ON THE THEORY OF HARMONIC FUNCTIONS OF SEVERAL VARIABLES

I. The theory of H^p-spaces

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1. Introduction

This paper is the first of series concerned with certain aspects of the theory of harmonic functions of several variables. Our particular interest will be to extend to n variables some of the deeper properties known to hold in the case of two variables.

The study of the more fundamental properties of harmonic functions of two real variables is linked, by the notion of the conjugate harmonic function, to the study of analytic functions of one complex variable. Therefore, the investigation of the deeper properties of harmonic functions of several variables appears, at first sight, to be connected with either the theory of analytic functions of several complex variables or with an appropriate extension of the notion of conjugate harmonic function. The theory of analytic functions of several complex variables, though widely studied, does not seem to have direct applications to the theory of harmonic functions of several real variables. On the other hand, there are known notions of "conjugacy" of harmonic functions which seem to us to be both more natural and more fruitful for the development of the latter theory. It is these notions that form the starting point for our investigation. We begin by sketching their background.

Let us first consider a function $u = u(re^{i\theta})$ which is harmonic in the interior of the unit circle $0 \leqslant r < 1$. Suppose that, for $0 \leqslant r < 1$ and $p \geqslant 1$,

$$\int_0^{2\pi} |u(re^{i\theta})|^p \, d\theta \leqslant A < \infty. \tag{1.1}$$

This condition is sufficient to guarantee the existence of boundary values $u(e^{i\theta})$ such that $u(re^{i\theta})\to u(e^{i\theta})$, as $r\to 1$, in an appropriate sense. In fact, as is well known, when $p>1$, (1.1) implies that u is the Poisson integral of a function in $L^p(0, 2\pi)$, and $u(re^{i\theta})$ converges to this function almost everywhere and in the L^p-norm. On the other hand, when $p=1$, (1.1) implies that u is a Poisson integral of a finite Lebesgue-Stieltjes measure, in which case the boundary values of u exist almost everywhere.

However, if we weaken the assumption $p\geqslant 1$ to, say, $p>0$, we no longer have these conclusions on the existence of boundary values. Under this weaker restriction, progress has been made only by considering together with u its harmonic conjugate, v, and, thus the (unique, up to an additive constant) analytic function $F(z)$ whose real part is u. More precisely, the study of the existence of boundary values has been shifted to the case of analytic functions, $F(z)$, of the "Hardy class" H^p, $p>0$, for which

$$\int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta \leqslant A < \infty, \quad 0\leqslant r<1; \tag{1.2}$$

or, even more generally, to the "Nevanlinna class" defined by the condition

$$\int_0^{2\pi} \log^+ |F(re^{i\theta})| \, d\theta \leqslant A < \infty, \quad 0\leqslant r<1. \tag{1.2'}$$

It is well known that, under condition (1.2), $F(e^{i\theta})$ exists such that

$$\int_0^{2\pi} |F(re^{i\theta}) - F(e^{i\theta})|^p \, d\theta \to 0 \text{ and } F(re^{i\theta})\to F(e^{i\theta})$$

almost everywhere as $r\to 1$. If only (1.2′) holds then the pointwise convergence almost everywhere is the best that can be concluded.

These results on the existence of boundary values can be obtained by either of two methods. Both of them reduce the problem from the case of analytic functions satisfying (1.2) for $p>0$, or (1.2′), to the case of harmonic functions satisfying (1.1) with $p\geqslant 1$. The basic tool of the first method is the construction of the "Blaschke product", $B(z)$, which carries the zeroes of $F(z)$ (see [18], Chapter VIII). The second method is based on the important fact, which has been of use in the study of functions of several complex variables (see [9] and [19]), that $\log|F(z)|$ and, hence, $|F(z)|^p$, $p>0$, is subharmonic whenever $F(z)$ is analytic. It is our intention to extend this last method to higher dimensions. Before describing this extension, however, we must introduce the suitable notion of conjugacy.

It is well known that (at least locally) two harmonic functions, $u(x, y)$ and $v(x, y)$, satisfy the Cauchy-Riemann equations in a region

$$u_x = v_y, \quad u_y = -v_x, \tag{1.3}$$

if and only if there exists a harmonic function, $h(x, y)$, such that the pair (v, u) is the gradient of the function h; i.e. $v = h_x$ and $u = h_y$. Thus, analytic functions of one complex variable are in a natural one-to-one correspondence with gradients of harmonic functions of two variables. We may take this correspondence as a motivation for the notion of conjugacy we now introduce. We say that an n-tuple, $F = (u_1, u_2, \ldots, u_n)$, of (real valued) harmonic functions of n variables, $X = (x_1, x_2, \ldots, x_n)$, forms a system of *conjugate harmonic functions* (in the sense of M. Riesz), in a heighborhood of a point, if, in this neighborhood, it is the gradient of a harmonic function $h(X)$; i.e. $u_i(X) = \partial h / \partial x_i$ (see, for example, [6].) (1) Thus, such an n-tuple, F, may be thought of as an extension of the notion of an analytic function of one complex variable—that is, two real variables. This extension is by no means completely satisfactory (for example, the fact that an analytic function of an analytic function is analytic is no longer true for $n \geqslant 3$) but does have, as will be seen, several interesting properties.

As in the case of two variables, we may characterize, at least locally, a system of conjugate harmonic functions in terms of a system of differential equations. More precisely, the n-tuple $F = (u_1, u_2, \ldots, u_n)$ of harmonic functions forms a system of conjugate harmonic functions if and only if it satisfies the analogue of the Cauchy-Riemann equations

$$\sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, \; i \neq j. \tag{1.4}$$

This can be written in the more compact form

$$\operatorname{div} F = 0, \quad \operatorname{curl} F = 0. \tag{1.4'}$$

Let us now return to the boundary value problem discussed above. We first must find a result that will replace the two-dimensional result that $\log |F(z)|$, and, hence, $|F(z)|^p$, $p > 0$, is subharmonic when $F(z)$ is analytic. Let $F = (u_1, u_2, \ldots, u_n)$ be a system of conjugate harmonic functions and denote by $|F|$ the norm $(u_1^2 + u_2^2 + \cdots + u_n^2)^{\frac{1}{2}}$. We thus begin by asking the question: *Is the function $|F|^p$ a subharmonic function of the variables $x_1, x_2, \ldots, x_n$?*

(1) We shall adhere to this convention of using capital letters for vectors and small latters for scalars throughout the paper.

It is an easy thing to check that if $p \geq 1$, the answer is *yes* (and holds for arbitrary harmonic functions $u_1, u_2, \ldots, u_n$, not necessarily related by the Cauchy-Riemann equations (1.4)). It turns out that for certain values of $p < 1$, the answer is still *yes* (but the result now depends on the generalized Cauchy-Riemann equations). More precisely, we will show (in the second section):

THEOREM A. *$|F|^p$ is subharmonic if* $p \geq \frac{n-2}{n-1}$.

Very simple examples show that this result is best possible.

This property of a general system of conjugate functions is, then the basic tool we will use in constructing a theory of H^p spaces of functions of several variables.

Instead of extending the more familiar case of H^p spaces of functions defined in the interior of the unit disc we will generalize the somewhat more diffecult case of functions defined in the upper half-plane. That is, we will extend to n dimensions the boundary-value results known for functions $F(z)$, $z = x + iy$, analytic for $y > 0$ and satisfying

$$\int_{-\infty}^{\infty} |F(x+iy)|^p \, dx \leq A < \infty$$

for all $y > 0$ (see [5] and [7]). As in the case of the circle, boundary values $F(x) = \lim_{y \to 0} F(x+iy)$ exist, both in the norm and almost everywhere. (¹) In this situation the roles played by the variables x and y are obviously different. This difference persists in higher dimensions and we now change our notation slightly in order to reflect better these distinct roles. We shall consider $n+1$ variables, $(X, y) = (x_1, x_2, \ldots, x_n, y)$, and, if the system of conjugate harmonic functions $F(X, y)$ arises as the gradient of the harmonic function $h(X, y)$, we shall denote by $u(X, y)$ the partial derivative of h with respect to the distinguished variable y and by V the vector $(\partial h/\partial x_1, \partial h/\partial x_2, \ldots, \partial h/\partial x_n)$. This notation, then, reflects the fact that we consider the function $F(X, y)$ as defined in the region $y > 0$ and the boundary values $F(X, 0) = F(X)$ will be assumed in the hyperplane $y = 0$. We also write v_k instead of $\partial h/\partial x_k$, $k = 1, 2, \ldots, n$, and refer to $v_1, v_2, \ldots, v_n$ as the *n conjugates* of u. (²) Thus, we see that the notation $F(X, y) = (u(X, y)\ V(X, y))$ is a natural extension of that used in the two-dimensional case.

(¹) If $p < 1$, $\|f\|_p = (\int |f|^p)^{1/p}$ is no longer a norm since Minkowski's inequality fails. Nevertheless, we will still refer to it as a norm as is usually done in the theory of H^p-spaces. We remind the reader that, in this case, $d(f, g) = \|f-g\|_p^p$ is a metric.

(²) Suppose $p(X)$ is a harmonic function of the n variables $X = (x_1, \ldots, x_n)$, then $\bar{v}_i(X, y) = v_i(X, y) + \frac{\partial p}{\partial x_i}(X)$, $i = 1, 2, \ldots, n$, is, clearly, another set of n conjugates of $u(X, y)$. Conversely, With this notation, the generalized Cauchy-Riemann equations become

$$\frac{\partial u}{\partial y}+\sum_{i=1}^{n}\frac{\partial v_i}{\partial x_i}=0,\quad \frac{\partial u}{\partial x_i}=\frac{\partial v_i}{\partial y},\quad i=1,2,\ldots,n, \tag{1.5}$$
$$\frac{\partial v_i}{\partial x_j}=\frac{\partial v_j}{\partial x_i},\quad i\neq j,\ 1\leqslant i,\ j\leqslant n.$$

These equations are assumed to hold in the region $y>0$.

We now define the H^p spaces, $p>0$, to be the classes of systems of conjugate harmonic functions, $F(X, y)$, satisfying

$$\int_{E_n} |F(X,y)|^p\, dX \leqslant A < \infty$$

for $0<y<\infty$, where E_n denotes Euclidean n-dimensional space.

Using the subharmonic character of $|F|^p$ we obtain the extension of the basic theorem of the classical H^p spaces to the n-dimensional spaces just defined, whenever $p\geqslant(n-1)/n=([n+1]-2)/([n+1]-1)$. That is, we will show that there exist boundary values $F(X)=F(X,0)$ such that $F(X,y)\to F(X)$ as $y\to 0$, in the norm, for $p>(n-1)/n$, and almost everywhere for $p\geqslant(n-1)/n$. This will be done in Section 4. There we will use properties of "least harmonic majorants" of powers of $|F|$, which will reduce the problem to known facts about Poisson integrals of functions in L^p, $p>1$, or of finite Lebesgue-Stieltjes measures. These properties will be developed in the third section. While the ideas of this reduction are simple (and have been used before) there are some novel technical complications. This is due to the fact that our underlying space, an Euclidean half-space, is unbounded. An extension of the theory of H^p spaces to, say, spheres would have avoided this technical complication.

Sections 5 and 6 are devoted to applications of the theory of H^p spaces. The background of the first application is the following.

Let $f(X)$ be a function in $L^p(E_n)$, $p>1$, and let $u(X, y)$, $y>0$, be the Poisson integral of $f(X)$. Then, n conjugates $v_1(X,y)$, $v_2(X,y)$, $\ldots$, $v_n(X,y)$ of $u(X,y)$ can be obtained as "conjugate Poisson integrals":

$$v_i(X,y)=\frac{1}{c_n}\int_{E_n}\frac{z_i}{(|Z|^2+y^2)^{\frac{1}{2}(n+1)}}f(X-Z)\,dZ,$$

$i=1, 2, \ldots, n.$

any two sets of n conjugates of u differ by the gradient of a harmonic function of X alone. This is easily deduced from (1.5), which shows that this difference must satisfy (1.4) and the partial derivatives with respect to y must all vanish. In case F belongs to the H^p-space defined below, however, the set of n conjugates of u is unique.

It is known that $\lim_{y\to 0} v_i(X, y) = \tilde{f}_i(X)$ exists almost everywhere and is a function in $L^p(E_n)$, and the $v_i(X, y)$'s are, in turn, Poisson integrals of the $\tilde{f}_i(X)$'s. The transformations $R_i: f(X) \to \tilde{f}_i(X)$, $i = 1, 2, \ldots, n$, are called the n M. Riesz transforms of f and reduce to the classical Hilbert transform when $n = 1$. These transforms are bounded transformations on $L^p(E_n)$, $1 < p$; furthermore, they satisfy the identity

$$R_1^2 + R_2^2 + \cdots + R_n^2 = -I,$$

where I is the identity transformation (see [6]).

By the use of these and other known facts it is not difficult to establish an "isomorphism" between the theory of H^p spaces for $p > 1$ and that of L^p spaces of functions defined on E_n. Thus, in considering H^p spaces for $p > 1$, we have not really gained much over the study of L^p spaces. On the other hand, for $p \leqslant 1$ there are essential differences between H^p and L^p. For example, the M. Riesz transforms of functions in L^p, $p \leqslant 1$, even when defined, need not belong to L^p. A positive result in this direction, when $p = 1$ and $n = 1$, is the celebrated theorem of F. and M. Riesz. Our first application is the n-dimensional generalization of this theorem which we state (somewhat unprecisely for the moment) as follows:

Let μ be a finite Lebesgue-Stieltjes measure on E_n. Suppose that its n M. Riesz transforms are also finite Lebesgue-Stieltjes measures. Then each of the $n+1$ measures in question is absolutely continuous.

Finally, in Section 6 we extend the classical theorem on fractional integrals of functions in H^p to be n-dimensional case introduced in this paper. We now sketch what is, perhaps, the most interesting special case of this result.

If $f(X)$ is in $L^p(E_n)$, we define the operator of *fractional integration* (or *M. Riesz potential*, [10]), I_α, by letting

$$I_\alpha(f) = \frac{1}{\gamma_\alpha} \int_{E_n} \frac{f(X-Y)}{|Y|^{n-\alpha}}\, dY, \quad 0 < \alpha < n,$$

where

$$\gamma_\alpha = \pi^{\frac{1}{2}n}\, 2^\alpha \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \quad (^1)$$

If $1 \leqslant p < n/\alpha$ the above integral converges for almost every X.

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(1) With γ_α so defined, the fractional integration operators satisfy the semigroup property $I_\alpha I_\beta = I_{\alpha+\beta}$.

When $1 < p$ Soboleff, [13], proved the following generalization of a classical theorem of Hardy and Littlewood:

The operator I_α is a bounded operator from L^p to L^q when $1/q = 1/p - \alpha/n$.

This theorem is best possible in the sense that it is not extensible to the case $p = 1$. That is, even though $I_\alpha(f)$ is finite almost everywhere if f is in $L^1(E_n)$, it is not a bounded operator from L^1 to L^q, where $1/q = 1 - \alpha/n$. In fact, it is easy to construct functions, f, in L^1 such that $I_\alpha(f)$ is no longer in L^q. The theory of H^p-spaces developed here, however, allows us to obtain a substitute result for this case.

Recalling the n M. Riesz transforms discussed earlier we can state part of this result as follows:

Suppose that f is in $L^1(E_n)$ and that its Riesz transforms $R_k(f)$, $k = 1, 2, \ldots, n$, are also in $L^1(E_n)$. (1) *Then*

$$I_\alpha(f),\ I_\alpha(R_1(f)),\ \ldots,\ I_\alpha(R_n(f))$$

are all in $L^q(E_n)$, where $1/q = 1 - \alpha/n$, $0 < \alpha < n$. (2)

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2. Proof of Theorem A

If $F(X) = (u_1(X), \ldots, u_n(X))$ is a system of conjugate harmonic functions in some region $\mathcal{R} \subset E_n$ we must show that $|F|^p$ is subharmonic if $p \geqslant \frac{n-2}{n-1}$. Thus, if $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator, it suffices to show that $\Delta(|F|^p) \geqslant 0$ (see [8], Chapter III). Toward this end, we begin by calculating $\Delta(|F|^p)$ and expressing our result in vector notation. In the following, if $G = (v_1, v_2, \ldots, v_n)$ is another function mapping $\mathcal{R}$ into another region of E_n, we let

$$F \cdot G = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

be the *inner product* of F and G. We note that $F \cdot G = G \cdot F$. For $k = 1, 2, \ldots, n$ we let

(1) The exact sense in which "$R_k(f)$ is in $L^1(E_n)$" is used will be defined later.

(2) Hardy and Littlewood obtained this result, as well as the more general ones discussed in the sixth section, in the special case $n = 1$ (see [18], Chapter XII).

$$G_{x_k} = \left(\frac{\partial v_1}{\partial x_k}, \ldots, \frac{\partial v_n}{\partial x_k}\right).$$

It is then easy to check that

$$\frac{\partial}{\partial x_k}(G \cdot F) = G_{x_k} \cdot F + G \cdot F_{x_k}.$$

Thus, $$\frac{\partial}{\partial x_k} |F|^p = \frac{\partial}{\partial x_k}(F \cdot F)^{\frac{1}{2}p} = p\,|F|^{p-2}(F_{x_k} \cdot F);$$

hence, $$\frac{\partial^2}{\partial x_k^2}|F|^p = p(p-2)\,|F|^{p-4}(F_{x_k} \cdot F)^2 + p\,|F|^{p-2}\{|F_{x_k}|^2 + (F \cdot F_{x_k x_k})\}$$

for $k = 1, 2, \ldots, n$.

Summing over k and taking into account that the components of F are harmonic, we obtain

$$\Delta(|F|^p) = p(p-2)\,|F|^{p-4}\sum_{k=1}^{n}(F_{x_k} \cdot F)^2 + p\,|F|^{p-2}\sum_{k=1}^{n}|F_{x_k}|^2. \tag{2.1}$$

We see, therefore, that $\Delta(|F|^p)$ fails to be defined only when $F(X) = 0$ (for $p < 4$). But, if $F(X) = 0$ at some point X, since $|F|^p \geqslant 0$, the mean value property of subharmonic functions (see [8], Chapter II) must hold at X. Thus, in order to establish the subharmonicity of $|F|^p$ it suffices to show $\Delta(|F|^p) \geqslant 0$ whenever the latter is defined (that is, whenever $F(X) \neq 0$). Thus, we may assume that F is never the zero vector.

The assertion, made in the last section, that $|F|^p$ is subharmonic for $p \geqslant 1$ is, then, an easy consequence of (2.1). For, if $1 \leqslant p \leqslant 2$ (note that $\Delta(|F|^p) \geqslant 0$ is obvious for $p \geqslant 2$), using Schwarz's inequality, $(F_{x_k} \cdot F)^2 \leqslant |F_{x_k}|^2 |F|^2$, we have

$$\Delta(|F|^p) \geqslant p(p-2)\,|F|^{p-4}\sum_{k=1}^{n}|F_{x_k}|^2 |F|^2 + p\,|F|^{p-2}\sum_{k=1}^{n}|F_{x_k}|^2 \tag{2.2}$$

$$= p(p-1)\,|F|^{p-2}\sum_{k=1}^{n}|F_{x_k}|^2 \geqslant 0.$$

Since the derivation of (2.1) does not depend on the Cauchy-Riemann equations (1.4) (only the fact that each u_k is harmonic is used), we see that this result holds for any set of n harmonic functions. The deeper result, that $|F|^p$ is subharmonic for values of p less than 1, depends on the following lemma.

LEMMA (2.2). *Suppose that*

$$\mathfrak{M} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

is a symmetric matrix with trace $\left(=\sum_{i=1}^{n} a_{ii}\right)$ *zero. Let* $\|\mathfrak{M}\|$ *be the norm of* $\mathfrak{M}$ *and* $|||\mathfrak{M}||| = \sqrt{\sum_{i,j} |a_{ij}|^2}$ *the Hilbert-Schmidt norm of* $\mathfrak{M}$. (1) *Then*

$$\|\mathfrak{M}\|^2 \leqslant \frac{n-1}{n} |||\mathfrak{M}|||^2. \text{ (2)} \tag{2.3}$$

Proof. Since $\|\mathfrak{M}\|$ and $|||\mathfrak{M}|||$ are unitary invariants and $\mathfrak{M}$ is symmetric, we may assume that $\mathfrak{M}$ is a diagonal matrix. Thus, we have

$$\mathfrak{M} = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix},$$

$$\|\mathfrak{M}\|^2 = \max \{\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2\} \text{ and } |||\mathfrak{M}|||^2 = \sum_{i=1}^{n} \lambda_i^2.$$

Since the trace of a matrix is invariant under a change of coordinates, we also have

$$\sum_{i=1}^{n} \lambda_i = 0. \tag{2.4}$$

We now show, that if (2.4) is satisfied, then, for $k = 1, 2, \dots, n$

$$\lambda_k^2 \leqslant \frac{n-1}{n} \left(\sum_{i=1}^{n} \lambda_i^2 \right), \tag{2.5}$$

which is equivalent to (2.3).

By Schwarz's inequality:

$$\left| \sum_{i \neq k} \lambda_i \right| = \left| \sum_{i \neq k} 1 \cdot \lambda_i \right| \leqslant (n-1)^{1/2} \left(\sum_{i \neq k} \lambda_i^2 \right)^{1/2}. \tag{2.6}$$

Thus, by (2.4) and (2.6),

(1) By the norm of $\mathfrak{M}$ we mean the number $\|\mathfrak{M}\| = \sup |\mathfrak{M}A|$, where the supremum is taken over all vectors $A = (a_1, a_2, \dots, a_n)$ such that $|A| = (|a_1|^2 + \dots + |a_n|^2)^{\frac{1}{2}} \leqslant 1$.

(2) The inequality $\|\mathfrak{M}\|^2 \leqslant |||\mathfrak{M}|||^2$ is true for any matrix $\mathfrak{M}$.

$$\lambda_k^2 = (\sum_{i \neq k} \lambda_i)^2 \leqslant (n-1) \sum_{i \neq k} \lambda_i^2 = \left\{ (n-1) \sum_{i=1}^{n} \lambda_i^2 \right\} - (n-1) \lambda_k^2.$$

Adding $(n-1)\lambda_k^2$ to both sides of this inequality and then dividing both sides by n we obtain (2.5), and the lemma is proved.

We now return to the proof of Theorem A. It remains to be shown that $\Delta(|F|^p) \geqslant 0$ for $1 > p \geqslant (n-2)/(n-1)$. This last fact is derived from the lemma in the following way: Letting

$$\mathfrak{M} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \cdots & \frac{\partial u_n}{\partial x_n} \end{pmatrix}$$

equation (2.1) becomes

$$\Delta(|F|^p) = p(p-2)|F|^{p-4}|\mathfrak{M}F|^2 + p|F|^{p-2}|||\mathfrak{M}|||^2. \tag{2.7}$$

Thus, the inequality $\Delta(|F|^p) \geqslant 0$ is equivalent to $|F|^{p-2}|||\mathfrak{M}|||^2 \geqslant (2-p)|F|^{p-4}|\mathfrak{M}F|^2$; which, in turn, reduces to

$$|\mathfrak{M}F|^2 \leqslant \frac{1}{2-p}|||\mathfrak{M}|||^2|F|^2.$$

This last inequality, on the other hand, is certainly true if

$$||\mathfrak{M}||^2 \leqslant \frac{|||\mathfrak{M}|||^2}{2-p}. \tag{2.8}$$

Clearly, if (2.8) holds for some value of $p < 2$, it will hold for all higher values of $p < 2$. Thus, it suffices to show (2.8) for $p = (n-2)/(n-1)$. That is, we must prove

$$||\mathfrak{M}||^2 \leqslant \frac{n-1}{n}|||\mathfrak{M}|||^2.$$

But, by lemma (2.2), this is the case if $\mathfrak{M}$ is symmetric and has trace zero. On the other hand, these two conditions on $\mathfrak{M}$ are exactly the generalized Cauchy-Riemann equations (1.4). This proves the theorem.

The following simple example shows that this result is best possible: Let

$$F(X) = \left(\frac{x_1}{r^n}, \frac{x_2}{r^n}, \ldots, \frac{x_n}{r^n}\right),$$

where $r = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$ and $n \geqslant 3$. Then F is the gradient of the harmonic function $r^{2-n}/(2-n)$, and, thus, (1.4) is satisfied by F. A simple computation yields

$$\Delta(|F|^p) = (1-n)\,p\,[(n-2)+(1-n)\,p]\,r^{p(1-n)-2}.$$

Thus, the condition $\Delta(|F|^p) \geqslant 0$, for $p > 0$, becomes $(2-n)+p\,(n-1) \geqslant 0$ which reduces to

$$p \geqslant \frac{n-2}{n-1}.$$

3. Harmonic Majorization of Certain Subharmonic Functions and Some Maximal Functions

As was mentioned in the introduction, we shall reduce most of the results on H^p spaces either to theorems about the Poisson integral of a function in $L^q(E_n)$, $q > 1$, or to properties of the Poisson-Stieltjes integral of a finite measure on E_n. Thus, we begin this section by defining these integrals and stating the known facts about them that we shall use. We refer the reader to [1], [6] and [15] (¹) for the proofs of (ii)–(vi).

The *Poisson kernel* for the half-space

$$E_{n+1}^+ = \{(X, y) : X \text{ in } E_n,\ y > 0\}$$

is the function
$$P(X, y) = \frac{y}{c_n\,(|X|^2 + y^2)^{\frac{1}{2}(n+1)}},$$

where
$$c_n = \frac{\pi^{\frac{1}{2}(n+1)}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

This function has the following three basic properties:

(i) $P(X, y) \geqslant 0$;

(ii) $\int_{E_n} P(X, y)\,dX = 1$ *for all* $y > 0$;

(iii) *if* $r > 0$, *then* $\int_{|X| \geqslant r} P(X, y)\,dX \to 0$ *as* $y \to 0$.

It is easy to see that if f is a function in $L^q(E_n)$, $q \geqslant 1$, then, for each (X, y) in E_{n+1}^+, $f(Z)\,P(X-Z, y)$ and $f(X-Z)\,P(Z, y)$ are integrable functions of Z in E_n. (²) Thus, the function

(¹) This last reference deals only with the one-dimensional case but (i) – (vi) are immediate generalizations of this case.

(²) We consider E_n as embedded in E_{n+1} by identifying it with $\{(X, y) \text{ in } E_{n+1} : y = 0\}$.

$$m(X, y) = \int_{E_n} f(Z)\, P(X - Z, y)\, dZ = \int_{E_n} f(X - Z)\, P(Z, y)\, dZ$$

is defined in E^+_{n+1}. This function is called the *Poisson integral* of f.

If μ is a finite (signed) Lebesgue-Stieltjes measure on E_n, then $P(X - Z, y)$ is also easily seen to be integrable with respect to μ. The function

$$m(X, y) = \int_{E_n} P(X - Z, y)\, d\mu(Z)$$

is called the *Poisson-Stieltjes integral* of μ.

(iv) *If $m(X, y)$ is the Poisson integral of a function in L^q, $q \geq 1$, or a Poisson-Stieltjes integral, then $m(X, y)$ is harmonic in E^+_{n+1}.*

(v) *If $m(X, y)$ is the Poisson integral of the function $f(X)$ in $L^q(E_n)$, $q \geq 1$, then*

$$\sup_{y>0} \int_{E_n} |m(X, y)|^q\, dX = \int_{E_n} |f(X)|^q\, dX,$$

and $\lim_{y \to 0} m(X, y) = f(X)$, both in the L^q norm and almost everywhere.

(vi) *If $m(X, y)$ is the Poisson-Stieltjes integral of the measure μ, $\lim_{y \to 0} m(X, y)$ exists for almost every X in E_n. Furthermore,*

$$\sup_{y>0} \int_{E_n} |m(X, y)|\, dX = \int_{E_n} |d\mu|.$$

We recall that, for each $p > 0$, we defined the class H^p to consist of all those systems of conjugate harmonic functions (1)

$$F(X, y) = (u(X, y),\ v_1(X, y),\ \dots,\ v_n(X, y)),$$

defined on E^+_{n+1}, satisfying

$$\int_{E_n} |F(X, y)|^p\, dX \leq A < \infty, \tag{3.1}$$

for $0 < y < \infty$. Since F satisfies the system of equations (1.5), Theorem A guarantees that $|F|^r$, for $r \geq (n-1)/n$, is subharmonic. We shall exploit (3.1) and this property, for appropriate values of r, to obtain harmonic majorants of $|F|^r$. It will be shown that these harmonic majorants are Poisson integrals or Poisson-Stieltjes integrals. Then, an application of two basic theorems, one due to N. Wiener, the other to A. P. Calderón, will yield the main result on H^p spaces—theorem B of Section 4.

(1) That is, the system satisfies the Cauchy-Riemann equations (1.5).

We begin with a chain of lemmas leading up to the existence of these harmonic majorants.

LEMMA (3.2). *Let* $s(X, y) \geqslant 0$ *be a subharmonic function defined in the region* E_{n+1}^+ *satisfying*

$$\int_{E_n} [s(X, y)]^q \, dX \leqslant C^q < \infty, \tag{3.3}$$

where $1 \leqslant q < \infty$ *and* C *is independent of* $y > 0$. *Then*

$$s(X, y) \leqslant C y^{-(n/q)}. \tag{3.4}$$

Furthermore, if $0 < \varepsilon \leqslant y \leqslant 1/\varepsilon$, $s(X, y) \to 0$ *uniformly in* y *as* $|X| \to \infty$.

Proof. We first observe that s^q, being a convex function of a subharmonic function, is subharmonic. Thus, letting ω be the volume of the unit sphere in E_{n+1}, we have, for (X, y) in E_{n+1}^+,

$$\begin{aligned}
[s(X, y)]^q &\leqslant \frac{1}{\omega y^{n+1}} \int_{|X-Z|^2+(y-t)^2<y^2} [s(Z, t)]^q \, dZ\, dt \\
&\leqslant \frac{1}{\omega y^{n+1}} \int_{0<t<2y} [s(Z, t)]^q \, dZ\, dt \\
&= \frac{1}{\omega y^{n+1}} \int_0^{2y} \left\{ \int_{E_n} [s(Z, t)]^q \, dZ \right\} dt \leqslant \frac{C^q\, 2y}{\omega y^{n+1}} = \frac{2}{\omega} C^q y^{-n} \leqslant C^q y^{-n},
\end{aligned}$$

and (3.4) is established.

In order to prove the last part of the lemma we observe that, if

$$I_k = \left\{ (X, y) : k-1 \leqslant |X| < k,\ 0 \leqslant y \leqslant \frac{1}{\varepsilon} + \varepsilon \right\},$$

$k = 1, 2, 3, \dots$, then

$$\int_{I_k} [s(Z, t)]^q \, dZ\, dt \to 0 \tag{3.5}$$

as $k \to 0$.

This is clear since

$$\sum_{k=1}^{\infty} \int_{I_k} [s(Z, t)]^q \, dZ\, dt = \int_0^{\varepsilon+1/\varepsilon} \left\{ \int_{E_n} [s(Z, t)]^q \, dZ \right\} dt \leqslant \frac{C^q (1+\varepsilon^2)}{\varepsilon} < \infty.$$

If (X, y) satisfies $\varepsilon \leqslant y < 1/\varepsilon$ then it belongs to I_k for some k. It follows that the sphere about (X, y) of radius ε is contained in $I_{k-1} \cup I_k \cup I_{k+1}$ (where I_0 is the null set). Thus

$$[s(X, y)]^q \leqslant \frac{1}{\omega\,\varepsilon^{n+1}} \int\limits_{|Z|^2+t^2<\varepsilon^2} [s(X+Z, y+t)]^q \, dZ\, dt$$

$$\leqslant \frac{1}{\omega\,\varepsilon^{n+1}} \sum_{j=k-1}^{k+1} \int\limits_{I_j} [s(Z, t)]^q \, dZ\, dt.$$

But, by (3.5), the last term tends to 0 as $k \to \infty$, and the last conclusion of the lemma follows.

LEMMA (3.6). *Let $m(X, y)$ be a harmonic function defined in E_{n+1}^+ satisfying*

$$\int\limits_{E_n} |m(X, y)|^q \, dX \leqslant C^q \tag{3.7}$$

for all $y > 0$, where $q \geqslant 1$. Then

a) *if $q > 1$, $m(X, y)$ is the Poisson integral of a function f in $L^q(E_n)$ such that $\|f\|_q \leqslant C$;*

b) *If $q = 1$, $m(X, y)$ is the Poisson-Stieltjes integral of a finite (signed) Lebesgue-Stieltjes measure μ such that $\int\limits_{E_n} |d\mu| \leqslant C$.*

Proof. Let us first assume that $q > 1$. Condition (3.7) asserts that the family of functions $m(X, y)$, parametrized by $y > 0$, is uniformly bounded in the norm of $L^q(E_n)$. Thus, we may select a sequence $\{y_k\}$, with $y_k \to 0$, such that $m(X, y_k)$ converges weakly to a function $f(X)$ in $L^q(E_n)$. That is, if $1/q + 1/q' = 1$, we have

$$\int\limits_{E_n} m(Z, y_k)\, g(Z)\, dZ \to \int\limits_{E_n} f(Z)\, g(Z)\, dZ$$

as $k \to \infty$, for each g in $L^{q'}(E_n)$. In particular, if we put $g(Z) = P(X - Z, y)$ we have

$$w_k(X, y) = \int\limits_{E_n} m(Z, y_k)\, P(X-Z, y)\, dZ \to \int\limits_{E_n} f(Z)\, P(X-Z, y)\, dZ.$$

By (iv), $w_k(X, y)$ is harmonic in E_{n+1}^+. We shall now show that $w_k(X, y) = m(X, y + y_k)$. Toward this end, we first show that $w_k(X, y) \to m(X, y_k)$ uniformly in X as $y \to 0$. We have, using (ii),

$$w_k(X, y) - m(X, y_k) = \int_{E_n} [m(Z, y_k) - m(X, y_k)] P(X-Z, y)\, dZ$$
$$= \Big(\int_{|X-Z|<r} + \int_{|X-Z|\geq r} \Big) [m(Z, y_k) - m(X, y_k)] P(X-Z, y)\, dZ$$
$$= I_1 + I_2.$$

On the other hand, since $m(X, y)$ is harmonic, $|m(X, y)|$ is subharmonic and, by (3.7), it satisfies the assumptions in lemma (3.2). A particular consequence of the last part of this lemma is, then, that $m(X, y_k)$ is uniformly continuous in E_n, for each $y_k > 0$. From this it follows that, if r is small enough, I_1 is uniformly small.

On the other hand, using (3.4) and (3.7) we also have

$$|I_2| \leq \int_{|X-Z|\geq r} (|m(Z, y_k)| + |m(X, y_k)|) P(X-Z, y)\, dZ$$
$$\leq 2\, C y_k^{-(n/q)} \int_{|X-Z|\geq r} P(X-Z, y)\, dZ.$$

But, by (iii), the last integral tends to zero as $y \to 0$. This shows that $w_k(X, y) \to m(X, y_k)$ uniformly in X as $y \to 0$. Consequently, for $\varepsilon > 0$ small enough $|w_k(X, \varepsilon) - m(X, y_k)|$ is small.

We see by (v) and (vi) that $|w_k(X, y)|$ satisfies condition (3.3). Furthermore, by (iv), $w_k(X, y)$ is harmonic and, hence, $|w_k(X, y)|$ is subharmonic. Thus, both $|w_k(X, y)|$ and $|m(X, y_k)|$ satisfy the assumptions of lemma (3.2). Hence, by (3.4), for y large enough, say y_0, $|w_k(X, y_0) - m(X, y_0 + y_k)|$ is small. Finally, the last part of lemma (3.2) implies that, if $\varepsilon \leq y \leq y_0$, $|w_k(X, y) - m(X, y + y_k)|$ is small for $|X|$ large, say $|X| = r$.

Summing up, we see that on the boundary of a region $\mathcal{D} = \{(X, y) : |X| \leq r, \varepsilon \leq y \leq y_0\}$, the harmonic function $w_k(X, y) - m(X, y + y_k)$ is small in absolute value. By the maximum principle for harmonic functions, therefore, it must be as small throughout $\mathcal{D}$. By expanding $\mathcal{D}$ we then obtain $m(X, y + y_k) = w_k(X, y)$.

Thus, since $y_k \to 0$ as $k \to \infty$, we obtain

$$m(X, y) = \int_{E_n} f(Z) P(X-Z, y)\, dZ$$

and part a) of the lemma is established if we show that $\|f\|_q \leq C$. But this last fact is a consequence of (3.7) and (v).

Let us now pass to the case $q = 1$. We consider the family of (signed) measures, $\{\mu_y\}$, where, for any measurable $S \subset E_n$,

$$\mu_y(S) = \int_S m(X, y)\, dX.$$

Inequality (3.7) then asserts that the total measures of the members of this family are uniformly bounded. Thus, as in the previous case, there exists a sequence $\{y_k\}$, with $y_k \to 0$, such that μ_{y_k} converge weakly to a finite measure μ. That is, if g is any continuous function vanishing at infinity, then

$$\int_{E_n} g(Z)\, d\mu_{y_k}(Z) \to \int_{E_n} g(Z)\, d\mu(Z)$$

as $k \to \infty$. Letting g be the Poisson kernel and repeating the above argument we obtain

$$m(X, y) = \int_{E_n} P(X-Z, y)\, d\mu(Z)$$

and, with the aid of (vi), the lemma is proved.

LEMMA (3.8). *If $s(Z, y) \geqslant 0$ is continuous and satisfies the conditions of lemma (3.2) it has a harmonic majorant, $m(X, y)$, in E_{n+1}^+. Furthermore,*

a) *if $q > 1$, $m(X, y)$ is a Poisson integral of a function f in $L^q(E_n)$ such that $f(X) = \lim_{y \to 0} m(X, y)$ both in the norm and almost everywhere and $\|f\|_q \leqslant C$;*

b) *if $q = 1$, $m(X, y)$ is a Poisson-Stieltjes integral of a finite Lebesgue-Stieltjes measure on E_n.*

Proof. For each $\varepsilon > 0$ we define

$$m_\varepsilon(X, y) = \int_{E_n} s(X-Z, \varepsilon)\, P(Z, y)\, dZ = \int_{E_n} s(Z, \varepsilon)\, P(X-Z, y)\, dZ.$$

By (v) we have

$$\int_{E_n} [m_\varepsilon(X, y)]^q\, dX \leqslant C^q. \tag{3.9}$$

By an almost verbatim repetition of the argument in the last proof, that showed $w_k(X, y) \to m(X, y_k)$ uniformly in X as $y \to 0$, we obtain the similar result:

$$|m_\varepsilon(X, y) - s(X, \varepsilon)| \to 0 \tag{3.10}$$

uniformly in X as $y \to 0$.

We can now show that $m_\varepsilon(X, y)$ is a majorant of $s(X, y+\varepsilon)$ in E_{n+1}^+ (the argument being very similar to the one in the previous proof showing that $w_k(X, y) = m(X, y + y_k)$). We first note that $s(X, y+\varepsilon)$ is uniformly continuous since it is continuous by assumption and, as seen from lemma (3.2), it vanishes at infinity. From this and (3.10) we see that for y small enough, say y_0, the difference $s(X, y_0+\varepsilon) - m_\varepsilon(X, y_0)$ is small. On the other hand, by (3.4), for y large enough, say y_1, $s(X, y_1+\varepsilon)$

is small. Finally, by the last part of lemma (3.2), $s(X, y+\varepsilon)$, in the region $y_0 \leqslant y \leqslant y_1$, is small for $|X|$ large enough, say $|X| \geqslant r$. Thus, since $m_\varepsilon(X, y)$ is non-negative, on the boundary of a region $\mathfrak{R} = \{(X, y) : y_0 \leqslant y \leqslant y_1, \ |X| \leqslant r\}$ the function $w(X, y) = s(X, y+\varepsilon) - m_\varepsilon(X, y)$ is bounded above by a small positive number. On the other hand, w, being the sum of two subharmonic functions, is subharmonic. Thus, by the maximum principle for subharmonic functions, w is bounded above by this small positive number throughout $\mathfrak{R}$. By expanding $\mathfrak{R}$ we obtain $w(X, y) \leqslant 0$ for $y > 0$. That is

$$s(X, y+\varepsilon) \leqslant m_\varepsilon(X, y) \tag{3.11}$$

for $y > 0$.

Now suppose $q > 1$. From (3.3) we see that $\{s(X, \varepsilon)\}$ is a uniformly bounded family of functions in the norm of $L^q(E_n)$. From this we can deduce, as we did in the proof of lemma (3.6), that there exists a function $f(X)$ in $L^q(E_n)$ and a null sequence $\{\varepsilon_k\}$ such that $\{s(X, \varepsilon_k)\}$ converges weakly to $f(X)$. Thus, in particular, letting $m(X, y)$ be the Poisson integral of f we have, for each (X, y) in E_{n+1}^+,

$$m_{\varepsilon_k}(X, y) = \int_{E_n} s(Z, \varepsilon_k) P(X-Z, y)\, dZ \to \int_{E_n} f(Z) P(X-Z, y)\, dZ = m(X, y)$$

as $k \to \infty$.

On the other hand, by (3.11),

$$s(X, y+\varepsilon_k) \leqslant m_{\varepsilon_k}(X, y).$$

Thus, letting $k \to \infty$, since the left hand side tends to $s(X, y)$ and the right hand side to $m(X, y)$, we obtain

$$s(X, y) \leqslant m(X, y).$$

This shows that m is a majorant of s.

The fact that $f(X) = \lim_{y \to 0} m(X, y)$ both in the norm and almost everywhere then follows from (v).

Furthermore, since $m_{\varepsilon_k}(X, y) \to m(X, y)$ as $k \to \infty$, an application of Fatou's lemma and (3.9) show that

$$\int_{E_n} [m(X, y)]^q \, dX \leqslant C^q.$$

Thus, we must have $\|f\|_q \leqslant C$. This completes the proof of part a).

Part b) follows from a similar argument. The only change that is needed, as in the proof of lemma (3.6), is to replace the weak convergence of a sequence of elements of the family $\{s(X, \varepsilon)\}$ to a function with the weak convergence of such a sequence to a measure. Consequently, we obtain a Poisson-Stieltjes integral majorizing $s(X, y)$. Thus, the lemma is proved.

We remark that the function $m(X, y)$ is not only a majorant of $s(X, y)$ but is the *least harmonic majorant* of $s(X, y)$. That is, if $\tilde{m}(X, y) \geqslant s(X, y)$ in E_{n+1}^{+} and $\tilde{m}$ is harmonic, then $m(X, y) \leqslant \tilde{m}(X, y)$ in E_{n+1}^{+}. This follows easily from the maximum principle for subharmonic functions and our construction of m. This fact, however, will not be needed in this paper.

Before stating the result of N. Wiener that was mentioned earlier we have to introduce the n-dimensional generalization of the Hardy-Littlewood maximal function: If $f(X)$ is in $L^q(E_n)$, $q \geqslant 1$, we define

$$f^*(X) = \sup_{r>0} \frac{1}{r^n} \int_{|Z|<r} |f(X+Z)|\, dZ.$$

The function f^* is then called the *maximal function* of f.

The basic properties of f^* that we shall need are stated in the following lemma (see [9], [12] and [17]):

LEMMA (3.12). *The function f^* is finite almost everywhere if f is in $L^q(E_n)$. Furthermore, if $q > 1$, there exists A_q, independent of f in $L^q(E_n)$, such that*

$$\|f^*\|_q \leqslant A_q \|f\|_q.$$

We will need lemma (3.12) and a substitute result for the maximal function of a measure. More precisely, if μ is any finite measure on E_n we define

$$\mu^*(X) = \sup_{r>0} \frac{1}{r^n} \int_{|Z|<r} |d\mu(X+Z)|$$

and we say that μ^* is the *maximal function of the measure* μ. We then have

LEMMA (3.13). *The function μ^* is finite almost everywhere.*

This fact is an immediate consequence of the well-known theorem on the differentiability of a measure (see [11]).

These two lemmas will be needed to obtain similar properties for another type of maximal function. More precisely, we shall prove the following generalization of an estimate of Hardy and Littlewood (see [18], Chapter IV):

LEMMA (3.14). *Let $\Gamma_\alpha(X) \subset E_{n+1}^{+}$ be the conical region, with vertex X, of all points (Z, y) satisfying $|X-Z| < \alpha y$. Suppose that $m(X, y)$ is harmonic in E_{n+1}^{+} and, for $q \geqslant 1$,*

$$\int_{E_n} |m(X, y)|^q \, dX \leqslant C^q < \infty$$

for all $y > 0$.

Let
$$m_\alpha^*(X) = \sup |m(Z, y)|,$$

the supremum being taken over all (Z, y) *in* $\Gamma_\alpha(X)$.

Then,

a) *if* $q > 1$, $m_\alpha^*(X)$ *is in* $L^q(E_n)$ *and*

$$\|m_\alpha^*\|_q \leqslant A\, C,$$

where A *depends only on* α, q *and the dimension* n;

b) *if* $q = 1$, $m_\alpha^*(X) < \infty$ *almost everywhere.*

Proof. The proof makes use of the idea that for Poisson integrals the approach to a boundary point along a cone (i.e. the, so-called, non-tangential approach) is essentially dominated by the approach along the direction normal to the boundary surface.

Let us first assume that $q > 1$. By lemma (3.6), $m(X, y)$ is the Poisson integral of a function $f(X)$ in $L^q(E_n)$ with $\|f\|_q \leqslant C$. Part a) of the present lemma will then be a consequence of the first part of Lemma (3.12) if we show

$$m_\alpha^*(X) \leqslant B f^*(X), \tag{3.15}$$

where B depends only on α and the dimension n.

Toward this end, we first observe that, for $|Z - X| < \alpha y$,

$$\frac{y}{(|W - Z|^2 + y^2)^{\frac{1}{2}(n+1)}} \leqslant D \frac{y}{(|W - X|^2 + y^2)^{\frac{1}{2}(n+1)}}, \tag{3.16}$$

where
$$D^{2/(n+1)} = \max\{1 + 2\alpha^2, 2\}.$$

By considering the positive and negative parts of f separately we may assume that $f(X) \geqslant 0$. Then, an immediate consequence of (3.16) is the inequality

$$m(Z, y) \leqslant D\, m(X, y)$$

for (Z, y) in $\Gamma_\alpha(X)$. Thus,

$$m_\alpha^*(X) \leqslant D\{\sup_{y>0} m(X, y)\}.$$

Hence, inequality (3.15) will be established if we show that, for each $y>0$,

$$m(X, y) \leqslant K f^*(X), \tag{3.17}$$

where K depends only on the dimension n. This inequality is known (see, for example, [9]), but for completeness we include its proof:

$$\begin{aligned}
m(X, y) &= \frac{y}{c_n} \int_{E_n} \frac{f(X-Z)}{(|Z|^2+y^2)^{\frac{1}{2}(n+1)}} dZ \\
&= \frac{y}{c_n}\left(\int_{|Z|\leqslant y} + \int_{|Z|>y} \right) \frac{f(X-Z)}{(|Z|^2+y^2)^{\frac{1}{2}(n+1)}} dZ \\
&\leqslant \frac{1}{c_n y^n} \int_{|Z|\geqslant y} f(X-Z)\, dZ + \frac{y}{c_n} \int_{|Z|>y} \frac{f(X-Z)}{|Z|^{n+1}} dZ.
\end{aligned}$$

The first term is clearly majorized by a constant multiple of $f^*(X)$. Thus, the proof of (3.17) is completed by the following chain of inequalities:

$$\begin{aligned}
y \int_{|Z|>y} \frac{f(X-Z)}{|Z|^{n+1}} dZ &= y \sum_{k=0}^{\infty} \int_{2^{k+1}y \geqslant |Z| > 2^k y} \frac{f(X-Z)}{|Z|^{n+1}} dZ \\
&\leqslant \sum_{k=0}^{\infty} y^{2-(n+1)} y^{-(n+1)} \int_{2^{k+1}y \geqslant |Z|} f(X-Z)\, dZ \\
&= 2^n \sum_{k=0} 2^{-k} (2^{k+1} y)^{-n} \int_{2^{k+1}y \geqslant |Z|} f(X-Z)\, dZ \\
&\leqslant 2^n \sum_{k=0}^{\infty} 2^{-k} f^*(X) = 2^{n+1} f^*(X).
\end{aligned}$$

The proof of the second part of this lemma follows similar lines. From lemma (3.6) we see that $m(X, y)$ is the Poisson-Stieltjes integral of a finite measure μ. As before, we obtain part b) from lemma (3.13) if we show

$$m_\alpha^*(X) \leqslant B \mu^*(X), \tag{3.18}$$

where B depends only on α and n. But we see that (3.15) and (3.18) are established in exactly the same way if we note that the proof of the former depends only on estimates on the Poisson kernel. Thus, lemma (3.14) in proved.

We will need one more result. This is the following special case of a basic theorem of A. P. Calderón [2] that we here state as a lemma:

LEMMA (3.19) *Let $w(X, y)$ be harmonic in E_{n+1}^{+}. Suppose that for a measurable set $S \subset E_n$*

$$|w(Z, y)| \leqslant M < \infty$$

for (Z, y) in $\Gamma_\alpha(X)$, X in S. Then, for almost every X in S, $\lim_{y\to 0} w(X, y)$ exists. (1)

4. H^p-spaces

Before we state and prove our results in the theory of H^p spaces we introduce some notation. If

$$F(X, y) = (u(X, y),\ V(X, y)) = (u(X, y),\ v_1(X, y),\ v_2(X, y),\ \ldots,\ v_n(X, y))$$

is in H^p we let

$$\mathfrak{M}_p(y) = \mathfrak{M}_p(y; F) = \Big(\int_{E_n} |F(X, y)|^p \, dX\Big)^{1/p},$$

for $y > 0$.

In case there exists a vector-valued function

$$G(X) = (w_0(X),\ w_1(X),\ \ldots,\ w_n(X)),$$

defined on E_n, such that

$$\|F(X, y) - G(X)\|_p = \Big(\int_{E_n} |F(X, y) - G(X)|^p \, dX\Big)^{1/p} \to 0$$

as $y \to 0$, we say that $G(X)$ is *the limit in the norm of* $F(X, y)$, as $y \to 0$. Similarly, we say

$$\lim_{y\to 0} F(X, y) = G(X)$$

for almost every X in E_n if $u(X, y) \to w_0(X)$ and $v_k(X, y) \to w_k(X)$, $k = 1, 2, \ldots, n$, for almost every X in E_n. In either case, we write

$$G(X) = F(X, 0),\ \ w_0(X) = u(X, 0),\ \ w_k(X) = v_k(X, 0),$$

for $k = 1, 2, \ldots, n$.

The main theorem in the theory of H^p spaces can then be stated as follows:

---

(1) Calderón actually proves that for almost every X in S lim $w(Z, y)$ exists when the point (Z, y) tends to X along any path in E_{n+1}^{+} that is not tangent to E_n. These non-tangential limits will also exist for the members of H^p. We restrict ourselves, however, to making the above simpler statement.

THEOREM B. *Suppose $F(X, y)$ is in H^p, $p \geqslant (n-1)/n$, then*

$$\lim_{y\to 0} F(X, y) = F(X, 0)$$

exists for almost every X in E_n.[1] *In case $p > (n-1)/n$, $F(X, 0)$ is also the limit in the norm of $F(X, y)$.*

Proof. Suppose F is in H^p, then, by assumption, there exists a constant K such that

$$\mathfrak{M}_p(y; F) \leqslant K < \infty \tag{4.1}$$

for all $y > 0$. Furthermore, by theorem A, since F satisfies the generalized Cauchy-Riemann equations (1.5), $|F|^{(n-1)/n}$ is subharmonic. Thus, if we let $q = \frac{n}{n-1} p$ (then $q \geqslant 1$) and $s(X, y) = |F(X, y)|^{(n-1)/n}$, the function $s(X, y)$ satisfies the hypotheses of lemma (3.8). Thus, by this lemma, there exists a harmonic function $m(X, y) \geqslant s(X, y)$ such that, if $q = 1$ (or, equivalently, $p = (n-1)/n$), it is a Poisson-Stieltjes integral of a finite measure μ, and, if $q > 1$ (that is, $p > (n-1)/n$), it is a Poisson integral of a function f in $L^p(E_n)$.

Let $w(X, y)$ be one of the components $u(X, y), v_1(X, y), \ldots, v_n(X, y)$ of $F(X, y)$. Then, for (Z, y) in $\Gamma_\alpha(X)$ ($=$ the conical region defined in lemma (3.14)) we have

$$|w(Z, y)|^{(n-1)/n} \leqslant m(Z, y) \leqslant m_\alpha^*(X).$$

But, by lemma (3.14), $m_\alpha^*(X)$ is finite almost everywhere. Hence, if we let $S_k \subset E_n$, $k = 1, 2, \ldots,$ be the set of all X in E_n such that $m_\alpha^*(X) \leqslant k$, $E_n - \bigcup_{k=1}^{\infty} S_k$ has measure zero.

On the other hand, for each k, the harmonic function $w(X, y)$ satisfies the conditions of lemma (3.19) with $S = S_k$ and $M = k^{n/(n-1)}$. Thus, by this lemma, $\lim_{y\to 0} w(X, y) = w(X, 0)$ exists for almost every X in S_k. Since $E_n - \bigcup_{k=1}^{\infty} S_k$ has measure zero it then follows that $\lim_{y\to 0} w(X, y) = w(X, 0)$ exists for almost every X in E_n.

This proves that

$$\lim_{y\to 0} F(X, y) = F(X, 0) \tag{4.2}$$

exists for almost every X in E_n.

The fact that for $p > (n-1)/n$

$$\| F(X, y) - F(X, 0) \|_p \to 0 \tag{4.3}$$

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[1] In fact, the non-tangential limit, described in the footnote on page 45, exists for almost every X in E_n.

as $y \to 0$ now follows from a simple argument. We first note that

$$\begin{aligned} |F(X, y) - F(X, 0)|^p &\leqslant 2^p (|F(X, y)|^p + |F(X, 0)|^p) \\ &\leqslant 2^p ([m_\alpha^*(X)]^q + [m_\alpha^*(X)]^q) \\ &= 2^{p+1} [m_\alpha^*(X)]^q. \end{aligned}$$

Since, by (4.2), for almost every X

$$|F(X, y) - F(X, 0)|^p \to 0$$

as $y \to 0$, and the function $[m_\alpha^*(X)]^q$ is integrable ober E_n, (4.3) is a consequence of the Lebesgue dominated convergence theorem. This completes the proof of Theorem B.

The following theorems are the n-dimensional generalization of other well-known results in the classical theory of H^p spaces.

THEOREM C. *If $p \geqslant (n-1)/n$ and F is in H^p the function*

$$\psi(y) = \{\mathfrak{M}_p(y; F)\}^{(n-1)/n}$$

is convex and decreasing. If $p > (n-1)/n$, $\psi(y)$ decreases to zero as $y \to \infty$.

Proof. Using the notation of the previous proof, $s(X, y) = |F(X, y)|^{(n-1)/n}$ and $q = pn/(n-1)$, we have

$$\psi(y) = \left\{ \int_{E_n} [s(X, y)]^q \, dX \right\}^{1/q}. \tag{4.4}$$

Since F is in H^p the function $\mathfrak{M}_p(y; F)$ is bounded and, thus, so is $\psi(y)$. Hence, if we show that the latter is convex, it must be decreasing.

It is easily checked that in order to establish the convexity of $\psi(y)$ it suffices to prove that the function $\phi(Z, y) = \psi(y)$ is subharmonic in E_{n+1}^+. This can be done in the following way: Let us take for granted, momentarily, that $\psi(y)$ is continuous. Fix a point (Z_0, y_0) in E_{n+1}^+ and let S_r be the sphere about this point of radius $r > y_0$. Suppose $g(X)$ is the function satisfying

$$\left\{ \int_{E_n} [s(Z_0 + X, y_0)]^q \, dX \right\}^{1/q} = \int_{E_n} g(X) \, s(Z_0 + X, y_0) \, dX. \ (^1)$$

Then, using the subharmonicity of $s(X, y)$,

(1) Note that $g(X) \geqslant 0$ and $\|g\|_{q'} = 1$, where $1/q + 1/q' = 1$.

$$\phi(Z_0, y_0) = \left\{ \int_{E_n} [s(Z_0 + X, y_0)]^q \, dX \right\}^{1/q} = \int_{E_n} g(X) \, s(Z_0 + X, y_0) \, dX$$

$$\leqslant \int_{E_n} g(X) \left\{ \frac{1}{\omega r^{n+1}} \int_{S_r} s(X + Z, y) \, dZ \, dy \right\} dX$$

$$= \frac{1}{\omega r^{n+1}} \int_{S_r} \left\{ \int_{E_n} g(X) \, s(X + Z, y) \, dX \right\} dZ \, dy$$

$$\leqslant \frac{1}{\omega r^{n+1}} \int_{S_r} \left\{ \int_{E_n} [s(X + Z, y)]^q \, dX \right\}^{1/q} dZ \, dy$$

$$= \frac{1}{\omega r^{n+1}} \int_{S_r} \phi(Z, y) \, dZ \, dy,$$

the last inequality being a consequence of Hölder's inequality. Thus,

$$\phi(Z_0, y_0) \leqslant \frac{1}{\omega r^{n+1}} \int_{S_r} \phi(Z, y) \, dZ \, dy.$$

But this is the mean value property characterizing subharmonic functions.

If $p > (n-1)/n$ or, equivalently, $q > 1$, lemma (3.8) guarantees that $s(X, y)$ is majorized by a Poisson integral of a function $f(X)$ in $L^q(E_n)$:

$$m(X, y) = \frac{y}{c_n} \int_{E_n} \frac{f(X - Z)}{(|Z|^2 + y^2)^{\frac{1}{2}(n+1)}} \, dZ.$$

We note that $m(X, y) \to 0$ as $y \to \infty$; for, letting $1/q + 1/q' = 1$ and using Hölder's inequality, we have

$$m(X, y) \leqslant \frac{y}{c_n} \left\{ \int_{E_n} \frac{dZ}{(|Z|^2 + y^2)^{\frac{1}{2}q'(n+1)}} \right\}^{1/q'} \|f\|_q$$

$$= \frac{y}{c_n} \left\{ \int_{E_n} \frac{dZ}{y^{(n+1)q'} (1 + |Z|^2/y^2)^{\frac{1}{2}q'(n+1)}} \right\}^{1/q'} \|f\|_q$$

$$= \frac{1}{c_n y^{n/q}} \left\{ \int_{E_n} \frac{dX}{(|X|^2 + 1)^{\frac{1}{2}q'(n+1)}} \right\}^{1/q'} \|f\|_q$$

which clearly tends to zero as y increases.

On the other hand, as was shown in the proof of lemma (3.14), we have $m(X, y)$ majorized by a constant multiple of $f^*(X)$ (see (3.17)). But the maximal function $f^*(X)$ is in $L^q(E_n)$ together with $f(X)$ (see lemma (3.12)).

We have shown, therefore, that the members of the family of functions $\{[s(X, y)]^q\}$, parametrized by y, are dominated by the integrable function $[f^*(X)]^q$ and $\lim_{y\to\infty} [s(X, y)]^q = 0$. Thus, by the Lebesgue dominated convergence theorem,

$$\int_{E_n} [s(X, y)]^q \, dX \to 0$$

as $y \to \infty$. But, by (4.4), this implies $\psi(y) \to 0$ as $y \to \infty$. We observe that this argument also proves the continuity of $\psi(y)$. Thus, the proof of the theorem is complete.

As a consequence of Theorem C we have

$$\sup_{y>0} \mathfrak{M}_p(y; F) = \lim_{y\to 0} \mathfrak{M}_p(y; F).$$

As is usually done in the case of the classical H^p spaces, we call this limit the *norm* of F and use the notation

$$\|F\|_p = \lim_{y\to 0} \mathfrak{M}_p(y; F).$$

In case $p > (n-1)/n$, an immediately corollary of Theorems B and C is the fact

$$\|F\|_p = \Big(\int_{E_n} |F(X, 0)|^p \, dX \Big)^{1/p}. \tag{4.5}$$

THEOREM D. *Suppose $p_1 > (n-1)/n$ and $p_2 \geqslant (n-1)/n$, $F(X, y)$ is in H^{p_1} and $|F(X, 0)|$ is in $L^{p_2}(E_n)$, then $F(X, y)$ is in H^{p_2}.*

Proof. By lemma (3.8), the subharmonic function $|F(X, y)|^{(n-1)/n}$ has a harmonic majorant, $m(X, y)$, in E^+_{n+1} such that $m(X, 0) = \lim_{y\to 0} m(X, y)$ exists both in the L^{q_1} norm and almost everywhere, where $q_1 = p_1 n/(n-1)$. Furthermore, it also follows from this lemma that

$$\|m(X, 0)\|_{q_1} = \Big(\int_{E_n} [m(X, 0)]^{q_1} \, dX \Big)^{1/q_1} \leqslant \|F\|_{p_1}^{(n-1)/n}. \tag{4.6}$$

But, by (4.5)

$$\|F\|_{p_1}^{(n-1)/n} = \Big(\int_{E_n} |F(X, 0)|^{q_1(n-1)/n} \, dX \Big)^{1/q_1}.$$

On the other hand, letting $y \to 0$ in the inequality $|F(X, y)|^{(n-1)/n} \leqslant m(X, y)$, we obtain

$$|F(X, 0)|^{(n-1)/n} \leqslant m(X, 0) \tag{4.7}$$

for almost every X.

Consequently, by (4.6) and (4.7), we must have

$$|F(X, 0)|^{(n-1)/n} = m(X, 0) \tag{4.8}$$

for almost every X.

The assumption $|F(X, 0)|$ in $L^{p_2}(E_n)$, however, is equivalent to the condition $m(X, 0) = |F(X, 0)|^{(n-1)/n}$ in $L^{q_2}(E_n)$, where $q_2 = p_2 n/(n-1)$. Since $m(X, y)$, by lemma (3.8), is the Poisson integral of $m(X, 0)$ it then follows from (c) of section 3 that

$$\int_{E_n} [m(X, y)]^{q_2} dX \leqslant \int_{E_n} [m(X, 0)]^{q_2} dX$$

for all $y > 0$.

The proof of the theorem is now complete if we notice that the left hand side of this inequality majorizes

$$\int_{E_n} |F(X, y)|^{p_2} dX.$$

Remarks. 1) We note that in the statements of the theorems of this section less was concluded in the case $p = (n-1)/n$ than in the cases $p > (n-1)/n$. For example, in Theorem B, for $p = (n-1)/n$ we proved only that $\lim_{y \to 0} F(X, y)$ exists for almost every X in E_n and stated nothing about convergence in the norm. Similarly, in Theorem C, we concluded that $\psi(y)$ decreases to zero as $y \to \infty$ only if $p > (n-1)/n$. Furthermore, nothing has been said about the properties of functions in H^p for $p < (n-1)/n$.

Whether or not these theorems are best possible is an open question. On the other hand, the known facts for the classical H^p spaces (when $n = 1$) indicate that the above theorems cannot be extended to other values of p and that the space $H^{(n-1)/n}$ is atypical.

An explicit example illustrating this situation is the following. Let us make the observation that the analogue of the space $H^{(n-1)/n}$, when $n > 1$, in the one dimensional case is the Nevanlinna class N of analytic functions $F(z) = F(x + iy)$ defined in the upper half plane $y > 0$ such that

$$\int_{-\infty}^{\infty} \log^+ |F(x + iy)| \, dx \leqslant A < \infty$$

for all $y > 0$. (This observation is motivated by the fact that the one dimensional analogue of the fact that $|F|^{(n-1)/n}$ is subharmonic is that $\log |F(z)|$ is subharmonic.) As was mentioned in the introduction, it is known that, for functions in this space, $\lim_{y\to 0} F(x+iy) = F(x)$ exists almost everywhere. On the other hand, even though there exists a natural "norm" on N [1], there are functions F in N for which $\lim_{y\to 0} F(x+iy) = F(x)$ in the norm is false.

2) The basic tool used in the proof of Theorem C is the fact that $s(X, y)$ is subharmonic. As was observed in the second section, $|F(X, y)|^p$ is subharmonic when $p \geq 1$ even if we only assume that the components of F are harmonic (that is, we do not assume F to be a system of conjugate harmonic functions). Theorem C, therefore, extends to $(n+1)$-tuplets, $F(X, y)$, of harmonic functions satisfying

$$\int_{E_n} |F(X, y)|^p \, dy \leq A < \infty$$

for all $y > 0$, for $p \geq 1$.

3) In most of the proofs of the results of the last two sections we have tacitly assumed that $n > 1$. This assumption was clearly necessary when we operated with the function $|F|^{(n-1)/n}$. On the other hand, very simple alterations of the arguments used above will include this classical case. For example, if F is in H^p, $p > 0$, when $n = 1$, instead of considering $|F|^{(n-1)/n}$ we can form the function $s(x, y) = |F(x, y)|^{p-\delta}$, for $0 < \delta < p$. Then $s(x, y)$ satisfies the conditions of lemma (3.2) with $q = p/(p-\delta) > 1$; having done this, the arguments based on this lemma are unchanged. Furthermore, as was indicated by the first remark, in the statements of the theorems of this section the space $H^{(n-1)/n}$ should be replaced by the Nevanlinna class N when $n = 1$.

5. An n-dimensional generalization of the theorem of F. and M. Riesz.

In order to state the next theorem we must introduce the notion of the M. Riesz transforms of functions (and measures) that was briefly discussed in the introduction. For $k = 1, 2, \ldots, n$, the kth *M. Riesz transform* of the function f in $L^p(E_n)$ in usually defined as the following Cauchy principal-value integral (see [6])

$$(R_k(f))(X) = \lim_{\varepsilon\to 0} \frac{1}{c_n} \int_{|X-Y|\geq\varepsilon} \frac{x_k - y_k}{|X-Y|^{n+1}} f(Y)\, dY, \tag{5.1}$$

---

(1) This norm in discussed, for the case of analytic functions defined in the unit disc, in remark (iii) on page 47 of [16].

where c_n is the constant occurring in the Poisson kernel (see the beginning of the third section). It is known that this limit exists for almost every X in E_n provided f is in $L^p(E_n)$, $1<p<\infty$. In fact, the limit exists almost everywhere even if we replace $f(Y)\,dY$ by $d\mu(Y)$, where μ is a finite Lebesgue-Stieltjes measure on E_n. Furthermore, as a transformation on $L^p(E_n)$, $1<p<\infty$, each R_k is bounded and maps into $L^p(E_n)$. That is, there exists A_p, independent of f in $L^p(E_n)$ such that

$$\| R_k(f)\|_p \leqslant A_p \|f\|_p,$$

$k=1, 2, \ldots, n$.

We shall also need the following anti-symmetric property of R_k (see [3], where the symmetry of the operator $i R_k$ is proved, which is equivalent to (5.2) below)

$$\int_{E_n} (R_k f)\, g\, dX = -\int_{E_n} f\,(R_k g)\, dX \tag{5.2}$$

for f in $L^p(E_n)$ and g in $L^q(E_n)$, where $1/p+1/q=1$.

Although the limit in (5.1) exists almost everywhere when f is in L^1 (or, as was mentioned above, when $f(Y)\,dY$ is replaced by $d\mu(Y)$), the resulting function may fail to be locally integrable. It is therefore convenient to define R_k on L^1, or on the class of finite Lebesgue-Stieltjes measures, in a different sense (the so-called *weak sense*). Thus, for f in L^1, or for a measure μ, we define $R_k(f)$, or $R_k(d\mu)$, $k=1, 2, \ldots, n$, as a distribution in the following way:

Let ϕ be a testing function; i.e. ϕ is in the class C^∞ and vanishes outside a compact subset of E_n. Thus, $R_k(\phi)$ is well defined by (5.1). In fact, it is not hard to see that $R_k(\phi)$ is a bounded and continuous function (since, by integrating by parts, we see that $R_k(\phi)$ is the convolution of the testing function $(\partial\phi)/(\partial x_k)$ with $(1-n)/c_n|X|^{n-1}$). (1) It is then justified to define $R_k(f)$ and $R_k(d\mu)$ as linear functionals on the space of testing functions by letting

$$[R_k(f)](\phi) = -\int f\, R_k(\phi)\, dX$$

and

$$[R_k(d\mu)](\phi) = -\int R_k(\phi)\, d\mu(X).$$

We note that if f is in $L^p(E_n)$, $1<p$, then the distribution $R_k(f)$ is represented by the function which is given by the usual pointwise limit (5.1).

If μ is a measure we then say that the *M. Riesz transform* $R_k(d\mu)$ *is a measure* ν_k, if for each testing function ϕ

(1) If $n=1$ we replace this function by a constant multiple of $\log|X|$.

$$\int_{E_n} \phi \, d\nu_k = - \int_{E_n} R_k(\phi)\, d\mu. \tag{5.3}$$

Having made these definitions, we can now state the following generalization of the theorem of F. and M. Riesz:

THEOREM E. *Let μ be a finite Lebesgue-Stieltjes measure on E_n having the property that each of its M. Riesz transforms, $\nu_k = R_k(d\mu)$, $k = 1, 2, \ldots, n$ is also a finite Lebesgue-Stieltjes measure on E_n. Then all of these measures, $\mu, \nu_1, \nu_2, \ldots, \nu_n$, are absolutely continuous with respect to Lebesgue measure.* (1)

Proof. Let

$$u(X, y) = \int_{E_n} P(X - Z, y)\, d\mu(Z)$$

$$v_k(X, y) = \int_{E_n} P(X - Z, y)\, d\nu_k(Z), \quad k = 1, 2, \ldots, n.$$

If we can show that these $n+1$ Poisson-Stieltjes integrals form a system of conjugate harmonic functions, it is then immediate that

$$F(X, y) = (u(X, y), v_1(X, y), \ldots, v_n(X, y))$$

is in H^1 (in fact, $\|F\|_1$ is majorized by the sum of the total measures of $\mu, \nu_1, \ldots, \nu_n$). Thus, by Theorem B, the boundary values $u(X, 0)$, $v_1(Z, 0), \ldots, v_n(X, 0)$ are in $L^1(E_n)$; furthermore,

$$\|u(X, y) - u(X, 0)\|_1 \to 0$$

and

$$\|v_k(X, y) - v_k(X, 0)\|_1 \to 0, \quad k = 1, 2, \ldots, n,$$

as $y \to 0$.

On the other hand, the components of $F(X, y)$ must be the Poisson integrals of their boundary values. This is easily established in the following way: If we let $G(X, y)$ be the system of harmonic functions obtained by taking the Poisson integrals of $u(X, 0)$, $v_1(X, 0), \ldots, v_n(X, 0)$, then, by (v) of the third section,

$$\lim_{y \to 0} \int_{E_n} |G(X, y) - F(X, 0)|\, dX = 0.$$

Thus, since

$$\gamma(y) = \int_{E_n} |G(X, y) - F(X, y)|\, dX$$

is a decreasing function of y (see Remark 2) in the fourth section), we must have

(1) For a different n-dimensional generalization of the theorem of F. and M. Riesz, see [4].

$$\int_{E_n} |G(X, y) - F(X, y)|\, dX = 0$$

for all $y > 0$. Hence, $G(X, y) = F(X, y)$ on E^+_{n+1}.

Let us restate this fact in a slightly different way. Putting,

$$d\zeta_0(Z) = u(Z, 0)\, dZ - d\mu(Z),\ d\zeta_k(Z) = v_k(Z, 0)\, dZ - d\nu_k(Z),\ k = 1, 2, \ldots, n,$$

we have

$$\int_{E_n} P(X - Z, y)\, d\zeta_j(Z) = 0 \tag{5.4}$$

for $j = 0, 1, \ldots, n$.

But (5.4) implies that each of the measures ζ_j is the zero measure [1], which is equivalent to saying that each of the measures $\mu, \nu_1, \ldots, \nu_n$ is absolutely continuous with respect to Lebesgue measure and has the Radon-Nikodym derivative $u(Z, 0)$, $v_1(Z, 0), \ldots, v_n(Z, 0)$, respectively.

Thus, all that remains to be done in the proof of Theorem E is to verify that $F(X, y) = (u(X, y), v_1(X, y), \ldots, v_n(X, y))$ is a system of conjugate harmonic functions. Toward this end we first observe that by a simple limiting argument we can extend (5.3) to hold for $\phi(Z) = P(X - Z, y)$, where (X, y) is any point of E^+_{n+1}. In doing this, we must first compute the kth M. Riesz transform of $\phi(Z) = P(X - Z, y)$. This is easily done and, as is well known (see [6]), we obtain the *conjugate Poisson kernel*

$$Q_n(X - Z, y) = \frac{-1}{c_n} \frac{x_k - z_k}{(|X - Z|^2 + y^2)^{\frac{1}{2}(n+1)}}.$$

Thus, from this extension of (5.3) and the expression for $v_n(X, y)$, we have

$$-v_k(X, y) = \int_{E_n} Q_k(X - Z, y)\ d\mu(Z), \quad k = 1, 2, \ldots, n.$$

From these formulas for the $v_k(X, y)$'s, together with

$$u(X, y) = \int_{E_n} P(X - Z, y)\, d\mu(Z) = \frac{y}{c_n} \int_{E_n} \frac{1}{(|X - Z|^2 + y^2)^{\frac{1}{2}(n+1)}}\, d\mu(Z),$$

---

(1) This is a consequence of the fact that (5.4) immediately implies that $\int_{E_n} h(X, y)\, d\zeta_j(X) = 0$ whenever $h(X, y)$ is the Poisson integral of a function in C_0 (= class of all continuous functions vanishing at infinity) and that such Poisson integrals are uniformly dense in C_0.

we see by straight-forward differentiation that $(u(X, y), v_1(X, y), \ldots, v_n(X, y)$ is the gradient

$$\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \cdot h(X, y),$$

where h is the harmonic function

$$h(X, y) = \frac{1}{c_n(1-n)} \int_{E_n} \frac{1}{(|X-Z|^2 + y^2)^{\frac{1}{2}(n-1)}} d\mu(Z).$$

This shows that $F(X, y)$ is a system of conjugate harmonic functions and the theorem is proved.

It is sometimes useful to rephrase this theorem in the language of Fourier transforms:

THEOREM E′. *Let μ be a Lebesgue-Stieltjes measure on E_n and*

$$\mathfrak{M}(X) = \int_{E_n} e^{iX \cdot Y} d\mu(Y)$$

its Fourier transform. Assume that the n functions $i(x_1/|X|)\mathfrak{M}(X), \ldots, i(x_n/|X|)\mathfrak{M}(X)$ are also Fourier transforms of measures $\nu_1, \nu_2, \ldots, \nu_n$. Then, each of the measures $\mu, \nu_1, \nu_2, \ldots, \nu_n$ is absolutely continuous with respect to Lebesgue measure.

Proof. This theorem follows from the previous one if we verify that, for each testing function ϕ,

$$\int_{E_n} \phi\, d\nu_k = -\int_{E_n} R_k(\phi)\, d\mu. \tag{5.5}$$

Let

$$\Phi(X) = \int_{E_n} e^{iX \cdot Y} \phi(Y)\, dY.$$

Then, by the Fourier inversion formula,

$$\phi(Y) = \frac{1}{(2\pi)^n} \int_{E_n} \Phi(X)\, e^{-iX \cdot Y} dX$$

(both integrals are absolutely convergent since ϕ is a testing function).
Thus,

$$\int_{E_n} \phi(Y)\, d\nu_k(Y) = \frac{1}{(2\pi)^n} \int_{E_n} \left\{ \int_{E_n} e^{-iX\cdot Y}\, \Phi(X)\, dX \right\} d\nu_k(Y)$$

$$= \frac{1}{(2\pi)^n} \int_{E_n} \Phi(X) \left\{ \int_{E_n} e^{-iX\cdot Y}\, d\nu_k(Y) \right\} dX$$

$$= -\frac{1}{(2\pi)^n} \int_{E_n} \Phi(X)\, i \frac{x_k}{|X|} \mathfrak{m}(-X)\, dX.$$

On the other hand, if $\Phi_k(X) = \int e^{iX\cdot Y} [R_k(\phi)](Y)\, dY$, then

$$\Phi_k(X) = i \frac{x_k}{|X|} \Phi(X)$$

(see [6]), and

$$[R_k(\phi)](X) = \frac{1}{(2\pi)^n} \int_{E_n} e^{-iX\cdot Y}\, \Phi_k(X)\, dX.$$

Thus,

$$\int_{E_n} R_k(\phi)\, d\mu = \frac{1}{(2\pi)^n} \int_{E_n} \left\{ \int_{E_n} e^{-iX\cdot Y}\, \Phi_k(X)\, dX \right\} d\mu(Y)$$

$$= \frac{1}{(2\pi)^n} \int_{E_n} \Phi_k(X)\, \mathfrak{m}(-X)\, dX$$

$$= \frac{1}{(2\pi)^n} \int_{E_n} \Phi(X) \frac{i x_k}{|X|} \mathfrak{m}(-X)\, dX.$$

Hence, (5.5) holds and the theorem is proved.

6. Fractional integrals defined on H^p-spaces

We begin by recalling some facts about fractional integrals of functions in $L^p(E_n)$, $1 \leqslant p < \infty$. For such a function, $f(X)$, its *fractional integral of order* α is the convolution

$$[I_\alpha(f)](X) = \frac{1}{\gamma_\alpha} \int_{E_n} \frac{f(X-Y)}{|Y|^{n-\alpha}}\, dY, \tag{6.1}$$

where

$$\gamma_\alpha = \frac{\pi^{\frac{1}{2}n}\, 2^\alpha\, \Gamma(\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}[n-\alpha])}. \tag{6.2}$$

It is easy to check that the integral in (6.1) converges for almost every X provided $0 < \alpha < n/p$.

If a linear transformation is defined on a class of functions containing $L^p(E_n)$ and, when restricted to this space, is a bounded transformation mapping into $L^q(E_n)$ $(p, q > 0)$, we say that it is of *type* (p, q). A fundamental result of Soboleff (see [13]) can then be stated in the following way:

THEOREM (Soboleff). *I_α is of type (p, q) whenever $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$.*

We assume the validity of this result which will be used for our extension of fractional integration to H^p spaces. In fact, this theorem and the following lemma are the basic facts in the theory of fractional integrals of functions in L^p spaces that we shall need.

LEMMA (6.2). *Let $f(X)$ be a function in $L^p(E_n)$, $1 \leqslant p < \infty$, and $u(X, y)$ its Poisson integral. Then,*

$$[I_\alpha(f)](X) = \frac{1}{\Gamma(\alpha)} \int_0^\infty u(X, y)\, y^{\alpha-1}\, dy, \tag{6.3}$$

where $0 < \alpha < n/p$.

Furthermore,

$$u_\alpha(X, y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty u(X, y+s)\, s^{\alpha-1}\, ds \tag{6.4}$$

is the Poisson integral of $I_\alpha(f)$.

Proof. The function $|u(X, y)|$, being the absolute value of the Poisson integral of a function in L^p, satisfies the hypotheses of lemma (3.2) with $q = p$ (see (v) of the third section of this paper). Thus, by (3.4), the integrand in (6.3) is absolutely integrable.

By decomposing f into its positive and negative parts we can reduce the proof of the lemma to the case $f \geqslant 0$. With this restriction on f, our various applications of Fubini's theorem are justified.

Since
$$u(X, y) = \int_{E_n} P(Z, y)\, f(X - Z)\, dZ,$$

we have
$$\int_0^\infty u(X, y)\, y^{\alpha-1}\, dy = \int_{E_n} \left\{ \int_0^\infty P(Z, y)\, y^{\alpha-1}\, dy \right\} f(X - Z)\, dZ.$$

But
$$\int_0^\infty P(Z, y)\, y^{\alpha-1}\, dy = \frac{1}{c_n} \int_0^\infty \frac{y}{(|Z|^2 + y^2)^{\frac{1}{2}(n+1)}}\, y^{\alpha-1}\, dy$$
$$= \frac{1}{c_n} |Z|^{\alpha-n} \int_0^\infty \frac{y^\alpha}{(1 + y^2)^{\frac{1}{2}(n+1)}}\, dy,$$

(the last equality following from the change of variables $s = y/|Z|$ and then replacing y for s).

Hence,
$$\int_0^\infty u(X, y)\, y^{\alpha-1}\, dy = \frac{1}{c_n}\left\{\int_0^\infty \frac{y^\alpha}{(1+y^2)^{\frac{1}{2}(n+1)}}\, dy\right\} \int_{E_n} \frac{f(X-Z)}{|Z|^{n-\alpha}}\, dZ$$
$$= \Gamma(\alpha)\,[I_\alpha(f)](X).\ (^1)$$

Equation (6.4) is, then, an immediate consequence of the "semigroup property" of the Poisson integral transform:

$$u(X, y+s) = \int_{E_n} u(Z, s)\, P(X-Z, y)\, dZ,$$

for all $s, y > 0$. [2]

Lemma (6.2) motivates the following definition: If $F(X, y)$ is a system of conjugate harmonic functions in E_{n+1}^+ we define its (vector-valued) *fractional integral of order* α, $\alpha > 0$, to be

$$F_\alpha(X, y) = [I_\alpha(F)](X, y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty F(X, y+s)\, s^{\alpha-1}\, ds \tag{6.5}$$

whenever this integral exists. [3]

Some of the formal properties of $I_\alpha(F)$ are contained in the following theorem:

THEOREM F. (a) *The integral in* (6.5) *converges absolutely for each* (X, y) *in* E_{n+1}^+ *provided* F *is in* H^p *and* $(n-1)/n \leqslant p < n/\alpha$ *(thus, the fractional integral may exist even if* $n \leqslant \alpha$*);*

(b) *Under the same hypotheses,* $F_\alpha(X, y)$ *is a system of conjugate harmonic functions;*

(c) *If* F *is in* H^p, $\alpha > 0$, $\beta > 0$ *and* $(n-1)/n \leqslant p < n/(\alpha+\beta)$, *then* $I_\alpha(I_\beta(F)) = I_{\alpha+\beta}(F)$.

Proof. (a) Let $m(X, y)$ be the harmonic majorant of $|F(X, y)|^{(n-1)/n}$ obtained in lemma (3.8). Using the notation $q = pn/(n-1)$, we then have

$$\int_{E_n} [m(X, y)]^q\, dX \leqslant C^q < \infty.$$

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(1) The fact that $\gamma_\alpha^{-1} = \dfrac{1}{\Gamma(\alpha)\, c_n} \displaystyle\int_0^\infty \frac{y^\alpha}{(1+y^2)^{\frac{1}{2}(n+1)}}\, dy$ follows from the formulas on pages 56 and 57 of [14].

(2) This semigroup property is well known when $n = 1$. Its proof for general n is essentially contained in the proof of lemma (3.6) (see, in particular, the proof that $w_k(X, y) = m(X, y+y_k)$).

(3) No confusion should arise from the fact that I_α will be used to denote both the operator acting on L^p-spaces as well as the operator acting on H^p-spaces.

Thus, by (3.4), $m(X, y) \leqslant C y^{-n/q}$. This, however, implies

$$|F(X, y)| \leqslant C^{n/(n-1)} y^{-n/p}.$$

Hence,
$$\int_0^\infty |F(X, y+s)| s^{\alpha-1} ds \leqslant C^{n/(n-1)} \int_0^\infty (y+s)^{-n/p} s^{\alpha-1} ds.$$

On the other hand, since $p < n/\alpha$ and $0 < \alpha$, the last integral is finite.

(b) It is easy to see that about each point of E_{n+1}^+ we can find a neighborhood, contained in E_{n+1}^+, and a sequence of Riemann sums that converges to the integral in (6.5), uniformly in (X, y) belonging to this neighborhood. But any such Riemann sum is, clearly, a system of conjugate harmonic functions. Thus, each such Riemann sum satisfies the system of equations (1.5). On the other hand, the above uniform convergence implies that (in a possibly smaller neighborhood) the derivatives of the members of the sequence converge uniformly (since the components are harmonic functions). Thus

$$\int_0^\infty F(X, y+s) s^{\alpha-1} ds,$$

the limit of this sequence, satisfies (1.5).

(c) We must show that

$$\frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty F(X, y+s) s^{\alpha+\beta-1} ds = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty r^{\alpha-1} \left\{ \int_0^\infty F(X, y+r+t) t^{\beta-1} dt \right\} dr.$$

On the other hand, the last (iterated) integral is equal to

$$\int_0^\infty r^{\alpha-1} \left\{ \int_r^\infty F(X, y+t)(t-r)^{\beta-1} dt \right\} dr = \int_0^\infty F(X, y+t) \left\{ \int_0^t (t-r)^{\beta-1} r^{\alpha-1} dr \right\} dt.$$

Thus, we need only verify that

$$\frac{1}{\Gamma(\alpha+\beta)} s^{\alpha+\beta-1} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^s (s-r)^{\beta-1} r^{\alpha-1} dr.$$

But this identity is well known (see [14], p. 56).

The various applications of Fubini's theorem are justified because of the absolute convergence of the integrals in question (see the argument for part (a)).

The main result of this section is the following theorem (in which the notion of type (p, q) is extended, in the obvious way, to transformations acting on H^p spaces).

THEOREM G. *If* $(n-1)/n < p < n/\alpha$ *(in particular,* α *may be greater than* n*) and* $1/q = 1/p - \alpha/n$*, then* $I_\alpha : F \to F_\alpha$ *is of type* (p, q).

Proof. If $p > 1$ and $F(X, y) = (u(X, y), v_1(X, y), \ldots, v_n(X, y))$ is in H^p, then, as was mentioned in the introduction, the components $u(X, y)\, v_1(X, y), \ldots, v_n(X, y)$ are the Poisson integrals of their boundary values. On the other hand, $v_1(X, 0), \ldots, v_n(X, 0)$ are the M. Riesz transforms of $u(X, 0)$. Thus, putting together various results, mentioned and derived in this paper (e.g. see, in particular, the beginning of the third anf fifth sections), we see that there exists A_p such that

$$\|F\|_p \leqslant A_p \|u(X, 0)\|_p.$$

On the other hand, it is trivially true that

$$\|u(X, 0)\|_p \leqslant \|F\|_p.$$

In view of the inequalities, the theorem of Soboleff and lemma (6.2), we see that the case $p > 1$ reduces to the case of fractional integrals of functions in $L^p(E_n)$. Thus, we assume $(n-1)/n < p \leqslant 1$ and that F is in H^p.

Let us first consider the case $n = 2$ and thus, restrict ourselves to $\frac{1}{2} < p \leqslant 1$. In addition, let us assume that $0 < \alpha < 1$. Applying lemma (3.8) we obtain a harmonic majorant, $m(X, y)$, of the subharmonic function $|F(X, y)|^{\frac{1}{2}}$. Furthermore, again by this lemma, $m(X, y)$ can be chosen to be the Poisson integral of a function $m(X)$ in L^{2p}, $1 < 2p < 2$, where $\|m\|_{2p}^2 = \|F\|_p$ (see (4.8)).

We thus have, using inequality (3.17),

$$\begin{aligned} \Gamma(\alpha)\,|F_\alpha(X, y)| &\leqslant \int_0^\infty |F(X, y+s)|\, s^{\alpha-1}\, ds \\ &\leqslant \int_0^\infty m^2(X, y+s)\, s^{\alpha-1} ds \leqslant \sup_{y>0} m(X, y) \int_0^\infty m(X, y+s)\, s^{\alpha-1}\, ds \\ &\leqslant K\, m^*(X)\, [I_\alpha(m)](X, y). \end{aligned} \tag{6.6}$$

Thus, a constant multiple of $\int |F_\alpha(X, y)|^q\, dX$ is majorized by

$$\int \{m^*(X) \cdot [I_\alpha(m)](X, y)\}^q\, dX.$$

On the other hand, since $1/q = 1/p - \frac{1}{2}\alpha$, $2p/q = 2 - \alpha p$. But $p \leqslant 1$ and $\alpha < 1$, by assumption, thus $2p/q > 1$. Let r be the exponent conjugate to $2p/q$ (that is, $1/r + q/2p = 1$). Thus, by Hölder's inequality

$$\int |F_\alpha(X, y)|^q \, dX \leqslant (\text{const.}) \left\{ \int \left\{ m^*(X) \right\}^{2p} dX \right\}^{q/2p} \left\{ \int \left\{ [I_\alpha(m)](X, y) \right\}^{qr} dX \right\}^{1/r}. \quad (6.7)$$

A simple calculation shows that

$$\frac{1}{qr} = \frac{1}{2p} - \frac{\alpha}{2}.$$

Thus, by Soboleff's theorem,

$$\left\{ \int \left\{ [I_\alpha(m)](X, y) \right\}^{qr} dX \right\}^{1/r} \leqslant (\text{const.}) \left\{ \int \left\{ m(X) \right\}^{2p} dX \right\}^{q/2p}.$$

Substituting this in (6.7) and applying lemma (3.12) to the function m^*, we obtain

$$\int |F_\alpha(X, y)|^q \, dX \leqslant (\text{const.}) \left\{ \int \left\{ m(X) \right\}^{2p} dX \right\}^{q/p}$$
$$= (\text{const.}) \, \|F\|_p^q.$$

But, since $\|F_\alpha\|_q = \sup_{y<0} \left\{ \int |F_\alpha(X, y)|^q \, dX \right\}^{1/q}$, this proves that I_α is of type (p, q).

The restriction $0 < \alpha < 1$ can be dropped by making use of the "semigroup property"

$$I_\alpha(I_\beta(F)) = I_{\alpha+\beta}(F).$$

For $n > 2$ the proof remains essentially the same, but technically more complicated. The necessary changes are the following: The restriction $0 < \alpha < 1$ is replaced by $0 < \alpha < (n-1)$, the harmonic majorant $m(X, y)$ is the Poisson integral of a function $m(X)$ in $L^{pn/(n-1)}$, in (6.6) we have $[m(X, y+s)]^{n/(n-1)}$ (instead of $m^2(X, y+s)$) and this function is majorized by $[m^*(X)]^{1/(n-1)} m(X, y+s)$. Once these changes have been made the above proof goes through without change.

The following theorem, mentioned in the introduction, is now an easy corollary of Theorem G.

THEOREM H. *Suppose f is a function in $L^1(E_n)$ such that each of its n M. Riesz transforms are also in $L^1(E_n)$.* (1) *Then*

$$I_\alpha(f), \, I_\alpha(R_1(f)), \, I_\alpha(R_2(f)), \, \ldots, \, I_\alpha(R_n(f))$$

are all in $L^q(E_n)$ whenever $1/q = 1/p - \alpha/n$, $0 < \alpha < n$.

Proof. Once we show the vector-valued function $F(X, y)$, whose components are the Poisson integrals of the functions $f, R_1(f), \ldots, R_n(f)$, is a member of H^1, this

(1) In the weak sense discussed in the fifth section of this paper.

theorem is an immediate consequence of theorem G. But this was shown in the proof of Theorem E (in fact, there we assumed only that the Riesz transforms of f were measures).

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