0. Chaque sommet de tout polygone est le point de rencontre de 3 polygones.

I. Les sommets des polygones coïncident (dans le même sens que dans 1°);

II. Tout polygone a 6 côtés au plus.

Fixons dans tout polygone un point intérieur arbitraire et joignons les points des polygones contigus par des lignes qui ne se croisent pas. Nous parviendrons ainsi à un réseau de triangles (pouvant être curvilignes) qui satisfait aux conditions 1° et 2°. Les noeuds défectifs de ce réseau correspondent aux polygones dont le nombre de côtés est inférieur à 6. Convenons, pour le moment, de dire qu'un polygone a d défauts lorsqu'il a $6-d$ côtés. D'après le théorème démontré, *tout parquetage du plan par des polygones, satisfaisant aux conditions* 0, I *et* II *peut avoir au plus* 6 *défauts*.

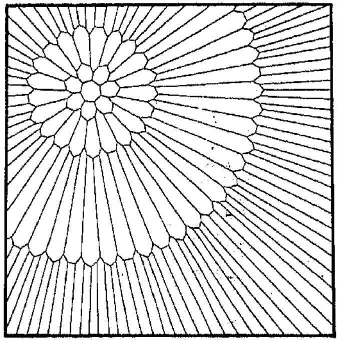

Fig. 8

En particulier, *le plan ne peut pas être recouvert par des pentagones* (conformément aux conditions 0 et I). Il est cependant intéressant qu'il peut être recouvert par des heptagones, mêmes convexes, comme on le voit sur la figure 8.

ON THE THEORY OF GRAPHS[1]

BY

P. TURÁN (BUDAPEST)

Let us consider a finite set of distinct elements $A_1, A_2, \ldots, A_n$ and a ϱ-relation which holds between some of them. If the ϱ-relation holds, for instance between A_1 and A_2, we denote it by $A_1 \varrho A_2$. We suppose only that this relation is symmetrical, *i. e.* with $A_1 \varrho A_2$ also $A_2 \varrho A_1$ holds, and antireflexive in the sense that $A_\nu \varrho A_\nu$ never holds. We obtain a more illustrative representation of the situation representing the elements A_ν by different points P_ν in the three-space and connecting P_μ and P_ν by a line if and only if $A_\mu \varrho A_\nu$ holds. The lines can obviously be chosen so that the only points they can have in common are the points P_j. Antireflexivity means in this representation that no P_j is connected with itself; further, that any two P_j-points are connected by one line at most. The figure so obtained we call a *graph* P (in a little more restricted sense than usual), the points P_j the *vertices*, the connecting lines the *edges* of the graph. A vertex not connected with any other vertex can occur in the graph. By a *subgraph* of P is meant a graph all of whose vertices and edges occur among those of the graph P. A graph P is called *complete* if *all* vertices are connected with each other. Having a graph P, we call a graph *complementary* to P and denote it by $\overline{P}$ if it consists of all the vertices of P and of all edges *not* belonging to P. The *order of a graph* is the number of its vertices, and the *degree of a vertex* P_j is the number of edges starting from P_j.

To show the applicability of the results offered by the theory of graphs we mention first a theorem of E. Marczewski[2]). According to that theorem to an arbitrary graph (finite or infinite) corresponds always a family of sets, one and only one set to each vertex, so that two vertices are connected by an edge in the graph if and only if the corresponding sets have no elements in common. Hence every theorem on graphs gives at the same time a theorem on families of sets. Another instance is concerned with Dirichlet's *Schubfachprinzip*. If we have a set of $(n+1)$ elements each of which has exactly one of n given properties, this prin-

[1]) Lecture delivered at the Mathematical Institute of the Polish Academy of Sciences in Warsaw on October 18, 1952, and in Wrocław on October 27, 1952.

[2]) E. Szpilrajn-Marczewski, *Sur deux propriétés des classes d'ensembles*, Fundamenta Mathematicae 33 (1945), p. 303-307, especially p. 305.

ciple asserts the existence of at least two different elements having a property in common. Let a point correspond to each element and let us connect two different points if and only if they have one of the given properties in common (of course no point is connected with itself). Then the principle asserts the existence of at least one edge in the corresponding graph. If l is an integer and we have $(ln+1)$ elements instead of $(n+1)$, then at least $(l+1)$ elements have a common property, or, in other words, the corresponding graph contains a *complete subgraph of order* $(l+1)$. But perhaps we can draw the same conclusion also from other assumptions made upon the graph; hence we are led to the question of ascertaining the existence of complete subgraphs of possibly high order from various hypotheses upon the graph. Speaking a little more generally, the question with which the lecture is mainly concerned, is the structural problem. By this we mean the problems which arise when, making various assumptions on the graph P, we try to deduce structural information about it or about $\bar{P}$. Looking back we must say that perhaps the first general result in this direction is due to Ramsey[3]). He asserts that there is an $f(k,l)$ such that if a graph P has at least $f(k,l)$ vertices, then either P has a complete subgraph of order k or the graph $\bar{P}$ a complete subgraph of order l. As to its applicability I only mention that Erdös and Szekeres[4]) rediscovered it when giving the first proof of the geometrical theorem according to which there is a $g(k)$ such that given $g(k)$ different arbitrary points on the plane one can select k out of them so that the arising k-gon is convex. I found in 1940 the theorem[5]) according to which if a graph of order n contains "too many" edges then it contains a complete subgraph of "high" order. The problem can also be formulated as an extremum problem: given the integers $(3 \leqslant) k \leqslant n$, how can we find the maximum number $M(n,k)$ of edges in a graph of order n without complete subgraphs of order k? The exact solution of this problem is as follows:

We define the integers t and r uniquely by

(1) $$n = (k-1)t + r \qquad (1 \leqslant r \leqslant k-1).$$

Then the required maximum number $M(n,k)$ of the edges is

(2) $$\frac{k-2}{2(k-1)}(n^2 - r^2) + \binom{r}{2}.$$

[3]) F. P. Ramsey, *Collected papers*, p. 82-111.

[4]) P. Erdös and G. Szekeres, *A combinatorial problem in geometry*, Compositio Mathematica 2 (1935), p. 463-470.

[5]) P. Turán, Matem. és Physikai Lapok 48 (1941), p. 436-452.

Further, the following question arises: by which types of graphs can this maximum value (2) be realised or what is the structure of the extremum graphs? I showed that, apart from isomorphisms, the following graph — subsequently called $D(n,k)$ — is the *only* extremal graph. We split up the vertices in $(k-1)$ classes, each of the first r contains $(t+1)$ vertices, each of the remaining classes only t vertices, and any two pairs of vertices of *different* classes are connected by an edge, but *no* two pairs of vertices of the same class are so connected. We can immediately see that $D(n,k)$ does not contain a complete subgraph of order k. For if it did, then according to Dirichlet's principle at least one of the classes would contain two of its vertices which cannot be connected according to the definition of $D(n,k)$ by an edge. As the late D. König remarked, the graph $D(n,k)$ can be formed numbering the vertices by $1,2,\ldots,n$ and connecting those and only those by an edge which are incongruent mod $(k-1)$. Graphs with such property that their vertices can be divided into *two* classes so that all such and only such pairs of vertices are connected by an edge which belong to *different* classes, have been the object of many investigations[6]). As D. König remarked in a conversation, a formal generalisation of this concept, replacing *two* classes by another number of classes, was proposed already by Saint-Lagüe, who did not name a single property of that type of graphs. Thus the property formulated above is the first property of the graphs of Saint-Lagüe-type.

In 1947 P. Erdös went a great deal further concerning the above mentioned theorem of Ramsey. We consider the graph $D(n,[\sqrt{n}]+2)$ according to the former definition (where $[x]$ denotes the integral part of x). Then it can be observed that neither $D(n,[\sqrt{n}]+2)$ nor $\bar{D}(n,[\sqrt{n}]+2)$ contains a complete subgraph of order $[\sqrt{n}]+2$. That $D(n,[\sqrt{n}]+2)$ does not contain such a subgraph, follows simply from the fact that it is the sum of $[\sqrt{n}]+1$ separate complete graphs G_i, each of order d_i satisfying the inequality

$$d_i \leqslant 1 + \left[\frac{n}{[\sqrt{n}]+1}\right] \leqslant 1 + [\sqrt{n}].$$

That $D(n,[\sqrt{n}]+2)$ cannot contain a complete subgraph of order $[\sqrt{n}]+2$, had been proved before (even that $D(n,k)$ does not contain a complete subgraph of order k). I had conjectured for a while that the graph $D(n,[\sqrt{n}]+2)$ is the extremum graph of the problem, and hence every graph P of order n or its complementary graph $\bar{P}$ contains a complete

[6]) See the book of D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig 1936.

subgraph of order "about $\sqrt{n}$". Now denoting by $h(n)$ the maximum integer l such that in every graph P of order n or in its complementary graph $\bar{P}$ there is a complete subgraph of order l, Erdös[7]) showed that for $n \geqslant 64$

$$h(n) \leqslant \frac{2}{\log 2} \log n, \tag{3}$$

i. e. the order of $h(n)$ is much less than expected. His proof is purely an existence proof; the construction of an explicit example "nearly" realising the estimation (3) would be of interest (**P111**). The exact solution of the corresponding extremum problem seems to be very difficult.

Starting from a topological problem Erdös and A. H. Stone[8]) came in 1946 to another question of structural type. They asked whether or not a graph R having "not too many" edges always contains "many" subgraphs, all of "comparatively high" order so that vertices belonging to *different* subgraphs are not connected in R by edges. They proved indeed that there is an $n_0 = n_0(r, \varepsilon)$ such that in an arbitrary graph R of order $n > n_0$ whose number m of edges satisfies the inequality

$$m < \left(\frac{1}{2(r-1)} - \varepsilon\right) n^2 \tag{4}$$

there are always r subgraphs R_ν without common vertices, each of order d_ν, satisfying the inequality

$$d_\nu \geqslant \sqrt{\log_{r-1} n}, \tag{5}$$

where $\log_{r-1}$ means the $(r-1)$-times iterated logarithm, and having such property that no two vertices taken from *different* R_ν-subgraphs are connected in R by an edge. Here the complete solution of the corresponding extremum problem has not been achieved (**P112**) so far, though the factor $2(r-1)$ cannot be replaced by a greater number.

A theorem of the same type as my result mentioned above was found in 1947 by K. Zarankiewicz[9]). He also tried to ascertain the existence of a complete subgraph of possibly high degree, by fixing not the number of edges, but another number associated with the graph, which may be called the *connection number* of the graph. By this he means the greatest integer l with such property that each vertex of the graph is a starting point of at least l edges. He proved that if the connection number l of a graph of order n satisfies the inequality

$$l > \frac{k-2}{k-1} n, \tag{6}$$

then it contains a complete subgraph of order k, and this is not true any more if

$$l \leqslant \frac{k-2}{k-1} n. \tag{7}$$

Since both Zarankiewicz's results and my own give criteria for the existence of a complete subgraph of order k, it is natural to try to compare those two criteria, called in the sequel, respectively, criterion I and criterion II. On one hand it is clear that criterion II can be fulfilled without the fulfilment of Zarankiewicz's criterion I, since one can easily construct a graph of order n having one more edges as required in (2) and one vertex of which is the starting point of only one edge. On the other hand we show that if criterion I is fulfilled then so is criterion II, consequently criterion II is of wider applicability than criterion I. In order to prove the second part of our assertion we remark that if every vertex is the starting-point of l edges at least, then the total-number m of edges of a graph of order n is obviously such that

$$m \geqslant \frac{nl}{2}, \tag{8}$$

i. e. if Zarankiewicz's condition (6) is satisfied then

$$m > \frac{k-2}{k-1} \cdot \frac{n^2}{2}.$$

If we can show that

$$\frac{k-2}{k-1} \cdot \frac{n^2}{2} > \frac{k-2}{2(k-1)} (n^2 - r^2) + \binom{r}{2},$$

then our statement follows from my above-mentioned theorem. But this is equivalent to

$$\binom{r}{2} < \frac{k-2}{2(k-1)} r^2 \quad \text{or} \quad r-1 < \frac{k-2}{k-1} r$$

which is indeed true owing to (1). Thus our criterion includes that of Zarankiewicz.

[7]) P. Erdös, *Some remarks on the theory of graphs*, Bulletin of the American Mathematical Society 53 (1947), p. 292-294.

[8]) P. Erdös and A. H. Stone, *On the structure of linear graphs*, Bulletin of the American Mathematical Society 52 (1946), p. 1087-1091.

[9]) K. Zarankiewicz, *Sur les relations symétriques dans l'ensemble fini*, Colloquium Mathematicum 1 (1947), p. 10-14.

The above remarks raise an interesting extremal problem. What is the minimum number of edges in a graph of order n if all vertices have a degree not less than l, where of course

$$l+1 \leqslant n. \tag{9}$$

Denoting this minimum number by $m(n,l)$ we shall show that

$$m(n,l) = \left[\frac{ln+1}{2}\right]. \tag{10}$$

The *inequality*

$$m(n,l) \geqslant \left[\frac{ln+1}{2}\right],$$

follows immediately from (8); hence the whole difficulty lies in exhibiting graphs of the required type for which equality can be attained (10). Contrarily to the above mentioned extremal problem we cannot list here at a given couple (n,l) *all* types of extremal graphs; the determination of those seems to be a difficult problem in general. But even the existence of extremal graphs with (10) is trivial only in the case of $n \equiv 0 \bmod (l+1)$; then for the graph consisting of $n/(l+1)$ distinct complete $(l+1)$-gons the equality is attained in (10).

For $l=1$ it is trivial that equality can be attained in (10) in the case of even n if and only if the graph consists of $n/2$ separate edges; when n is odd, the only type of extremal graphs is given by the vertices

$$P_1, P_2, \ldots, P_n$$

and edges

$$(P_1P_2),\ (P_3P_4),\ \ldots,\ (P_{n-4}P_{n-3}),\ (P_{n-2}P_{n-1}),\ (P_{n-1}P_n). \tag{11}$$

Now let $l=2$. Then $m(n,2)=n$ can obviously be attained if and only if all vertices are of degree 2. We fix an extremal graph. Starting from an edge P_1P_2 we can choose a notation $P_2P_3, P_3P_4, \ldots$ which covers all edges of our extremal graph. If $P_{\nu-1}P_\nu$ is the first for which $P_\nu = P_j$, $1 \leqslant j < \nu$, then owing to the above remark we have $P_\nu = P_1$, *i. e.* the edges form a "circle". If we repeat this argument, the graph is split into "circles" without common edges and vertices, and, conversely, every such graph is evidently an extremal graph. Hence in the case of $l=1$ and $l=2$ we know *all* extremal graphs, and while the type of them is uniquely determined for $l=1$, in the case of $l=2$ there are many different types of extremal graphs. If $n-1 \geqslant l \geqslant 3$, we have no complete view upon *all* extremal graphs and we must be satisfied with exhibiting some extremal graphs to each (n,l)-couple. This can be done in various ways; one, which is simpler than my original proof, is due to T. Varga

and runs as follows. First, let n be even, $n=2n_1$. Let the points $P_1, P_2, \ldots, P_n$ be on the periphery of a circle and form the polygons

$$(P_1P_2P_3\ldots P_nP_1),$$
$$(P_1P_3P_5\ldots P_1),\quad (P_2P_4P_6\ldots P_2),$$
$$(P_1P_4P_7\ldots P_1),\quad (P_2P_5P_8\ldots P_2),\quad (P_3P_6P_9\ldots P_3),$$
$$(P_1P_{n_1}\ldots P_1),\quad (P_2P_{n_1+1}\ldots P_2),\quad \ldots,\quad (P_{n_1-1}\ldots P_{n_1-1}),$$
$$(P_1P_{n_1+1}P_1),\quad (P_2P_{n_1+2}P_2),\quad \ldots,\quad (P_{n_1}P_{2n_1}P_{n_1}),$$

where of course not all polygons have the same number of edges. The union S_ν of polygons of the ν-th row form a graph of order n where no two edges overlap and where the degree of each vertex is 2 for $1 \leqslant \nu < n_1$ and 1 for $\nu = n_1$. We may call $S_1, \ldots, S_{n_1-1}$ graphs of the first kind and S_{n_1} of the second kind. It is also evident that S_μ and S_ν have no common edge if $\mu \neq \nu$. If l is even, $l=2l'$, then $2l'=l \leqslant n-1$, $l'<n/2$, *i. e.* superposing *any* l' out of the graphs S_ν of the first kind we get extremal graphs for the couple (n,l). If $l=2l''+1$, then taking $S_{n/2}$ and any l'' out of the S_ν graphs of the first kind we again get extremal graphs for the couple (n,l). In both cases the degree of all vertices is l. If n is odd, $n=2n_1+1$, then we retain the S_ν-graphs of the first kind but replace the former graph of the second kind by the graph S' of order n consisting of n_1+1 edges,

$$P_1P_2,\quad P_3P_4,\quad \ldots,\quad P_{2n_1-1}P_{2n_1},\quad P_{2n_1}P_{2n_1+1},$$

where all vertices are of degree 1 except P_{2n_1} which is of degree 2. The fact that no two S_μ and S_ν have common edges if $\mu \neq \nu$, is no longer true, since S' is contained in S_1; but evidently, apart from this exception, for $\mu \neq \nu$, S_μ and S_ν have no common edges. Thus in the superposition we have to take care of graphs S_ν and S'. If l is even, $l=2l'$, then the superposition of any l' graphs of the first kind yields extremal graphs. Finally, if $l=2l'+1$, then $2l'+1 \leqslant n-1 = 2n_1$, $l' \leqslant n_1-1$, *i. e.* the superposition of *any* l' out of the graphs

$$S_2, S_3, \ldots, S_{n_1}$$

and S' gives extremal graphs, and thus the proof is finished. The degree of all vertices is again l if l is even; there is, however, a single vertex of order $l+1$ if l is odd.

Now let us mention some unsolved problems of this theory. In the group-theoretical investigations of G. Szekeres first occured an interesting restriction on a graph, namely that if P_1, P_2, P_3 are three arbitrary vertices, then at least one of the possible edges P_1P_2, P_1P_3, P_2P_3 actually occurs in the graph. Originally Szekeres had countably infinite graphs in mind and deduced from this hypothesis the existence of an infinite complete subgraph[10]). Now this sort of restriction can also be made upon a graph of finite order and the question arises how "large" a complete subgraph is certainly contained in a graph of order n. It can be shown that if s is an integer with $s(s+1)/2 \leqslant n$ then there is a complete subgraph of order s, but the exact solution of the corresponding extremal problem is unknown (**P 113**). Another sort of question arose from an interesting remark of H. Rademacher[11]). My theorem mentioned above (2) asserts that in the case of $n=2n', k=3$ if there are at least n'^2+1 edges in a graph of order $2n'$, then it contains a complete subgraph of order 3 or, in short, a triangle and this cannot be asserted in the case of only n'^2 edges. Now Rademacher asserts that having (n'^2+1) edges we have not only one but n' triangles at the same time. Erdös has proved[12]) for $k \leqslant 3$ that given (n'^2+k) edges in a graph of order $2n'$ we have kn' triangles; as he has pointed out, this is false for $k=n$ but for $4 \leqslant k < n$ the question is open, as well as all similar questions concerning the number of complete subgraphs of higher order.

Appendix

We have discussed the relationship of Zarankiewicz's theorem and mine [5]). Since the proof of my theorem has been given in a paper written in Hungarian, to make it more accessible I reproduce it here with some simplifications.

We are going to prove the following theorem:

In the class $A_k(n)$ of graphs of order n, which have no complete subgraphs of order k $(3 \leqslant k \leqslant n)$, the graph $D(n,k)$ (defined in the section after (2)) and only that graph has the maximum number of edges.

We denote this maximum number of edges in the class $A_k(n)$ of graphs by $M_k(n)$ and the number of edges in $D(n,k)$ by $d_k(n)$; then we have to prove

$$M_k(n) = d_k(n),$$

and equality is attained *only* for the graph $D(n,k)$.

We have seen before that

$$D(n,k) \in A_k(n). \tag{12}$$

[10]) See my paper [5]) where a proof of this and also a generalisation can be found, namely that the same conclusion follows when there is an integer d with the analogous property, *i. e.* at least two of *any* d vertices are connected in the graph by an edge. Szekeres's condition corresponds to the case $d=3$.

[11]) Communication of P. Erdös, see [5]).

[12]) See [5]).

Next we shall prove the almost trivial inequality

$$M_\nu(n) > M_{\nu-1}(n) \qquad (\nu = 3, 4, \ldots, n). \tag{13}$$

Let us consider a graph E of the class $A_{\nu-1}(n)$ with $M_{\nu-1}(n)$ edges. We add one more edge (this can be done owing to $\nu \leqslant n$) and investigate the new E'-graph with $M_{\nu-1}(n)+1$ edges. We assert that E' belongs to the class $A_\nu(n)$; by this (13) will be proved. Suppose — contrarily to our assertion — that E' contains a complete E'' subgraph of order ν with the vertices $P_1, \ldots, P_\nu$. There are two cases: either the new edge belongs to E'' or not. The second case is evidently impossible, since it would imply $E'' \in E \in A_{\nu-1}(n)$.

Consider the first case. We may suppose the new edge to be P_1P_2. Then the complete graph with the vertices $P_2, P_3, \ldots, P_\nu$ would belong to E, against its definition.

Now let us prove the inequality

$$M_k(n) \leqslant \binom{k-1}{2} + M_k(n-k+1) + (k-2)(n-k+1). \tag{14}$$

Consider an extremal graph B_1 of the class $A_k(n)$, *i. e.* one with $M_k(n)$ edges. According to (13) B_1 has a complete subgraph B_2 of order $(k-1)$; without loss of generality the vertices of B_2 are $P_1, P_2, \ldots, P_{k-1}$; we denote the set of the other vertices $P_k, \ldots, P_n$ by C. We observe that from every vertex of C at most $(k-2)$ edges can start to the vertices of B_2; since if, for instance, $(k-1)$ edges started from P_k towards B_2 then the vertices $P_1, P_2, \ldots, P_{k-1}, P_k$ would form a complete subgraph of order k in B_1, contrarily to the definition of B_1. Hence the number of edges connecting the vertices of C with those of B_2 is at most

$$(k-2)(n-k+1).$$

Since the number of edges connecting the vertices of B_2 is $\binom{k-1}{2}$ and the number of those connecting the vertices of C is at most $M_k(n-k+1)$, (14) is proved.

Now we can turn to the proof of our theorem. We fix $k \geqslant 3$ and proceed by induction from n to $n+k-1$. The possible values for n are grouped as follows:

$$\begin{matrix} k & 2k-1 & 3k-2 & \ldots \\ k+1 & 2k & 3k-1 & \ldots \\ \vdots & \vdots & \vdots & \\ 2k-2 & 3k-3 & 4k-4 & \ldots \end{matrix} \tag{15}$$

We have to prove our assertion for the n-values of the first column and then for each fixed row. However, we shall first assume that the theorem is already proved for the first column of (15), deduce from it the theorem for each row, and afterwards verify it for the n-values of the first column.

Thus we consider our theorem for

$$(16) \qquad n=(k-1)t+r_1 \qquad (1\leqslant r_1\leqslant k-1),$$

r_1 and k fixed, $t=1,2,\ldots$ We suppose that the theorem is already proved for $t\leqslant T$ $(T\geqslant 1)$ and investigate the case of $t=T+1$, *i. e.* for

$$(17) \qquad n_1=(k-1)(T+1)+r_1.$$

Then from (14), using also the induction hypothesis for $M_k(n_1-k+1)$, we get

$$(18) \qquad M_k(n_1)\leqslant\binom{k-1}{2}+d_k\big((k-1)T+r_1,k\big)+(k-2)(n_1-k+1).$$

We have to find out when equality can occur in (18). Owing to the deduction of (18) and the induction hypothesis (also concerning the *unicity* of the extremal graphs) an extremal graph B_1 has the following three properties:

I. From each vertex of C exactly $(k-2)$ edges start towards those of B_2.

II. After a suitable numbering of the vertices of C the edges connecting the vertices belonging to C form the graph

$$D\big((k-1)T+r_1,k\big).$$

III. The complete graph B_2 belongs to B_1.

Graphs of order n_1 satisfying I, II, and III can be realised in many different ways; but if we can show that the *only* one among them which does not contain a complete subgraph of order k is the graph $D\big((k-1)(T+1)+r_1,k\big)$ then first part of induction will be finished.

In order to show this we consider the part C of B_1. According to II and to the definition of graph $D\big((k-1)T+r_1,k\big)$ its vertices can be split into $(k-1)$-classes, each of the first r_1 classes containing $T+1$ vertices, and each of the remaining $(k-1-r_1)$-classes T vertices. Further, according to I, to each vertex P_ν of C there corresponds a uniquely determined vertex P_μ of B_2 with which it is *not* connected. P_μ will be called the *associate vertex*, or shortly the *associate of* P_ν. First we assert that the associates of the vertices of C belonging to *different* classes are distinct from one another. For, if they were not, there would be two vertices of C, say P_k and P_{k+1}, belonging to different classes and still having the same associate P_1. But according to the definition of the associate both P_k and P_{k+1} are connected with $P_2,P_3,\ldots,P_{k-1}$; P_k and P_{k+1} belonging to *different* classes, they are connected according to the definition of $D\big((k-1)T+r_1,k\big)$, and finally all pairs of the vertices of B_2 are connected according to III. Hence the complete graph with the k vertices $P_2,P_3,\ldots,P_k,P_{k+1}$ would be a subgraph of B_1 which is impossible. Hence the associates of vertices belonging to *different* classes are actually distinct from one another. This we shall quote in the sequel as the first observation. Next we assert that the associates of the vertices of C belonging *to the same* class are *identical*. For if they were not, P_k and P_{k+1} would be vertices of the first class of C with *different* associates, P_1 and P_2. Let $P_{k+2},P_{k+3},\ldots,P_{2k-1}$ be a representative system of the remaining $(k-2)$ classes of C; their associates, according to the first observation, are distinct from each other as well as from P_1 and P_2. As a matter of fact this would give k *different* associates which is impossible since B_2 has only $(k-1)$ vertices. Hence the associates of vertices belonging to the same class are identical indeed. This we shall quote as the second observation. From those two observations it follows that to each class of C belongs a uniquely determined associate. *Now we adjoin to each class of C the common associate of its vertices*; the new classes do now contain all vertices of the extremal graph B_1, there are again $(k-1)$ new classes, the first r_1 classes containing each $T+2$ vertices, the remaining $(k-1-r_1)$ classes each $T+1$ vertices. Two vertices of the same new class are never connected again, according to the definition of the associates. Hence, if we succeed in showing that the vertices of different new classes are *always* connected, then we shall have proved that the extremal graph B_1 is identical with the graph $D\big((k-1)(T+1)+r_1,k\big)$.

But it is easy to prove this last assertion. Let us consider two vertices belonging to *different* new classes. If both vertices originally belonged to C then the assertion follows from II and the definition of the graph $D\big((k-1)T+r_1,k\big)$. If both belonged to B_2 then they are obviously connected since B_2 is a complete subgraph of B_1. Finally, if one vertex, say P_k, belonged to C and the other one, say P_1, to B_2, then — according to the construction of the new classes — P_1 is not the associate of P_k, *i. e.* they are connected according to the definition of the associate. Hence the first part of the induction is completed.

We have still to prove the theorem for $T=1$, *i. e.* to investigate

$$M_k(k-1+r_1) \qquad (1\leqslant r_1\leqslant k-1).$$

From (14) we get

$$(19) \qquad \begin{aligned} M_k(k-1+r_1)&\leqslant\binom{k-1}{2}+M_k(r_1)+(k-2)r_1\\ &\leqslant\binom{k-1}{2}+\binom{r_1}{2}+(k-2)r_1. \end{aligned}$$

We have to investigate when the equality-sign can occur in (19). That is the case if and only if

I C is a complete subgraph of order r_1,

II each vertex of C is connected with exactly $(k-2)$ vertices of B_2.

Hence we can define the associate of a vertex of C as before and, analogously, it can be seen that the associates of different vertices of C are distinct from one another. But of course not all vertices of B_2 are now associates in general; with a suitable notation we can arrange that P_1 should be the associate of P_k, P_2 that of $P_{k+1}, \ldots, P_{r_1}$ that of P_{k-1+r_1}; if $r_1<k-1$, then $P_{r_1+1}, \ldots, P_{k-1}$ are *not* associates now. Now we form $(k-1)$ classes, the first class consisting of P_k and P_1, the second of P_{k+1} and $P_2, \ldots$, the r_1-th of P_{k+r_1-1} and P_{r_1}, and if $r_1<k-1$, each of the remaining $(k-1-r_1)$ classes consisting of the single vertices $P_{r_1+1}, \ldots, P_{k-1}$ respectively. In order to identify this graph G with $D(k-1+r_1, k)$ we only have to show that two vertices of different classes are always connected and two of the same class never. The second assertion follows immediately from the construction of the classes and from the notion of the associate. To see the first assertion it is sufficient to remark that, according to the construction, *all* pairs of vertices are connected by an edge except the ones in C with their associates. Hence the proof is completed.

ON THE SEPARABILITY OF TOPOLOGICAL SPACES

A SUPPLEMENT TO A PAPER OF R. SIKORSKI

BY

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R. Sikorski considers in a paper[1]) six properties of a topological space, marked as

$$(*) \qquad (B),\ (\overline{M}),\ (\underline{M}),\ (I),\ (D),\ (S).$$

It is known[2]) that the following implications are true:

$$(**) \qquad \begin{array}{ccccc} (\overline{M}) & \to & (I) & \to & (S) \\ \uparrow & & \uparrow & & \uparrow \\ (B) & \to & (\underline{M}) & \to & (D) \end{array}$$

The author's intention is to prove that these implications are the only true logical connections between the properties $(*)$. Of course, in order to prove this it suffices to show that for each conjunction

$$(*_*^*) \qquad (P_1)(P_2)(P_3)\ldots(P_6)$$

((P_k) being one of the properties $(*)$ or its negation), which is not false owing to the implications $(**)$, there is a topological space for which this conjunction is true.

Sikorski considers the following nine conjunctions:

(1) $(I)'(D)$,	(2) $(\overline{M})'(\underline{M})$,	(3) $(\overline{M})(D)'$,
(4) $(B)'(\overline{M})(\underline{M})$,	(5) $(\overline{M})(\underline{M})'(D)$,	(6) $(I)'(D)'(S)$,
(7) $(\overline{M})'(\underline{M})'(I)(D)$,	(8) (B),	(9) $(S)'$,

regarding them as all the cases not contradictory to $(**)$.

The last remark is not true. A conjunction not contradictory to $(**)$ was omitted by Sikorski.

Let $B, \overline{M}, \underline{M}, I, D, S$ be the classes of all topological spaces for which the respective properties $(*)$ are true. From the implications $(**)$ there follow certain inclusions between the classes $B, \ldots, S$.

Consider the following diagram where the largest square represents the class of all topological spaces, and the remaining squares represent the classes $B, \ldots, S$.

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[1]) R. Sikorski, *On the separability of topological spaces*, Colloquium Mathematicum 1 (1948), p. 279-284.

[2]) See E. Marczewski, *Séparabilité et multiplication cartésienne des espaces topologiques*, Fundamenta Mathematicae 34 (1947), p. 127-143.