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ON THE THEORY OF INCONSISTENT FORMAL SYSTEMS

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Introduction This is an expository work,* in which we shall treat some questions related to the theory of inconsistent formal systems. The exposition will be neither rigorous nor complete. For details, the reader may consult the works cited in the References. (With reference to the historical aspects of the theory, see specially [1].) In general, the terminology, the notations, etc., are those of Kleene's book [17], with evident adaptations.

A formal system (deductive system, deductive theory, . . .) **S** is said to be inconsistent if there is a formula A of **S** such that A and its negation, $\neg A$, are both theorems of this system. In the opposite case, **S** is called consistent. A deductive system **S** is said to be trivial if all its formulas are theorems. If there is at least one unprovable formula in **S**, it is called non-trivial.

If the underlying logic of a system **S** is the classical logic (the intuitionistic logic, . . .), then **S** is trivial if, and only if, it is inconsistent. Hence, employing such a category of logics, the inconsistent systems do not present any proper logico-mathematical interest. Usually, we try to change the inconsistent theories to transform them into consistent ones. It is clear that under this transformation, some characteristic properties of a given inconsistent theory must be preserved; for instance, the common formal systems of set theory preserve certain traits of inconsistent naive set theory.

Nonetheless, there are certain cases in which we might think of studying directly an inconsistent theory. For example, a set theory containing Russell's class (the class of all classes which are not members of themselves) as an existing set, or a theory whose aim be the systematization of Meinong's theory of objects.[1] Apparently, it would be as

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1. Meinong's theory is discussed, for example, by Russell (*cf.* [21] and the articles by Meinong, Ameseder and Mally cited there). One of the objections formulated by Russell against Meinong's theory is precisely that it implies a derogation of the principle of contradiction.

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interesting to study the inconsistent systems as, for instance, the non-eucledian geometries: we would obtain a better idea of the nature of certain paradoxes, could have a better insight on the connections amongst the various logical principles necessary to obtain determinate results, etc. But, if we intend to do this, then we must construct new types of logic. In fact, as we noted above, without the use of new logics, the inconsistent systems would lose their logico-mathematical importance.

It seems convenient to insist on the following point: given an inconsistent system **S**, our aim is not to eliminate the possible paradoxes or inconsistencies of **S**, but to derive in it as many paradoxes as it is convenient, to analyse and to study them. However, this does not mean that we wish each formula of **S** and its negation to be theorems. Intuitively speaking, in an inconsistent theory **S**, presenting real interest, there are "good" theorems, whose negations are not provable, and "bad" ones, whose negations are also theorems. In particular, if **S** is sufficiently strong to contain elementary arithmetic, then it seems rather natural to require that it must be arithmetically consistent (though it may be inconsistent), i.e.: supposing that A is a formula belonging (in a certain precise sense) to the elementary arithmetic of **S**, A and $\neg A$ cannot be, at the same time, theorems of **S**. An *antinomy* implies triviality. A paradox is not in general an antinomy. In the sequel, we shall see how it is possible to develop the theory of inconsistent systems.

1 The Calculi $\mathbf{C}_n$, $\mathbf{C}_n^*$, and $\mathbf{C}_n^=$ We introduce in this section certain propositional calculi $\mathbf{C}_n$, $1 \leqslant n \leqslant \omega$. Then, we proceed with the construction of the corresponding first order predicate calculi without equality, $\mathbf{C}_n^*$, $1 \leqslant n \leqslant \omega$, and with equality, $\mathbf{C}_n^=$, $1 \leqslant n \leqslant \omega$. ($\mathbf{C}_0$, $\mathbf{C}_0^*$, and $\mathbf{C}_0^=$ will denote respectively the classical propositional calculus, the classical (first order) predicate calculus without equality and the classical predicate calculus with equality.) These new calculi can be used as foundations for non-trivial inconsistent theories, as we shall see.

1.1 The Calculi $\mathbf{C}_n$ As $\mathbf{C}_n$, $1 \leqslant n \leqslant \omega$, are intended to serve as bases for non-trivial inconsistent theories, it seems natural that they satisfy the following conditions:

I) In these calculi the principle of contradiction, $\neg(A \,\&\, \neg A)$, must not be a valid schema; II) From two contradictory formulas, A and $\neg A$, it will not in general be possible to deduce an arbitrary formula B; III) It must be simple to extend $\mathbf{C}_n$, $1 \leqslant n \leqslant \omega$, to corresponding predicate calculi (with or without equality) of first order; IV) $\mathbf{C}_n$, $1 \leqslant n \leqslant \omega$, must contain the most part of the schemata and rules of $\mathbf{C}_0$, which do not interfere with the first conditions. (Evidently, the last two conditions are vague.)

1.1.1 The Calculus $\mathbf{C}_1$ To begin with, we introduce the calculus $\mathbf{C}_1$, which has the following postulates, where A° is an abbreviation for $\neg(A \,\&\, \neg A)$:

(1) $A \supset (B \supset A)$,

(2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$,

(3) $\dfrac{A \quad A \supset B}{B}$,

(4) $A \,\&\, B \supset A$,

(5) $A \,\&\, B \supset B$,

(6) $A \supset (B \supset A \,\&\, B)$,

(7) $A \supset A \vee B$,

(8) $B \supset A \vee B$,

(9) $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$,

(10) $A \vee \neg A$,

(11) $\neg\neg A \supset A$,

(12) $B^\circ \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$,

(13) $A^\circ \,\&\, B^\circ \supset (A \,\&\, B)^\circ$,

(14) $A^\circ \,\&\, B^\circ \supset (A \vee B)^\circ$,

(15) $A^\circ \,\&\, B^\circ \supset (A \supset B)^\circ$.

We have:

Theorem 1 *In* $\mathbf{C}_1$ *all rules of deduction of the propositional calculus of* Theorem 2 *of Kleene's book* [17] *are true, with the exception of the rule of reductio ad absurdum, which is* $\mathbf{C}_1$ *can be stated as follows:*

$$\text{If } \Gamma, A \vdash B^\circ,\ \Gamma, A \vdash B \text{ and } \Gamma, A \vdash \neg B, \text{ then } \Gamma \vdash \neg A.$$

Theorem 2 *Among others, the following schemata in* $\mathbf{C}_1$ *are not valid:*

$$\neg A \supset (A \supset B),\ \neg A \supset (A \supset \neg B),\ A \supset (\neg A \supset B),$$
$$A \supset (\neg A \supset \neg B),\ A \,\&\, \neg A \supset B,\ A \,\&\, \neg A \supset \neg B,$$
$$(A \supset B) \supset ((A \supset \neg B) \supset \neg A),\ A \supset \neg\neg A,\ (A \sim \neg A) \supset B,$$
$$(A \sim \neg A) \supset \neg B,\ \neg(A \,\&\, \neg A),\ (A \vee B) \,\&\, \neg A \supset B,$$
$$A \vee B \supset (\neg A \supset B),\ (A \supset B) \supset (\neg B \supset \neg A),\ A \sim \neg\neg A.$$

Proof: It suffices to employ the matrices

$A \,\&\, B$: A \ B	1	2	3
1	1	1	3
2	1	1	3
3	3	3	3

,

$A \vee B$: A \ B	1	2	3
1	1	1	1
2	1	1	1
3	1	1	3

,

$A \supset B$: A \ B	1	2	3
1	1	1	3
2	1	1	3
3	1	1	1

,

$\neg A$: A	$\neg A$
1	3
2	1
3	1

,

where 1 and 2 are the designated truth-values.

Theorem 3 *In* $\mathbf{C}_1$ *all schemata and rules of deduction of the classical positive propositional calculus are true, and if we adjoin to* $\mathbf{C}_1$ *the principle* *of contradiction, we obtain* $\mathbf{C}_0$. *In* $\mathbf{C}_1$ *we have also*[2]*:*

$$B^\circ, A \supset B \vdash \neg B \supset \neg A,\ B^\circ, A \supset \neg B \vdash B \supset \neg A,$$
$$B^\circ, \neg A \supset B \vdash \neg B \supset A,\ B^\circ, \neg A \supset \neg B \vdash B \supset A,$$
$$\vdash (A \supset \neg A) \supset \neg A,\ \vdash (\neg A \supset A) \supset A,$$
$$\vdash A^\circ \supset (\neg A)^\circ.$$

Theorem 4 *If* $A_1, A_2, \ldots, A_m$ *are the prime components of the formulas* Γ, A, *then a necessary and sufficient condition for* $\Gamma \vdash A$ *in* $\mathbf{C}_0$ *is that* $\Gamma, A_1^\circ, A_2^\circ, \ldots, A_m^\circ \vdash A$ *in* $\mathbf{C}_1$.

Definition 1 $\neg^* A =_{Def} \neg A \,\&\, A^\circ$.

$\neg^* A$ is the strong negation of A.

Theorem 5 *In* $\mathbf{C}_1$, $\neg^*$ *has all properties of the classical negation.*

For instance, we have:

$$\vdash A \vee \neg^* A,\ \vdash \neg^*(A \,\&\, \neg^* A),\ \vdash (A \supset B) \supset ((A \supset \neg^* B) \supset \neg^* A),$$
$$\vdash A \sim \neg^*\neg^* A,\ \vdash \neg^* A \supset (A \supset B),\ \vdash (A \sim \neg^* A) \supset B.$$

Theorem 6 $\mathbf{C}_1$ *is consistent.*

A non-trivial system $\mathbf{S}$ is said to be finitely trivializable if there is a formula (not a schema) F such that, adjoining F to $\mathbf{S}$ as a new axiom, the resulting system is trivial. For example, the intutionistic or classical implicative propositional calculi and the classical positive propositional calculus are not finitely trivializable; the classical predicate calculus is finitely trivializable.

Theorem 7 $\mathbf{C}_1$ *is finitely trivializable.*

Proof: Each formula of the type $A \,\&\, \neg^* A$ trivializes $\mathbf{C}_1$.

1.1.2 The Calculi $\mathbf{C}_n$, $n > 1$.

$\mathbf{C}_1$ is not the only propositional calculus that satisfy conditions I-IV. Leaving aside other possibilities, we shall describe a hierarchy of calculi $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n, \ldots, \mathbf{C}_\omega$, having properties similar to those of $\mathbf{C}_1$. To introduce $\mathbf{C}_n$, $1 < n < \omega$, it is convenient to abbreviate $A^{\circ\circ\cdots\circ}$, where the symbol $^\circ$ appears m times, $m \geq 1$, by A^m, and $A^1 \,\&\, A^2 \,\&\, \ldots A^m$ by $A^{(m)}$. The postulates of $\mathbf{C}_n$, $1 < n < \omega$, are those of $\mathbf{C}_1$, excepting the postulates (12)-(15), which are replaced by the following:

(12′) $B^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$,
(13′) $A^{(n)} \,\&\, B^{(n)} \supset (A \,\&\, B)^{(n)}$,
(14′) $A^{(n)} \,\&\, B^{(n)} \supset (A \vee B)^{(n)}$,
(15′) $A^{(n)} \,\&\, B^{(n)} \supset (A \supset B)^{(n)}$.

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2. M. Guillaume proved that $A^\circ \supset (\neg A)^\circ$ is a consequence of our postulates (in [7], $A^\circ \supset (\neg A)^\circ$ appears as a postulate of $\mathbf{C}_1$). More generally, he noted that $A^{(n)} \supset (\neg A)^{(n)}$ is a theorem of the calculus $\mathbf{C}_n (1 \leq n < \omega)$, which will be defined in the sequel. (For other propositional calculi serving the same purpose, the reader may consult [16] and [19].)

The postulates of $\mathbf{C}_\omega$ are (1)-(11) above.

Theorem 8 *Every calculus belonging to the hierarchy* $\mathbf{C}_n$, $0 \leqslant n < \omega$, *is finitely trivializable.* $\mathbf{C}_\omega$ *is not finitely trivializable.*

Theorem 9 *Every calculus of the hierarchy* $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_\omega$ *is strictly stronger than those which follow it.*

Theorem 10 (Arruda) *The calculi* $\mathbf{C}_n$, $1 \leqslant n \leqslant \omega$, *are not decidable by finite matrices.*

Theorem 11 (Fidel) *The calculi* $\mathbf{C}_n$, $0 \leqslant n \leqslant \omega$, *are decidable.*

$\mathbf{C}_{n+1}$ is weaker than $\mathbf{C}_n$, $n \geqslant 0$. Therefore, if we are trying to obviate triviality, it is safer to employ $\mathbf{C}_{n+1}$ to found a system, than to use $\mathbf{C}_n$; if we limit ourselves to choose a calculus of the hierarchy under consideration, the greatest security is attained employing $\mathbf{C}_\omega$.

Theorem 12 *The schemata of* Theorem 2 *above are not true in* $\mathbf{C}_n$, $1 \leqslant n \leqslant \omega$.

Theorem 13 *In* $\mathbf{C}_n$, $1 \leqslant n < \omega$, *one has:*

$$B^{(n)}, A \supset B \vdash \neg B \supset \neg A,\ B^{(n)}, A \supset \neg B \vdash B \supset \neg A,$$
$$B^{(n)}, \neg A \supset B \vdash \neg B \supset A,\ B^{(n)}, \neg A \supset \neg B \vdash B \supset A,$$
$$\vdash (A \supset \neg A) \supset \neg A,\ \vdash (\neg A \supset A) \supset A,$$
$$A^{(n)} \vdash (\neg A)^{(n)},\ \vdash A^{(n)(n)}.$$

Theorem 14 $\mathbf{C}_n$, $0 \leqslant n \leqslant \omega$, *are consistent.*

Theorem 15 *In* $\mathbf{C}_\omega$ *Peirce's law,* $((A \supset B) \supset A) \supset A$, *is not true.*

In general, the results valid for $\mathbf{C}_1$ can be adapted to apply to $\mathbf{C}_n$, $2 \leqslant n < \omega$.

Remark 1 It is possible to construct propositional calculi for inconsistent systems, in which the principle of contradiction is true (*cf.*, for instance, [4]).

1.2 The Predicate Calculi $\mathbf{C}_n^*$ $\mathbf{C}_1^*, \mathbf{C}_2^*, \ldots, \mathbf{C}_\omega^*$ are the first order predicate calculi corresponding to $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_\omega$.

1.2.1 The Calculus $\mathbf{C}_1^*$ First, we shall describe $\mathbf{C}_1^*$. The list of its postulates is that of $\mathbf{C}_1$, plus the following:

(I) $\dfrac{C \supset A(x)}{C \supset \forall x\, A(x)}$, (II) $\forall x\, A(x) \supset A(t)$,

(III) $A(t) \supset \exists x\, A(x)$, (IV) $\dfrac{A(x) \supset C}{\exists x\, A(x) \supset C}$,

(V) $\forall x(A(x))^\circ \supset (\forall x\, A(x))^\circ$, (VI) $\forall x(A(x))^\circ \supset (\exists x A(x))^\circ$,

(VII) *If A and B are congruent formulas,*[3] *or one is obtained from the other by the suppression of void quantifiers, then* $A \sim B$ *is an axiom.*

3. Kleene [17], p. 153.

The postulates (I)-(IV) are subjected to the usual restrictions.[4]

Theorem 16 *All rules of* Theorem 2 *of Kleene's book* [17] *are true for* C_1^*, *the rule of reductio ad absurdum being stated as in* Theorem 1 *above. The valid schemata of the classical positive predicate logic are also valid in* C_1^*. *If we add to this calculus the schema* $\neg(A \,\&\, \neg A)$ *as a new postulate, we get* C_0^*.

Theorem 17 Lemmas 6-10 *and* Theorems 13 *and* 15, *among others, of Kleene's book are true for* C_1^*.

Theorem 18 *If the prime components of the formulas* Γ, A *are* $A_1, A_2, \ldots, A_m$, *then a necessary and sufficient condition for* $\Gamma \vdash A$ *in* C_0^* *is that* $\Gamma, A_1^\circ, A_2^\circ, \ldots, A_m^\circ \vdash A$ *in* C_1^*.

Theorem 19 C_1^* *is undecidable.*

Proof: Consequence of the foregoing theorem and of the undecidability of C_0^*.

Theorem 20 C_1^* *is consistent.*

Theorem 21 *If* $\Gamma \vdash A$ *in* C_1^*, *then all of the k-transforms*[5] *of A are deducible from the k-transforms of the formulas* Γ *in* C_1.

Corollary 1 *If* $\vdash A$ *in* C_1^*, *then all of the k-transforms of A are provable in* C_1.

Corollary 2 *Suppose that A is a predicate letter formula containing only predicate letters with zero attached variables. Then,* $\vdash A$ *in* C_1^* *if and only if* $\vdash A$ *in* C_1.

Corollary 2 shows that a propositional schema which is not valid in C_1 cannot be valid in C_1^*. For example, the schemata $\neg(A \,\&\, \neg A)$, $\neg A \supset (A \supset B)$, $A \,\&\, \neg A \supset B$, and $(A \sim \neg A) \supset B$ are not provable in C_1^*. Corollary 1 clearly implies the non-validity in C_1^* of schemata such as $\neg(A(x) \,\&\, \neg A(x))$, $(A(x) \sim \neg A(x)) \supset B(x)$ and $\exists x(A(x) \sim \neg A(x)) \supset B(x)$.

Theorem 22 *In* C_1^* *the following schemata are not valid:*

$$\neg\exists x\, \neg A(x) \sim \forall x\, A(x),\ \neg\forall x\, \neg A(x) \sim \exists x\, A(x),$$
$$\neg\exists x\, A(x) \sim \forall x\, \neg A(x),\ \exists x\, \neg A(x) \sim \neg\forall x\, A(x).$$

Proof: It is sufficient to consider the 2-transforms of formulas of the indicated types and to apply Corollary 1.

1.2.2 The Predicate Calculi C_n^*, $n > 1$ The postulates of C_n^* are those of C_n, $2 \leqslant n < \omega$, plus the following: (I)-(IV) and (VII) introduced above and

$$(\mathrm{V}_n)\quad \forall x(A(x))^{(n)} \supset (\forall x\, A(x))^{(n)},$$
$$(\mathrm{VI}_n)\quad \forall x(A(x)^{(n)} \supset (\exists x\, A(x))^{(n)}.$$

4. Kleene [17], pp. 81-82.

5. Kleene [17], pp. 177-178.

The postulates of C_ω^* are (1)-(11) and (I)-(IV).

Theorem 23 *If the prime components of the formulas* Γ, A *are* $A_1, A_2, \ldots, A_m$, *then* $\Gamma \vdash A$ *in* C_0^* *if and only if* $\Gamma, A_1^{(n)}, A_2^{(n)}, \ldots, A_m^{(n)} \vdash A$ *in* C_n, $1 \leqslant n < \omega$.

Theorem 24 *The calculi* C_n^*, $0 \leqslant n \leqslant \omega$, *are undecidable.*

Theorem 25 *If* $\Gamma \vdash A$ *in* C_n^*, *then all of the k-transforms of A are deducible in* C_n, $0 \leqslant n \leqslant \omega$, *from the k-transforms of the formulas* Γ.

Theorem 26 *Let A denote a formula of* C_n; *then,* $\vdash A$ *in* C_n^* *if and only if* $\vdash A$ *in* C_n, $0 \leqslant n \leqslant \omega$.

Theorem 27 *Every calculus of the hierarchy* $C_0^*, C_1^*, C_2^*, \ldots, C_\omega^*$ *is strictly stronger than those following it.*

Corollary C_n^*, $0 \leqslant n \leqslant \omega$, *are consistent.*

Theorem 28 C_n^*, $0 \leqslant n < \omega$, *are finitely trivializable, but* C_ω^* *is not.*

Definition 2 $\neg^{(n)}A =_{Def} \neg A \;\&\; A^{(n)}$, $n \geqslant 1$.

In particular, $\neg^*$ and $\neg^{(1)}$ are abbreviations of $\neg A \;\&\; A^\circ$.

Theorem 29 *In* C_n *and* C_n^*, $1 \leqslant n < \omega$, $\neg^{(n)}$ *has all properties of the classical negation.*

1.3 The Calculi $C_n^=$ $C_1^=, C_2^=, \ldots, C_\omega^=$ are the predicate calculi with equality corresponding respectively to $C_1^*, C_2^*, \ldots, C_\omega^*$. $C_n^=$ is obtained from C_n, $1 \leqslant n \leqslant \omega$, as $C_0^=$ is constructed from C_0^*. In particular, we add the new postulates:

(I′) $x = x$, (II′) $x = y \supset (A(x) \supset A(y))$, where the schema (II′) is subjected to the usual restrictions.[6]

1.3.1 The Calculus $C_1^=$ A list of the most important results regarding $C_1^=$ follows.

Theorem 30 *We have in* $C_1^=$ (*with the same restrictions of* $C_0^=$):

$$\vdash x = x, \vdash x = y \supset y = x, \vdash x = y \;\&\; y = z \supset x = z,$$
$$\vdash x = y \supset (A(x) \sim A(y)), \vdash \forall y\, \exists x(x = y),$$
$$(A(t))^\circ \vdash {}^t A(x) \;\&\; \neg A(x) \supset x \neq y,$$
$$\vdash \forall y(F(y) \sim \exists x(x = y \;\&\; F(x)),$$
$$\vdash \exists x(F(x) \;\&\; \forall y(F(y) \supset x = y)) \sim \exists x\, \forall y(x = y \sim F(y)),$$
$$\vdash \exists x F(x) \;\&\; \forall x \forall y(F(x) \;\&\; F(y) \supset x = y) \sim \exists x(F(x) \;\&\; \forall y(F(y) \supset x = y)),$$
$$(F(x))^\circ, (x = y)^\circ \vdash \exists y \forall x(y = x \sim F(x)) \sim \exists x F(x) \;\&\; \neg \exists x(F(x) \;\&\; \exists y(x \neq y \;\&\; F(y))).$$

Theorem 31 $A_1, A_2, \ldots, A_m$ *are the prime components of the formulas* Γ, A. *Then,* $\Gamma \vdash A$ *in* $C_0^=$ *if and only if* $\Gamma, A_1^\circ, A_2^\circ, \ldots, A_m^\circ \vdash A$ *in* $C_1^=$.

6. Kleene [17], p. 399.

Theorem 32 $C_1^=$ *is undecidable, and adjoining to it the law of non-contradiction, we obtain* $C_0^=$.

Theorem 33 *If the symbol* $=$ *does not occur in the formula* A, *then* $\vdash A$ *in* $C_1^=$ *if and only if* $\vdash A$ *in* C_1^*.

The preceding theorem is important, because it shows that the non-valid schemata of C_1^* are also non-valid in $C_1^=$. For example, the schemata $\neg A \supset (A \supset B)$, $\neg(A \,\&\, \neg A)$, $(A(x) \sim \neg A(x) \supset B)$, and $\neg(A(x) \,\&\, \neg A(x))$ are not valid in $C_1^=$.

1.3.2 The Calculi $C_n^=$, $n > 1$ We can prove for $C_n^=$, $1 < n \leqslant \omega$, among others, the following results.

Theorem 34 *The schemata and formulas of* Theorem 30 *are true in* $C_n^=$, $1 < n < \omega$, *replacing the symbol* $^\circ$ *by* $^{(n)}$. *The schemata and formulas in which* $^\circ$ *does not occur are also true in* $C_\omega^=$.

Theorem 35 *Let* $A_1, A_2, \ldots, A_m$ *be the prime components of* Γ, A. *We have:* $\Gamma \vdash A$ *in* $C_0^=$ *if and only if* $\Gamma, A_1^{(n)}, A_2^{(n)}, \ldots, A_m^{(n)} \vdash A$ *in* $C_n^=$, $1 \leqslant n < \omega$.

Theorem 36 $C_n^=$, $0 \leqslant n \leqslant \omega$, *are consistent and undecidable.*

Theorem 37 *Every calculus of the hierarchy* $C_0^=, C_1^=, C_2^=, \ldots, C_\omega^=$ *is strictly stronger than those following it. The calculi* $C_n^=$, $0 \leqslant n < \omega$, *are finitely trivializable.* $C_\omega^=$ *is not finitely trivializable.*

Theorem 38 *In* $C_n^=$, $1 \leqslant n < \omega$, $\neg^{(n)}$ *has all properties of the classical negation.*

From the last result, we conclude that $C_0^=$ (respectively C_0^*, C_0) is contained (under a convenient translation) in $C_n^=$ (respectively C_n^*, C_n), $1 \leqslant n < \omega$. $C_n^=$ is also a conservative extension of C_n^*, $0 \leqslant n \leqslant \omega$. The replacement theorem[7] is not true for C_n, C_n^*, and $C_n^=$, $0 < n \leqslant \omega$. In particular, we do not have (in general) in these systems: $A \sim B \vdash \neg A \sim \neg B$.

2 The System $\mathbf{NF}_1$ The calculi constructed may be used, e.g., to examine the well-known paradoxes. Let us analyse informally the paradox of the liar. In an intuitive language $\mathcal{L}_n$ based on C_n, $0 \leqslant n \leqslant \omega$, one can accept that it is possible to talk about sentences like in ordinary language. (The languages $\mathcal{L}_n$, $0 \leqslant n \leqslant \omega$, are pressupposed to be strong enough from the point of view of the power of expression.) Taking $n = 1$ to fix our ideas, a simple formulation of the liar paradox is as follows:

α) *This sentence implies its negation.*

Reasoning as in the classical case ($n = 0$), one deduces that

$$\vdash \alpha \,\&\, \neg\alpha.$$

However, as $A \,\&\, \neg A \supset B$, $\neg A \supset (A \supset B)$, etc., are not valid schemas of

7. Kleene [17], pp. 115 and 151.

$\mathbf{C}_1$, α does not cause directly any difficulty in $\mathcal{L}_1$. But there is a strong form of the liar which cannot be surmounted by the device of using a language whose underlying propositional calculus is $\mathbf{C}_1$, namely:

β) *This sentence implies its strong negation.*

With an argument formally analogous to the classical one, it is possible to prove that

$$\vdash \beta \ \& \ \neg^* \beta.$$

In this case, since $\vdash A \ \& \ \neg^* A \supset B$ in $\mathbf{C}_1$, we have a real antinomy in $\mathcal{L}_1$: β makes our language trivial.[8] In a few words, to each $\mathcal{L}_n$, $0 \leqslant n < \omega$, there corresponds a convenient formulation of the liar paradox which makes it trivial. The same is not true with respect to $\mathcal{L}_\omega$. Hence, in languages of the category of $\mathcal{L}_n$, $0 \leqslant n < \omega$, some limitations on self-reference must be imposed. Nevertheless, this does not signify that we are compelled to eliminate all forms of self-referential statements. Even statements such as α need not be eliminated from $\mathcal{L}_1$ as senseless, though it is self-contradictory. We do not make any further comments on the non-mathematical paradoxes. Now, we shall study a system $\mathbf{NF}_1$ of set theory which is inconsistent and apparently non-trivial.

Remark 2 Related to the subject matter of the paradoxes there is the problem of the universe of discourse of a language. To exemplify, let us call U_1 the universe of discourse of $\mathcal{L}_1$. Depending on the properties of an "inconsistent" object σ (for instance, the round-square), it can belong to U_1: if the definition of σ does not imply a contradiction of the form $A \ \& \ \neg^* A$, then it may be legitimate to accept σ as a good element of U_1. Clearly, the same is not true of $\mathcal{L}_o$. Paraphrazing Quine, one could maintain that to be is to be the value of a variable, in a given language with a convenient logic. Loosely speaking, if one weakens the underlying logic of a certain language, the objects about which it is possible to talk become more numerous. Logic and ontology are intimately correlated.

2.1 Description of $\mathbf{NF}_1$ The underlying logic of $\mathbf{NF}_1$ is $\mathbf{C}_1^=$ and its only specific symbol is the binary predicate constant ϵ. Hence, the primitive symbols of $\mathbf{NF}_1$ are the individual variables, $\supset$, &, $\vee$, $\neg$, $\forall$, $\exists$, (,), =, and ϵ. The notions of formula, (formal) proof, etc., are defined as usual (*cf.* [17]).

Definition 3 If $F(x)$ and $G(x)$ are formulas whose variables are subjected to the common conditions, then:

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8. A variant of the liar which can be directly reproduced in $\mathcal{L}_n$, $0 < n \leqslant \omega$, is the following. Suppose that $\mathcal{L}_n (0 < n \leqslant \omega)$ contains propositional variables and a rule of substitution for these variables. (This is surely not the case of our calculus $\mathbf{C}_n (0 \leqslant n \leqslant \omega)$, as we defined it.) Then, the statement (p is a propositional variable):

This sentence implies p.

is easily seen to be antinomical in $\mathcal{L}_n$, $0 \leqslant n \leqslant \omega$.

$$x \epsilon \hat{y}F(y) =_{Def} \exists z(\forall y(y \epsilon z \sim F(y)) \;\&\; x \epsilon z),$$
$$\hat{y}F(y) \epsilon x =_{Def} \exists z(\forall y(y \epsilon z \sim F(y)) \;\&\; z \epsilon x),$$
$$\hat{x}F(x) \epsilon \hat{y}G(y) =_{Def} \exists t \exists z(\forall x(x \epsilon t \sim F(x)) \;\&\; \forall y(y \epsilon z \sim G(y)) \;\&\; t \epsilon z),$$
$$x = \hat{y}F(y) =_{Def} \exists z(\forall y(y \epsilon z \sim F(y)) \;\&\; x = z),$$
$$\hat{y}F(y) = x =_{Def} \exists z(\forall y(y \epsilon z \sim F(y)) \;\&\; z = x),$$
$$\hat{x}F(x) = \hat{y}F(n) =_{Def} \exists t \exists z(\forall x(x \epsilon t \sim F(x)) \;\&\; \forall y(y \epsilon z \sim G(y)) \;\&\; t = z),$$
$$\exists \hat{x}F(x) =_{Def} \exists t \; \forall x(x \epsilon t \sim F(x)),$$
$$\exists ! \hat{x}F(x) =_{Def} \exists x(F(x)) \;\&\; \forall y(F(y) \supset x = y).$$

Definition 4 A formula $F(x)$ is said to be normal if it is stratified[9] or if unstratified and neither (the symbolic abbreviation) $\neg^*$ nor $\supset$ occur in it.

Specific postulates of $\mathbf{NF}_1$:

(P_1) $\forall x(x \epsilon y \sim x \epsilon z) \supset z = y$.

(P_2) $\exists y \forall x(x \epsilon y \sim F(x))$, *where x and y are distinct variables, y does not occur free in $F(x)$ and this formula is normal.*

In the following postulates we do not make explicit the pertinent restrictions:

(P_3) $\hat{x}F(x) = \hat{y}G(y) \supset (A(\hat{x}F(x)) \sim A(\hat{y}G(y)))$.
(P_4) $\exists \hat{x}F(x) \;\&\; A(\hat{x}F(x)) \supset \exists tA(t)$.
(P_5) $\exists \hat{x}F(x) \;\&\; \forall tA(t) \supset A(\hat{x}F(x))$.

In the sequel, we shall use the terminology and notations of Rosser [20], with clear modifications.

Theorem 39 *In* $\mathbf{NF}_1$ *we have (under the usual restrictions):*

$\vdash \exists \hat{x}F(x) \sim \exists ! \hat{x}F(x)$, $\vdash \exists \hat{x}F(x) \sim \exists ! y \; \forall x(x \epsilon y \sim F(x))$,
$\vdash \exists \hat{x}F(x) \sim \hat{x}F(x) = \hat{x}F(x)$, $\vdash \exists \hat{x}F(x) \supset (\hat{x}F(x) = \hat{y}F(y))$,
$\vdash \hat{x}F(x) = \hat{y}G(y) \supset \exists \hat{x}F(x) \;\&\; \exists yG(y)$, $\vdash \forall x(F(x) \sim G(x)) \;\&\; \exists \hat{x}F(x) \supset \exists \hat{x}G(x)$,
$\vdash \hat{x}F(x) = \hat{y}G(y) \supset \hat{y}G(y) = \hat{x}F(x)$, $\vdash A(\hat{x}F(x)) \sim A(\hat{y}F(y))$,
$\vdash \hat{x}F(x) = \hat{y}G(y) \;\&\; \hat{y}G(y) = \hat{z}H(z) \supset \hat{x}F(x) = \hat{z}H(z)$,
$\vdash \exists \hat{x}F(x) \sim \forall x(x \epsilon \hat{x}F(x) \sim F(x))$, $\vdash \exists \Lambda$, $\vdash \exists \mathrm{V}$,
$\vdash \exists \hat{x}F(x) \supset \forall y(y = \hat{x}F(x)) \sim \forall x(x \epsilon y \sim F(x))$,
$\vdash \forall x(P(x) \sim Q(x)) \supset \hat{x}P(x) = \hat{x}Q(x)$,
$\vdash \forall y \; \forall x(x \epsilon \hat{x}(x \epsilon y) \sim x \epsilon y)$, $\vdash \forall y(y = \hat{x}(x \epsilon y))$,
$\vdash \exists \hat{x}(x \neq x \;\&\; (x = x)^\circ)$, $\vdash \forall y \; \forall z \; \exists \hat{x}(x \epsilon y \vee x \epsilon z)$,
$\vdash \forall y \; \forall z \; \exists \hat{x}(x \epsilon y \;\&\; x \epsilon z)$, $\vdash \forall y \; \exists \hat{x}(x \notin y)$,
$\vdash \forall x \; \forall y(x \epsilon \overline{y} \sim x \notin y)$, $\vdash x \cup x = x$, $\vdash x \cup y = y \cup x$,
$\vdash (x \cup y) \cup z = x \cup (y \cup z)$, $\vdash x \cup (x \cap y) = x$,
$\vdash x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$, $\vdash x \cap x = x$, $\vdash x \cap y = y \cap x$,
$\vdash (x \cap y) \cap z = x \cap (y \cap z)$, $\vdash x \cap (x \cup y) = x$, $\vdash x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$,
$\vdash x \cap \Lambda = \Lambda$, $\vdash \Lambda = \overline{\mathrm{V}}$, $\vdash x \cup \Lambda = x$, $\vdash x \cap \mathrm{V} = x$,
$\vdash \overline{\Lambda} \subseteq \mathrm{V}$, $\vdash x \cup \mathrm{V} = \mathrm{V}$, $\vdash \forall x \; \exists \mathrm{USC}(x)$, $\vdash \forall x \; \exists \mathrm{SC}(x)$.

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9. *Cf.*, for example, Rosser [20] and Quine [18].

Theorem 40 $\mathbf{NF}_1$ *is inconsistent* (*but apparently is not trivial*).

Proof: In $\mathbf{NF}_1$ we can derive Russell's paradox. That is, $\vdash \exists \hat{x}(x \notin x)$ and, consequently, $\vdash \hat{x}(x \notin x) \in \hat{x}(x \notin x)$ & $\hat{x}(x \notin x) \notin \hat{x}(x \notin x)$.

Remark 3 The class $R = \hat{x}(x \notin x)$ has many interesting properties. For example: $\vdash R \notin R$, $\vdash R \in R$, $\vdash R \in R$ & $R \notin R$, $\vdash R \cup \overline{R} = \mathsf{V}$, $\vdash R \in R \cap \overline{R}$, $\vdash \exists \mathsf{USC}(R)$ and $\vdash \exists \mathsf{SC}(R)$.

Remark 4 The restrictions imposed on postulate (P_2), the schema of separation, are necessary, because the existence of classes such that $\hat{x}(\neg^*(x \in x))$ and $\hat{x}(x \in x \supset B)$, where B is an arbitrary formula, would make $\mathbf{NF}_1$ trivial.[10]

2.2 $\mathbf{NF}_1$ and Quine's System NF

Definition 5 The $\neg^*$-transform of a formula F is the formula F^* obtained from F by replacing in it every occurrence of $\neg$ by an occurrence of $\neg^*$. If Γ is a sequence of formulas, then Γ^* is the corresponding sequence of the $\neg^*$-transforms of the formulas of Γ.

Theorem 41 *If* $\Gamma \vdash F$ *in* NF (Quine's system [18]), *then* $\Gamma^* \vdash F^*$ *in* $\mathbf{NF}_1$.

Proof: Consequence of Theorem 38 and of (the form of) the postulates of NF and $\mathbf{NF}_1$.

Corollary *If* $\vdash F$ *in* NF, *then* $\vdash F^*$ *in* $\mathbf{NF}_1$.

The previous corollary shows us that NF is *contained* in $\mathbf{NF}_1$. Hence, this last system is strictly stronger than Quine's. (Granting that NF is consistent.)

Theorem 42 *The non-triviality of* $\mathbf{NF}_1$ *implies the consistency of* NF.

To sum up, $\mathbf{NF}_1$ is an inconsistent and very strong system. Apparently, it is not trivial, but we cannot prove its non-triviality more or less in the same sense in which one is unable to prove the consistency of the traditional strong deductive systems. This fact is a consequence of a generalization of Gödel's theorems, which covers the case of certain inconsistent formal systems.[11]

Remark 5 $\mathbf{NF}_1$ can be reinforced in several ways. As an outcome, other paradoxes are derivable in it (see [12]). $\mathbf{NF}_1$ contains elementary arithmetic and is apparently arithmetically consistent in the sense referred to in the introduction to the present work. In an analogous way to that in which we have constructed $\mathbf{NF}_1$, it is possible to construct infinitely many inconsistent set theories $\mathbf{NF}_1, \mathbf{NF}_2, \ldots, \mathbf{NF}_\omega$, whose properties are similar

10. This question is connected with the Curry-Moh Schaw-Kwei paradox. For details, see [12].

11. See [3] for details.

to those of $\mathbf{NF}_1$, using as underlying logics respectively the calculi $\mathbf{C}_1^=, \mathbf{C}_2^=, \ldots, \mathbf{C}_\omega^=$.

Remark 6 Our starting point to construct $\mathbf{NF}_1$ was $\mathbf{NF}$. We could also employ instead any one of the extant set theories, such as the systems of Zermelo-Fraenkel and of von Neumann-Bernays-Gödel.

Concluding remarks The delineated theory seems important to us, among others, for the following reasons:

1) In several inconsistent (and apparently non-trivial) systems, such that $\mathbf{NF}_1$, we can develop, by various devices, most of the extant set theories. It is possible to define, in these systems, certain "paradoxical" sets, whose *existence* can be proved in them, but that do not exist in the classical theories. This fact does not cause any difficulty and from a precise point of view such "paradoxical" sets exist more or less as the usual sets of a classical theory are said to exist. (These new sets may be thought of as existing in a similar sense in which, for instance, the points at infinity exist in eucledian plane geometry.) All these questions of existence, to which we may add the questions concerning the *meanings* of the formulas that infringe the principle of contradiction and of the very nature of this law, originate interesting and important philosophical problems. Evidently, the theory of inconsistent systems can contribute to clarify such issues.
2) The relations between the schema of separation and various types of logics, weaker than the classical, were studied. Under suitable precise conditions, the schema of separation is proved to be incompatible with every one of a series of weak elementary logics. It was only possible to discover this fact by a direct study of inconsistent set theories and related topics.
3) A lot of problems of mathematical character have been originated by the construction of logical systems that are apt to function as foundations for inconsistent and non-trivial theories. We mention here only the question referring to the algebrization of certain propositional calculi, such as $\mathbf{C}_n$, $1 \leq n \leq \omega$.
4) Some parts of the theory of inconsistent systems are related to modal and intuitionistic logics. These connections, a little surprising at first sight, are also the starting point of important problems, deserving of serious research.
5) The semantical analysis of certain new calculi, in the theory of inconsistent systems, seems to be very promising, and numerous results have been obtained already. In particular, some new types of models have been defined and some classical results of model theory generalized. (See, for instance, Fidel [14] and [15].)
6) Dialectic logic is intimately connected with the theory of inconsistent systems. There are several conflicting conceptions of dialectic logic, and for most specialists it is neither formal, nor even in principle formalizable. Nonetheless, employing techniques used in the theory of inconsistent systems, it is apparently possible to formalize some of the

proposed dialectic logics. It is convenient to note that the formalizations we are talking about are analogous in nature to the formalizations presented for various parts of intuitionistic mathematics: we do not intend to found dialectic logic on given formalisms, but only try to make explicit certain "regularities" of the "dialectical movement." Thus, we may throw a new light on dialectical logic.

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