}{2}kq - \frac{1-\sigma^2}{4}\theta kq & \sigma q \\ \frac{1+\sigma}{2}kq - \frac{1-\sigma^2}{4}\theta kq \quad, & -k^2 - \frac{1-\sigma}{2}q^2 - \frac{1-\sigma^2}{4}\theta q^2, & -k \\ -\sigma q \quad, & k \quad, & 1+(1-\sigma^2)\theta q^2 \end{vmatrix}$$

Expanding and ignoring θ^2, we have

$$\frac{(1-\sigma)(1-\sigma^2)}{2}\left[q^4 + q^2\theta\left\{k^2(k^2+1) + 2k^2q^2 + \frac{1+\sigma}{2}q^2 + q^4\right\}\right] = 0.$$

It is assumed that k is not zero,* as this case has been already discussed.

* The forms assumed for u', v', w' are not valid if $k = 0$.

Accordingly, q^2 must now be small if a stress of practical interest is to cause collapse. As a first result then

$$\theta = -\frac{q^2}{k^2(k^2+1)}.$$

It is next necessary to find the various types of terms in the expansion of the determinant to order h^2; the coefficients are not in the first instance required. It then appears that the minimum value of θ for given k (which must be integral) corresponds with a value of q which is such that θ, q^2 and h are small quantities of the same order. The determinant contains no terms independent of, or linear in, these three quantities. A first approximation to θ is therefore obtained from an evaluation of the determinant to the second order of these quantities. The terms in q^4 and θq^2 are known, none in θh or $q^2 h$ can appear, so that only terms in θ^2 and h^2 remain to be evaluated. There are physical grounds for ignoring the terms in θ^2, and some mathematical grounds. Completely to determine them would of course require a revision of the preceding work, so that the shell equations should contain the relevant terms of the third-order. However, some of the terms in θ^2 already appear and are multiplied in every case by the square or higher power of q; this is also the case with those that arise in third-order shell equations which have been found in a special case for another purpose. It has therefore appeared impossible that these terms could affect the result, and they have not been calculated in full. The h^2 term is readily obtained. Writing the determinant

$$\left|\begin{array}{ccc}
-\frac{1-\sigma}{2}k^2-q^2-\frac{1-\sigma^2}{4}\theta k^2+\mathrm{A}\frac{h^2}{\rho^2}, & \frac{1+\sigma}{2}kq-\frac{1-\sigma^2}{4}\theta kq+\mathrm{B}\frac{h^2}{\rho^2}, & \sigma q+\mathrm{C}\frac{h^2}{\rho^2} \\
\frac{1+\sigma}{2}kq-\frac{1-\sigma^2}{4}\theta kq+\mathrm{D}\frac{h^2}{\rho^2}, & -k^2-\frac{1-\sigma}{2}q^2-\frac{1-\sigma^2}{4}\theta q^2+\mathrm{F}\frac{h^2}{\rho^2}, & -k+\mathrm{G}\frac{h^2}{\rho^2} \\
-\sigma q+\mathrm{H}\frac{h^2}{\rho^2}, & k+\mathrm{I}\frac{h^2}{\rho^2}, & 1+(1-\sigma^2)\theta q^2+\mathrm{J}\frac{h^2}{\rho^2}
\end{array}\right|,$$

where A, B, C, ... are constants, the term in h^2 is seen to be

$$-\frac{(1-\sigma)k^2}{2}\frac{h^2}{\rho^2}(\mathrm{F}-k\mathrm{G}+k\mathrm{I}-k^2\mathrm{J}).$$

In this expression only that part of F, for instance, independent of q is required, and this must come from the terms in v' not differentiated with regard to α in the second stability equation.

The final result is

$$\frac{1-\sigma}{2}\left[(1-\sigma^2)q^4+(1-\sigma^2)\theta q^2 k^2(k^2+1)+\frac{k^4(k^2-1)^2}{3}\frac{h^2}{\rho^2}\right]=0.$$

The minimum value of θ is given by

$$q^2 = \frac{k^2(k^2-1)}{\sqrt{3(1-\sigma^2)}}\frac{h}{\rho};$$

the corresponding minimum total thrust, S, that will cause instability in the mode defined by k is

$$S = 8\pi E h^2 \frac{k^2-1}{k^2+1}\sqrt{\frac{1}{3(1-\sigma^2)}},$$

verifying the result given by the 'Theory of Thin Shells.'*

On the Fluorescence and Channelled Absorption of Bismuth at High Temperatures.

By K. Rangadhama Rao, M.A., Madras University Research Scholar.

(Communicated by Lord Rayleigh, F.R.S.—Received January 2, 1925.)

[Plate 12.]

In a paper recently communicated to the Royal Society, experiments dealing with the absorption spectra of several metals were described, in which it was found that bismuth vapour shows both lines and bands in absorption. The banded spectrum consists of three groups of bands, each group consisting of a number of bands degraded towards the red, the group of bands in the visible region appearing at high temperatures.

In the above experiments it was hoped that by raising the temperature of the absorption chamber sufficiently high, and raising the absorption in the lines of the several bands, it might be possible to detect a fine structure in some of these bands. Accordingly, the author modified the furnace previously used so as to blow through it a larger quantity of compressed air, and succeeded finally by using coke and this furnace to obtain a temperature of about 1500° C. to 1600° C. At this temperature the vapour emitted a fluorescent radiation orange yellow in colour.

In these experiments the spectra were photographed by a Hilger glass spectrometer of the constant deviation type. To obtain the fluorescence and absorption spectra in juxtaposition, the slit of the collimator is provided with

* Southwell, *loc. cit.*, p. 235.