XXXIII. *On the Theory of Oscillatory Waves.* By G. G. Stokes, M.A., *Fellow of Pembroke College.*

[Read *March* 1, 1847.]

In the Report of the Fourteenth Meeting of the British Association for the Advancement of Science it is stated by Mr. Russell, as a result of his experiments, that the velocity of propagation of a series of oscillatory waves does not depend on the height of the waves*. A series of oscillatory waves, such as that observed by Mr. Russell, does not exactly agree with what it is most convenient, as regards theory, to take as the type of oscillatory waves. The extreme waves of such a series partake in some measure of the character of solitary waves, and their height decreases as they proceed. In fact it will presently appear that it is only an indefinite series of waves which possesses the property of being propagated with a uniform velocity, and without change of form: at least this is the case when the waves are such as can be propagated along the surface of a fluid which was previously at rest. The middle waves, however, of a series such as that observed by Mr. Russell agree very nearly with oscillatory waves of the standard form. Consequently, the velocity of propagation determined by the observation of a number of waves, according to Mr. Russell's method, must be very nearly the same as the velocity of propagation of a series of oscillatory waves of the standard form, and whose length is equal to the mean length of the waves observed, which are supposed to differ from each other but slightly in length.

On this account I was induced to investigate the motion of oscillatory waves of the above form to a second approximation, that is, supposing the height of the waves finite, though small. I find that the expression for the velocity of propagation is independent of the height of the waves to a second approximation. With respect to the form of the waves, the elevations are no longer similar to the depressions, as is the case to a first approximation, but the elevations are narrower than the hollows, and the height of the former exceeds the depth of the latter. This is in accordance with Mr. Russell's remarks at page 448 of his first Report†. I have proceeded to a third approximation in the particular case in which the depth of the fluid is very great, so as to find in this case the most important term, depending on the height of the waves, in the expression for the velocity of propagation. This term gives an increase in the velocity of propagation depending on the square of the ratio of the height of the waves to their length.

There is one result of a second approximation which may possibly be of practical importance. It appears that the forward motion of the particles is not altogether compensated by their backward motion; so that, in addition to their motion of oscillation, the particles have a progressive motion in the direction of propagation of the waves. In the case in which the depth of the fluid is very great, this progressive motion decreases rapidly as the depth of the particle considered increases. Now when a ship at sea is overtaken by a storm, and the sky remains overcast, so as to prevent astronomical observations, there is nothing to trust to for finding the ship's place but the dead reckoning. But the estimated velocity and direction of motion of the ship are her velocity and direction of motion relatively to the water. If then the whole of the water near the surface be moving in the direction of the waves, it is evident that the ship's estimated place will be erroneous. If, however, the velocity of the water can be expressed in terms of the length and height of the waves, both which can be observed approximately from the ship, the motion of the water can be allowed for in the dead reckoning.

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* Page 369 (note), and page 370.

† *Reports of the British Association*, Vol. VI.

As connected with this subject, I have also considered the motion of oscillatory waves propagated along the common surface of two liquids, of which one rests on the other, or along the upper surface of the upper liquid. In this investigation there is no object in going beyond a first approximation. When the specific gravities of the two fluids are nearly equal, the waves at their common surface are propagated so slowly that there is time to observe the motions of the individual particles. The second case affords a means of comparing with theory the velocity of propagation of oscillatory waves in extremely shallow water. For by pouring a little water on the top of the mercury in a trough we can easily procure a sheet of water of a small, and strictly uniform depth, a depth, too, which can be measured with great accuracy by means of the area of the surface and the quantity of water poured in. Of course, the common formula for the velocity of propagation will not apply to this case, since the motion of the mercury must be taken into account.

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1. IN the investigations which immediately follow, the fluid is supposed to be homogeneous and incompressible, and its depth uniform. The inertia of the air, and the pressure due to a column of air whose height is comparable with that of the waves are also neglected, so that the pressure at the upper surface of the fluid may be supposed to be zero, provided we afterwards add the atmospheric pressure to the pressure so determined. The waves which it is proposed to investigate are those for which the motion is in two dimensions, and which are propagated with a constant velocity, and without change of form. It will also be supposed that the waves are such as admit of being excited, independently of friction, in a fluid which was previously at rest. It is by these characters of the waves that the problem will be rendered determinate, and not by the initial disturbance of the fluid, supposed to be given. The common theory of fluid motion, in which the pressure is supposed equal in all directions, will also be employed.

Let the fluid be referred to the rectangular axes of x, y, z, the plane xz being horizontal, and coinciding with the surface of the fluid when in equilibrium, the axis of y being directed downwards, and that of x taken in the direction of propagation of the waves, so that the expressions for the pressure, &c. do not contain z. Let p be the pressure, ρ the density, t the time, u, v the resolved parts of the velocity in the directions of the axes of x, y; g the force of gravity, h the depth of the fluid when in equilibrium. From the character of the waves which was mentioned last, it follows by a known theorem that $udx + vdy$ is an exact differential $d\phi$. The equations by which the motion is to be determined are well known. They are

$$p = g\rho y - \rho\frac{d\phi}{dt} - \frac{\rho}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\}, \qquad (1);$$

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0, \qquad (2);$$

$$\frac{d\phi}{dy} = 0, \text{ when } y = h, \qquad (3);$$

$$\frac{dp}{dt} + \frac{d\phi}{dx}\frac{dp}{dx} + \frac{d\phi}{dy}\frac{dp}{dy} = 0, \text{ when } p = 0, \qquad (4);$$

where (3) expresses the condition that the particles in contact with the rigid plane on which the fluid rests remain in contact with it, and (4) expresses the condition that the same surface of particles continues to be the free surface throughout the motion, or, in other words, that there is no generation or destruction of fluid at the free surface.

If c be the velocity of propagation, u, v and p will be by hypothesis functions of $x - ct$ and y. It follows then from the equations $u = \dfrac{d\phi}{dx}$, $v = \dfrac{d\phi}{dy}$ and (1), that the differential coefficients of ϕ with respect to x, y and t will be functions of $x - ct$ and y; and therefore ϕ itself must be of the form $f(x - ct, y) + Ct$. The last term will introduce a constant into (1); and if this constant be expressed, we may suppose ϕ to be a function of $x - ct$ and y. Denoting $x - ct$ by x', we have

$$\frac{dp}{dx} = \frac{dp}{dx'}, \quad \frac{dp}{dt} = -c\frac{dp}{dx'},$$

and similar equations hold good for ϕ. On making these substitutions in (1) and (4), omitting the accent of x, and writing $-gk$ for C, we have

$$p = g\rho(y + k) + c\rho\frac{d\phi}{dx} - \frac{\rho}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\}, \ldots\ldots\ldots\ldots (5),$$

$$\left(\frac{d\phi}{dx} - c\right)\frac{dp}{dx} + \frac{d\phi}{dy}\frac{dp}{dy} = 0, \text{ when } p = 0. \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Substituting in (6) the value of p given by (5), we have

$$g\frac{d\phi}{dy} - c^2\frac{d^2\phi}{dx^2} + 2c\left(\frac{d\phi}{dx}\frac{d^2\phi}{dx^2} + \frac{d\phi}{dy}\frac{d^2\phi}{dx\,dy}\right)$$
$$- \left(\frac{d\phi}{dx}\right)^2\frac{d^2\phi}{dx^2} - 2\frac{d\phi}{dx}\frac{d\phi}{dy}\frac{d^2\phi}{dx\,dy} - \left(\frac{d\phi}{dy}\right)^2\frac{d^2\phi}{dy^2} = 0, \ldots\ldots\ldots (7),$$

when $$g(y + k) + c\frac{d\phi}{dx} - \tfrac{1}{2}\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\} = 0. \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The equations (7) and (8) are exact; but if we suppose the motion small, and proceed to the second order only of approximation, we may neglect the last three terms in (7), and we may easily eliminate y between (7) and (8). For putting ϕ', $\phi_{,}$, &c. for the values of $\dfrac{d\phi}{dx}$, $\dfrac{d\phi}{dy}$, &c. when $y = 0$, the number of accents above marking the order of the differential coefficient with respect to x, and the number below its order with respect to y, and observing that k is a small quantity of the first order at least, we have from (8)

$$g(y + k) + c(\phi' + \phi_{,}'y) - \tfrac{1}{2}(\phi'^2 + \phi_{,}^2) = 0,$$

whence $$y = -k - \frac{c}{g}\phi' + \frac{c}{g}\phi_{,}'\left(k + \frac{c}{g}\phi'\right) + \frac{1}{2g}(\phi'^2 + \phi_{,}^2).^* \ldots\ldots (9).$$

Substituting the first approximate value of y in the first two terms of (7), putting $y = 0$ in the next two, and reducing, we have

$$g\phi_{,} - c^2\phi'' - (g\phi_{,,} - c^2\phi_{,}'')\left(k + \frac{c}{g}\phi'\right) + 2c(\phi'\phi'' + \phi_{,}\phi_{,}') = 0. \ldots (10).$$

ϕ will now have to be determined from the general equation (2) with the particular conditions (3) and (10). When ϕ is known, y, the ordinate of the surface, will be got from (9), and k will then be determined by the condition that the mean value of y shall be zero. The value of p, if required, may then be obtained from (5).

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* The reader will observe that the y in this equation is the ordinate of the surface, whereas the y in (1) and (2) is the ordinate of any point in the fluid. The context will always show in which sense y is employed.

2. In proceeding to a first approximation we have the equations (2), (3) and the equation obtained by omitting the small terms in (10), namely,

$$g\frac{d\phi}{dy} - c^2\frac{d^2\phi}{dx^2} = 0, \text{ when } y = 0. \quad\ldots\ldots\ldots\ldots (11).$$

The general integral of (2) is

$$\phi = \Sigma A\epsilon^{mx+ny},$$

the sign Σ extending to all values of A, m and n, real or imaginary, for which $m^2 + n^2 = 0$: the particular values of ϕ, $Cx + C'$, $Dy + D'$, corresponding respectively to $n = 0$, $m = 0$, must also be included, but the constants C', D' may be omitted. In the present case, the expression for ϕ must not contain real exponentials in x, since a term containing such an exponential would become infinite either for $x = -\infty$, or for $x = +\infty$, as well as its differential coefficients which would appear in the expressions for u and v; so that m must be wholly imaginary. Replacing then the exponentials in x by circular functions, we shall have for the part of ϕ corresponding to any one value of m,

$$(A\epsilon^{my} + A'\epsilon^{-my})\sin mx + (B\epsilon^{my} + B'\epsilon^{-my})\cos mx,$$

and the complete value of ϕ will be found by taking the sum of all possible particular values of the above form and of the particular value $Cx + Dy$. When the value so formed is substituted in (3), which has to hold good for all values of x, the coefficients of the several sines and cosines, and the constant term must be separately equated to zero. We have therefore

$$D = 0, \quad A' = \epsilon^{2mh}A, \quad B' = \epsilon^{2mh}B;$$

so that if we change the constants we shall have

$$\phi = Cx + \Sigma\,(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)})\,(A\sin mx + B\cos mx), \ldots (12),$$

the sign Σ extending to all real values of m, A and B, of which m may be supposed positive.

3. To the term Cx in (12) corresponds a uniform velocity parallel to x, which may be supposed to be impressed on the fluid in addition to its other motions. If the velocity of propagation be defined merely as the velocity with which the wave form is propagated, it is evident that the velocity of propagation is perfectly arbitrary. For, for a given state of relative motion of the parts of the fluid, the velocity of propagation, as so defined, can be altered by altering the value of C. And in proceeding to the higher orders of approximation it becomes a question what we shall define the velocity of propagation to be. Thus, we might define it to be the velocity with which the wave form is propagated when the mean horizontal velocity of a particle in the upper surface is zero, or the velocity of propagation of the wave form when the mean horizontal velocity of a particle at the bottom is zero, or in various other ways. The following two definitions appear chiefly to deserve attention.

First, we may define the velocity of propagation to be the velocity with which the wave form is propagated in space, when the mean horizontal velocity *at each point of space occupied by the fluid* is zero. The term mean here refers to the variation of the time. This is the definition which it will be most convenient to employ in the investigation. I shall accordingly suppose $C = 0$ in (12), and c will represent the velocity of propagation according to the above definition.

Secondly, we may define the velocity of propagation to be the velocity of propagation of the wave form in space, when the mean horizontal velocity of the mass of fluid comprised between two very distant planes perpendicular to the axis of x is zero. The mean horizontal velocity of the mass means here the same thing as the horizontal velocity of its centre of gravity. This appears to be the most natural definition of the velocity of propagation, since in the case considered there is no current in the mass of fluid, taken as a whole. I shall denote the velocity of propagation according to this definition by c'. In the most important case to consider, namely, that in which the depth is infinite, it is easy to see that $c' = c$, whatever be the order of approximation. For when the depth becomes infinite, the velocity of the centre of gravity of the mass comprised between any two planes parallel to the plane yz vanishes, provided the expression for u contain no constant term.

4. We must now substitute in (11) the value of ϕ.

$$\phi = \Sigma\, (\epsilon^{m(h-y)} + \epsilon^{-m(h-y)})\, (A \sin mx + B \cos mx) \;\ldots\ldots\ldots\ldots\; (13);$$

but since (11) has to hold good for all values of x, the coefficients of the several sines and cosines must be separately equal to zero: at least this must be true, provided the series contained in (11) are convergent. The coefficients will vanish for any one value of m, provided

$$c^2 = \frac{g}{m} \frac{\epsilon^{mh} - \epsilon^{-mh}}{\epsilon^{mh} + \epsilon^{-mh}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (14).$$

Putting for shortness $2mh = \mu$, we have

$$\frac{d \log c^2}{d\mu} = -\frac{1}{\mu} + \frac{2}{\epsilon^{\mu} - \epsilon^{-\mu}},$$

which is positive or negative, μ being supposed positive, according as

$$2\mu >< \epsilon^{\mu} - \epsilon^{-\mu} >< 2\left(\mu + \frac{\mu^3}{1\,.\,2\,.\,3} + \ldots\ldots\right),$$

and is therefore necessarily negative. Hence the value of c given by (14) decreases as μ or m increases, and therefore (11) cannot be satisfied, for a given value of c, by more than one positive value of m. Hence the expression for ϕ must contain only one value of m. Either of the terms $A \cos mx$, $B \sin mx$ may be got rid of by altering the origin of x. We may therefore take, for the most general value of ϕ,

$$\phi = A\, (\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}) \sin mx \;\ldots\ldots\ldots\ldots\ldots\ldots\; (15).$$

Substituting in (8), we have for the ordinate of the surface

$$y = -\frac{mAc}{g} (\epsilon^{mh} + \epsilon^{-mh}) \cos mx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (16),$$

k being $= 0$, since the mean value of y must be zero. Thus everything is known in the result except A and m, which are arbitrary.

5. It appears from the above, that of all waves for which the motion is in two dimensions, which are propagated in a fluid of uniform depth, and which are such as could be propagated into fluid previously at rest, so that $udx + vdy$ is an exact differential, there is only one particular kind, namely, that just considered, which possesses the property of being propagated with a constant velocity, and without change of form; so that a solitary wave cannot be propagated in this manner. Thus the degradation in the height of such waves, which Mr. Russell observed, is not to be attributed wholly, (nor I believe chiefly,) to the imperfect fluidity of the fluid, and its adhesion to the sides and bottom of the canal, but it is an essential characteristic of a solitary wave. It is true that this conclusion depends on an investigation which applies strictly to indefinitely small motions only: but if it were true in general that a solitary wave could be propagated uniformly, without degradation, it would be true in the limiting case of indefinitely small motions; and to disprove a general proposition it is sufficient to disprove a particular case.

6. In proceeding to a second approximation we must substitute the first approximate value of ϕ, given by (15), in the small terms of (10). Observing that $k = 0$ to a first approximation, and eliminating g from the small terms by means of (14), we find

$$g\phi_{,} - c^2\phi'' - 6A^2m^3c \sin 2mx = 0 \;\ldots\ldots\ldots\ldots\; (17).$$

The general value of ϕ given by (13), which is derived from (2) and (3), must now be restricted to satisfy (17). It is evident that no new terms in ϕ involving $\sin mx$ or $\cos mx$ need be introduced, since such terms may be included in the first approximate value, and the only other term which can enter is one of the form $B\,(\epsilon^{2m(h-y)} + \epsilon^{-2m(h-y)}) \sin 2mx$. Substituting this term in (17), and simplifying by means of (14), we find

$$B = \frac{3m A^2}{c\,(\epsilon^{mh} - \epsilon^{-mh})^2}.$$

Moreover since the term in ϕ containing $\sin mx$ must disappear from (17), the equation (14) will give c to a second approximation.

If we denote the coefficient of $\cos mx$ in the first approximate value of y, the ordinate of the surface, by a, we shall have

$$A = -\frac{ga}{mc\,(\epsilon^{mh} + \epsilon^{-mh})} = -\frac{ca}{\epsilon^{mh} - \epsilon^{-mh}};$$

and substituting this value of A in that of ϕ, we have

$$\phi = -ac\,\frac{\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}}{\epsilon^{mh} - \epsilon^{-mh}} \sin mx + 3ma^2c\,\frac{\epsilon^{2m(h-y)} + \epsilon^{-2m(h-y)}}{(\epsilon^{mh} - \epsilon^{-mh})^4} \sin 2mx \;\ldots\; (18).$$

The ordinate of the surface is given to a second approximation by (9). It will be found that

$$y = a\cos mx - ma^2\,\frac{(\epsilon^{mh} + \epsilon^{-mh})(\epsilon^{2mh} + \epsilon^{-2mh} + 4)}{2\,(\epsilon^{mh} - \epsilon^{-mh})^3} \cos 2mx \;\ldots\ldots\ldots\; (19),$$

$$k = \frac{ma^2}{\epsilon^{2mh} - \epsilon^{-2mh}}.$$

7. The equation to the surface is of the form

$$y = a\cos mx - Ka^2 \cos 2mx \;\ldots\ldots\ldots\; (20),$$

where K is necessarily positive, and a may be supposed to be positive, since the case in which it is negative may be reduced to that in which it is positive by altering the origin of x by the quantity $\frac{\pi}{m}$ or $\frac{\lambda}{2}$, λ being the length of the waves. On referring to (20) we see that the waves are symmetrical with respect to vertical planes drawn through their ridges, and also with respect to vertical planes drawn through their lowest lines. The greatest depression of the fluid occurs when $x = 0$ or $= \pm\lambda$, &c., and is equal to $a - a^2K$: the greatest elevation occurs when $x = \pm\frac{\lambda}{2}$ or $= \pm\frac{3\lambda}{2}$, &c. and is equal to $a + a^2K$. Thus the greatest elevation exceeds the greatest depression by $2a^2K$. When the surface cuts the plane of mean level, $\cos mx - aK\cos 2mx = 0$. Putting in the small term in this equation the approximate value $mx = \frac{\pi}{2}$, we have $\cos mx = -aK = \cos\left(\frac{\pi}{2} + aK\right)$; whence $x = \pm\left(\frac{\lambda}{4} + \frac{aK\lambda}{2\pi}\right)$, $= \pm\left(\frac{5\lambda}{4} + \frac{aK\lambda}{2\pi}\right)$, &c. We see then that the breadth of each hollow, measured at the height of the plane of mean level, is $\frac{\lambda}{2} + \frac{aK\lambda}{\pi}$, while the breadth of each elevated portion of the fluid is $\frac{\lambda}{2} - \frac{aK\lambda}{\pi}$.

It is easy to prove from the expression for K, which is given in (19), that for a given value of λ or of m, K increases as h decreases. Hence the difference in form of the elevated and depressed portions of the fluid is more conspicuous in the case in which the fluid is moderately shallow than in the case in which its depth is very great compared with the length of the waves.

8. When the depth of the fluid is very great compared with the length of a wave, we may without sensible error suppose h to be infinite. This supposition greatly simplifies the expressions already obtained. We have in this case

$$\phi = -ac\epsilon^{-my}\sin mx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(21)},$$

$$y = a\cos mx - \tfrac{1}{2}ma^2\cos 2mx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(22)},$$

$$k = 0,\quad K = \frac{m}{2} = \frac{\pi}{\lambda},\quad c^2 = \frac{g\lambda}{2\pi},$$

the y in (22) being the ordinate of the surface.

It is hardly necessary to remark that the state of the fluid at any time will be expressed by merely writing $x - ct$ in place of x in all the preceding expressions.

9. To find the nature of the motion of the individual particles, let $x + \xi$ be written for x, $y + \eta$ for y, and suppose x and y to be independent of t, so that they alter only in passing from one particle to another, while ξ and η are small quantities depending on the motion. Then taking the case in which the depth is infinite, we have

$$\frac{d\xi}{dt} = u = -mac\epsilon^{-m(y+\eta)}\cos m(x+\xi-ct) = -mac\epsilon^{-my}\cos m(x-ct) + m^2ac\epsilon^{-my}\sin m(x-ct)\,.\,\xi$$
$$+\, m^2ac\epsilon^{-my}\cos m(x-ct)\,.\,\eta, \text{ nearly,}$$

$$\frac{d\eta}{dt} = v = mac\epsilon^{-m(y+\eta)}\sin m(x+\xi-ct) = mac\epsilon^{-my}\sin m(x-ct) + m^2ac\epsilon^{-my}\cos m(x-ct)\,.\,\xi$$
$$-\, m^2ac\epsilon^{-my}\sin m(x-ct)\,.\,\eta, \text{ nearly.}$$

To a first approximation

$$\xi = a\epsilon^{-my}\sin m(x-ct),\quad \eta = a\epsilon^{-my}\cos m(x-ct),$$

the arbitrary constants being omitted. Substituting these values in the small terms of the preceding equations, and integrating again, we have

$$\xi = a\epsilon^{-my}\sin m(x-ct) + m^2a^2ct\,\epsilon^{-2my},$$
$$\eta = a\epsilon^{-my}\cos m(x-ct).$$

Hence the motion of the particles is the same as to a first approximation, with one important difference, which is that in addition to the motion of oscillation the particles are transferred forwards, that is, in the direction of propagation, with a constant velocity depending on the depth, and decreasing rapidly as the depth increases. If U be this velocity for a particle whose depth below the surface in equilibrium is y, we have

$$U = m^2a^2c\,\epsilon^{-2my} = a^2\left(\frac{2\pi}{\lambda}\right)^{\frac{3}{2}} g^{\frac{1}{2}}\,\epsilon^{-\frac{4\pi y}{\lambda}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(23)}.$$

The motion of the individual particles may be determined in a similar manner when the depth is finite from (18). In this case the values of ξ and η contain terms of the second order, involving respectively $\sin 2m(x-ct)$ and $\cos 2m(x-ct)$, besides the term in ξ which is multiplied by t. The most important thing to consider is the value of U, which is

$$U = m^2a^2c\,\frac{\epsilon^{2m(y-h)} + \epsilon^{-2m(y-h)}}{(\epsilon^{mh} - \epsilon^{-mh})^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(24)}.$$

Since U is a small quantity of the order a^2, and in proceeding to a second approximation the velocity of propagation is given to the order a only, it is immaterial which of the definitions of velocity of propagation mentioned in Art. 3, we please to adopt.

10. The waves produced by the action of the wind on the surface of the sea do not probably differ very widely from those which have just been considered, and which may be regarded as the typical form of oscillatory waves. On this supposition the particles, in addition to their motion of oscillation, will have a progressive motion in the direction of propagation of the waves, and consequently in the direction of the wind, supposing it not to have recently shifted, and this progressive motion will decrease rapidly as the depth of the particle considered increases. If the pressure of the air on the posterior parts of the waves is greater than on the anterior parts, in consequence of the wind, as unquestionably it must be, it is easy to see that some such progressive motion must be produced. If then the waves are not breaking, it is probable that equation (23), which is applicable to deep water, may give approximately the mean horizontal velocity of the particles; but it is difficult to say how far the result may be modified by friction. If then we regard a ship as a mere particle, in the first instance, for the sake of simplicity, and put U_0 for the value of U when $y = 0$, it is easy to see that after sailing for a time t, the ship must be a distance $U_0 t$ to the lee of her estimated place. It will not however be sufficient to regard the ship as a mere particle, on account of the variation of the factor ϵ^{-2my}, as y varies from 0 to the greatest depth of the ship below the surface of the water. Let δ be this depth, or rather a depth something less, in order to allow for the narrowing of the ship towards the keel, and suppose the effect of the progressive motion of the water on the motion of the ship to be the same as if the water were moving with a velocity the same at all depths, and equal to the mean value of the velocity U from $y = 0$ to $y = \delta$. If U_1 be this mean velocity,

$$U_1 = \frac{1}{\delta}\int_0^\delta U dy = \frac{ma^2c}{2\delta}\left(1 - \epsilon^{-2m\delta}\right).$$

On this supposition, if a ship be steered so as to sail in a direction making an angle θ with the direction of the wind, supposing the water to have no current, and if V be the velocity with which the ship moves through the water, her actual velocity will be the resultant of a velocity V in the direction just mentioned, which, for shortness, I shall call the direction of steering, and of a velocity U_1 in the direction of the wind. But the ship's velocity as estimated by the log-line is her velocity relatively to the water at the surface, and is therefore the resultant of a velocity V in the direction of steering, and a velocity $U_0 - U_1$ in a direction opposite to that in which the wind is blowing. If then E be the estimated velocity, and if we neglect U^2,

$$E = V - (U_0 - U_1)\cos\theta.$$

But the ship's velocity is really the resultant of a velocity $V + U_1 \cos\theta$ in the direction of steering, and a velocity $U_1 \sin\theta$ in the perpendicular direction, while her estimated velocity is E in the direction of steering. Hence, after a time t, the ship will be a distance $U_0 t \cos\theta$ ahead of her estimated place, and a distance $U_1 t \sin\theta$ aside of it, the latter distance being measured in a direction perpendicular to the direction of steering, and on the side towards which the wind is blowing.

I do not suppose that the preceding formula can be employed in practice; but I think it may not be altogether useless to call attention to the importance of having regard to the magnitude and direction of propagation of the waves, as well as to the wind, in making the allowance for lee-way.

11. The formulæ of Art. 6 are perfectly general as regards the ratio of the length of the waves to the depth of the fluid, the only restriction being that the height of the waves must be sufficiently small to allow the series to be rapidly convergent. Consequently, they must apply to the limiting case, in which the waves are supposed to be extremely long. Hence long waves, of the kind considered, are propagated without change of form, and the velocity of propagation is independent of the height of the waves to a second approximation. These conclusions might seem, at first sight,

at variance with the results obtained by Mr. Airy for the case of long waves*. On proceeding to a second approximation, Mr. Airy finds that the form of long waves alters as they proceed, and that the expression for the velocity of propagation contains a term depending on the height of the waves. But a little attention will remove this apparent discrepancy. If we suppose mh very small in (19), and expand, retaining only the most important terms, we shall find for the equation to the surface

$$y = a \cos mx - \frac{3a^2}{4m^2h^3} \cos 2mx.$$

Now, in order that the method of approximation adopted may be legitimate, it is necessary that the coefficient of $\cos 2mx$ in this equation be small compared with a. Hence $\frac{a}{m^2h^3}$, and therefore $\frac{\lambda^2 a}{h^3}$ must be small, and therefore $\frac{a}{h}$ must be small compared with $\left(\frac{h}{\lambda}\right)^2$. But the investigation of Mr. Airy is applicable to the case in which $\frac{\lambda}{h}$ is very large; so that in that investigation $\frac{a}{h}$ is large compared with $\left(\frac{h}{\lambda}\right)^2$. Thus the difference in the results obtained corresponds to a difference in the physical circumstances of the motion.

12. There is no difficulty in proceeding to the higher orders of approximation, except what arises from the length of the formulæ. In the particular case in which the depth is considered infinite, the formulæ are very much simpler than in the general case. I shall proceed to the third order in the case of an infinite depth, so as to find in that case the most important term, depending on the height of the waves, in the expression for the velocity of propagation.

For this purpose it will be necessary to retain the terms of the third order in the expansion of (7). Expanding this equation according to powers of y, and neglecting terms of the fourth, &c. orders, we have

$$g\phi_{,} - c^2\phi'' + (g\phi_{,,} - c^2\phi_{,}'')\,y + (g\phi_{,,,} - c^2\phi_{,,}'')\,\frac{y^2}{2} + 2c\,(\phi'\phi'' + \phi_{,}\phi_{,}')$$
$$+ 2c\,(\phi_{,}'\phi'' + \phi'\phi_{,}'' + \phi_{,,}\phi_{,}' + \phi_{,}\phi_{,,}')\,y - \phi'^2\phi'' - 2\phi'\phi_{,}\phi_{,}' - \phi_{,}^2\phi_{,,} = 0. \;\ldots\ldots\; (25).$$

In the small terms of this equation we must put for ϕ and y their values given by (21) and (22) respectively. Now since the value of ϕ to a second approximation is the same as its value to a first approximation, the equation $g\phi_{,} - c^2\phi'' = 0$ is satisfied to terms of the second order. But the coefficients of y and $\frac{y^2}{2}$, in the first line of (25), are derived from the left-hand member of the preceding equation by inserting the factor ϵ^{-my}, differentiating either once or twice with respect to y, and then putting $y = 0$. Consequently these coefficients contain no terms of the second order, and therefore the terms involving y in the first line of (25) are to be neglected. The next two terms are together equal to $c\,\frac{d}{dx}(\phi'^2 + \phi_{,}^2)$. But

$$\phi'^2 + \phi_{,}^2 = m^2a^2c^2,$$

which does not contain x, so that these two terms disappear. The coefficient of y in the second line of (25) may be derived from the two terms last considered in the manner already indicated, and therefore the terms containing y will disappear from (25). The only small terms

* *Encyclopædia Metropolitana*, *Tides and Waves*, Articles 198, &c.

remaining are the last three, and it will easily be found that their sum is equal to $m^4a^3c^3\sin mx$, so that (25) becomes

$$g\,\phi_{,} - c^2\,\phi'' + m^4a^3c^3\sin mx = 0 \ldots\ldots\ldots\ldots\ldots\ldots (26).$$

The value of ϕ will evidently be of the form $A\epsilon^{-my}\sin mx$. Substituting this value in (26), we have

$$(m^2c^2 - mg)\,A + m^4a^3c^3 = 0.$$

Dividing by mA, and putting for A and c^2 their approximate values $-ac$, $\frac{g}{m}$ respectively in the small term, we have

$$mc^2 = g + m^2a^2g,$$

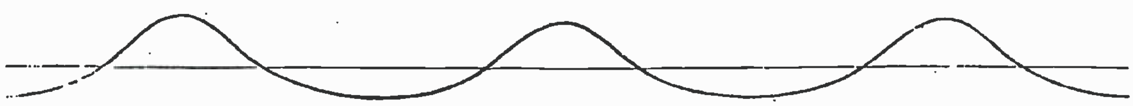
whence
$$c = \left(\frac{g}{m}\right)^{\frac{1}{2}}\left(1 + \tfrac{1}{2}m^2a^2\right) = \left(\frac{g\lambda}{2\pi}\right)^{\frac{1}{2}}\left(1 + \frac{2\pi^2a^2}{\lambda^2}\right).$$

The equation to the surface may be found without difficulty. It is

$$y = a\cos mx - \tfrac{1}{2}ma^2\cos 2mx + \tfrac{3}{8}m^2a^3\cos 3mx^{*}, \ldots\ldots\ldots\ldots\ldots (27):$$

we have also
$$k = 0,\quad \phi = -ac\left(1 - \tfrac{5}{8}m^2a^2\right)\epsilon^{-my}\sin mx.$$

The following figure represents a vertical section of the waves propagated along the surface of deep water. The figure is drawn for the case in which $a = \frac{7\lambda}{80}$. The term of the third order in (27) is retained, but it is almost insensible. The straight line represents a section of the plane of mean level.



13. If we consider the manner in which the terms introduced by each successive approximation enter into equations (7) and (8), we shall see that, whatever be the order of approximation, the series expressing the ordinate of the surface will contain only cosines of mx and its multiples, while the expression for ϕ will contain only sines. The manner in which y enters into the coefficient of cos rmx in the expression for ϕ is determined in the case of a finite depth by equations (2) and (3). Moreover, the principal part of the coefficient of cos rmx or sin rmx will be of the order a^r at least. We may therefore assume

$$\phi = \Sigma_1^{\infty}\, a^r A_r\left(\epsilon^{rm(h-y)} + \epsilon^{-rm(h-y)}\right)\sin rmx,$$

$$y = a\cos mx + \Sigma_2^{\infty}\, a^r B_r\cos rmx,$$

and determine the arbitrary coefficients by means of equations (7) and (8), having previously expanded these equations according to ascending powers of y. The value of c^2 will be determined by equating to zero the coefficient of sin mx in (7).

Since changing the sign of a comes to the same thing as altering the origin of x by $\frac{1}{2}\lambda$, it is plain that the expressions for A_r, B_r and c^2 will contain only even powers of a. Thus the values of each of these quantities will be of the form $C_0 + C_1a^2 + C_2a^4 + \ldots$

It appears also that, whatever be the order of approximation, the waves will be symmetrical with respect to vertical planes passing through their ridges, as also with respect to vertical planes passing through their lowest lines.

* It is remarkable that this equation coincides with that of the prolate cycloid, if the latter equation be expanded according to ascending powers of the distance of the tracing point from the centre of the rolling circle, and the terms of the fourth order be omitted. The prolate cycloid is the form assigned by Mr. Russell to waves of the kind here considered. *Reports of the British Association*, Vol. VI. p. 448. When the depth of the fluid is not great compared with the length of a wave, the form of the surface does not agree with the prolate cycloid even to a second approximation.

14. Let us consider now the case of waves propagated at the common surface of two liquids, of which one rests on the other. Suppose as before that the motion is in two dimensions, that the fluids extend indefinitely in all horizontal directions, or else that they are bounded by two vertical planes parallel to the direction of propagation of the waves, that the waves are propagated with a constant velocity, and without change of form, and that they are such as can be propagated into, or excited in the fluids supposed to have been previously at rest. Suppose first that the fluids are bounded by two horizontal rigid planes. Then taking the common surface of the fluids when at rest for the plane xz, and employing the same notation as before, we have for the under fluid

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0, \quad \text{.............................. (28)},$$

$$\frac{d\phi}{dy} = 0 \text{ when } y = h, \quad \text{...................... (29)},$$

$$p = C + g\rho y + c\rho\frac{d\phi}{dx},$$

neglecting the squares of small quantities. Let $h_{,}$ be the depth of the upper fluid when in equilibrium, and let $p_{,}$, $\rho_{,}$, $\phi_{,}$, $C_{,}$ be the quantities referring to the upper fluid which correspond to p, ρ, ϕ, C referring to the under: then we have for the upper fluid

$$\frac{d^2\phi_{,}}{dx^2} + \frac{d^2\phi_{,}}{dy^2} = 0 \quad \text{.. (30)};$$

$$\frac{d\phi_{,}}{dy} = 0 \text{ when } y = -h_{,} \quad \text{.. (31)},$$

$$p_{,} = C_{,} + g\rho_{,}y + c\rho_{,}\frac{d\phi_{,}}{dx}.$$

We have also, for the condition that the two fluids shall not penetrate into, nor separate from each other,

$$\frac{d\phi}{dy} = \frac{d\phi_{,}}{dy}, \text{ when } y = 0 \quad \text{.. (32)}.$$

Lastly, the condition answering to (11) is

$$g\left(\rho\frac{d\phi}{dy} - \rho_{,}\frac{d\phi_{,}}{dy}\right) - c^2\left(\rho\frac{d^2\phi}{dx^2} - \rho_{,}\frac{d^2\phi_{,}}{dx^2}\right) = 0 \quad \text{................ (33)},$$

when $$C - C_{,} + g(\rho - \rho_{,})\,y + c\left(\rho\frac{d\phi}{dx} - \rho_{,}\frac{d\phi_{,}}{dx}\right) = 0 \quad \text{........... (34)}.$$

Since $C - C'$ is evidently a small quantity of the first order at least, the condition is that (33) shall be satisfied when $y = 0$. Equation (34) will then give the ordinate of the common surface of the two liquids when y is put $= 0$ in the last two terms.

The general value of ϕ suitable to the present case, which is derived from (28) subject to the condition (29), is given by (13) if we suppose that the fluid is free from a uniform horizontal motion compounded with the oscillatory motion expressed by (13). Since the equations of the present investigation are linear, in consequence of the omission of the squares of small quantities, it will be sufficient to consider one of the terms in (13). Let then

$$\phi = A\left(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}\right)\sin mx \quad \text{.............................. (35)}.$$

The general value of $\phi_{,}$ will be derived from (13) by merely writing $-h_{,}$ for h. But in order that (32) may be satisfied, the value of $\phi_{,}$ must reduce itself to a single term of the same form as the second side of (35). We may take then for the value of $\phi_{,}$

$$\phi_{,} = A_{,}(\epsilon^{m(h_{,}+y)} + \epsilon^{-m(h_{,}+y)}) \sin mx \ldots\ldots (36).$$

Putting for shortness

$$\epsilon^{mh} + \epsilon^{-mh} = S, \quad \epsilon^{mh} - \epsilon^{-mh} = D,$$

and taking $S_{,}$, $D_{,}$ to denote the quantities derived from S, D by writing $h_{,}$ for h, we have from (32)

$$DA + D_{,}A_{,} = 0 \ldots\ldots (37),$$

and from (33)

$$\rho(gD - mc^2S)A + \rho_{,}(gD_{,} + mc^2S_{,})A_{,} = 0 \ldots\ldots (38)$$

Eliminating A and $A_{,}$ from (37) and (38), we have

$$c^2 = \frac{g}{m}\frac{(\rho - \rho_{,})DD_{,}}{\rho SD_{,} + \rho_{,}S_{,}D} \ldots\ldots (39).$$

The equation to the common surface of the liquids will be obtained from (34). Since the mean value of y is zero, we have in the first place

$$C_{,} = C \ldots\ldots (40).$$

We have then, for the value of y,

$$y = a \cos mx \ldots\ldots (41),$$

where

$$a = \frac{mc}{g}\frac{\rho_{,}A_{,}S_{,} - \rho AS}{\rho - \rho_{,}} = \frac{DD_{,}}{c}\frac{\rho_{,}A_{,}S_{,} - \rho AS}{\rho SD_{,} + \rho_{,}S_{,}D} \ldots\ldots (42).$$

Substituting in (35) and (36) the values of A and $A_{,}$ derived from (37) and (42), we have

$$\phi = -\frac{ac}{D}(\epsilon^{m(h-y)} + \epsilon^{-m(h-y)}) \sin mx \ldots\ldots (43),$$

$$\phi_{,} = \frac{ac}{D_{,}}(\epsilon^{m(h_{,}+y)} + \epsilon^{-m(h_{,}+y)}) \sin mx \ldots\ldots (44).$$

Equations (39), (40), (41), (43) and (44) contain the solution of the problem. It is evident that C remains arbitrary. The values of p and $p_{,}$ may be easily found if required.

If we differentiate the logarithm of c^2 with respect to m, and multiply the result by the product of the denominators, which are necessarily positive, we shall find a quantity of the form $P\rho + P_{,}\rho_{,}$, where P and $P_{,}$ do not contain ρ or $\rho_{,}$. It may be proved in nearly the same manner as in Art. 4, that each of the quantities P, $P_{,}$ is necessarily negative. Consequently c will decrease as m increases, or will increase with λ. It follows from this that the value of ϕ cannot contain more than two terms, one of the form (35), and the other derived from (35) by replacing $\sin mx$ by $\cos mx$, and changing the constant A: but the latter term may be got rid of by altering the origin of x.

The simplest case to consider is that in which both h and h' are regarded as infinite compared with λ. In this case we have

$$\phi = -ac\epsilon^{-my} \sin mx, \quad \phi_{,} = ac\epsilon^{my} \sin mx, \quad c^2 = \frac{\rho - \rho_{,}}{\rho + \rho_{,}}\frac{g}{m}, \quad y = a \cos mx,$$

the latter being the equation to the surface.

15. The preceding investigation applies to two incompressible fluids, but the results are applicable to the case of the waves propagated along the surface of a liquid exposed to the air, provided that in considering the effect of the air we neglect terms which, in comparison with those retained, are of the order of the ratio of the length of the waves considered to the length of a wave of sound of the same period in air. Taking then ρ for the density of the liquid, $\rho_{,}$ for that of the air at the time, and supposing $h_{,} = \infty$, we have

$$c^2 = \frac{g}{m}\frac{(\rho - \rho_{,})D}{\rho S + \rho_{,} D} = \frac{gD}{mS}\left\{1 - \left(1 + \frac{D}{S}\right)\frac{\rho_{,}}{\rho}\right\}, \text{ nearly.}$$

If we had considered the buoyancy only of the air, we should have had to replace g in the formula (14) by $\frac{\rho - \rho_{,}}{\rho_{,}} g$. We should have obtained in this manner

$$c^2 = \frac{g}{m}\frac{(\rho - \rho_{,})D}{\rho S} = \frac{gD}{mS}\left(1 - \frac{\rho_{,}}{\rho}\right).$$

Hence, in order to allow for the inertia of the air, the correction for buoyancy must be increased in the ratio of 1 to $1 + \frac{D}{S}$. The whole correction therefore increases as the ratio of the length of a wave to the depth of the fluid decreases. For very long waves the correction is that due to buoyancy alone, while in the case of very short waves the correction for buoyancy is doubled. Even in this case the velocity of propagation is altered by only the fractional part $\frac{\rho_{,}}{\rho}$ of the whole; and as this quantity is much less than the unavoidable errors of observation, the effect of the air in altering the velocity of propagation may be neglected.

16. There is a discontinuity in the density of the fluid mass considered in Art. 14, in passing from one fluid into the other; and it is easy to show that there is a corresponding discontinuity in the velocity. If we consider two fluid particles in contact with each other, and situated on opposite sides of the surface of junction of the two fluids, we see that the velocities of these particles resolved in a direction normal to that surface are the same; but their velocities resolved in a direction tangential to the surface are different. These velocities are, to the order of approximation employed in the investigation, the values of $\frac{d\phi}{dx}$ and $\frac{d\phi_{,}}{dx}$ when $y = 0$. We have then from (43) and (44), for the velocity with which the upper fluid slides along the under,

$$mac\left(\frac{S_{,}}{D_{,}} + \frac{S}{D}\right)\cos mx.$$

17. When the upper surface of the upper fluid is free, the equations by which the problem is to be solved are the same as those of Art. 14, except that the condition (31) is replaced by

$$g\frac{d\phi_{,}}{dy} - c^2\frac{d^2\phi_{,}}{dx^2} = 0, \text{ when } y = -h_{,}; \quad \ldots\ldots (45);$$

and to determine the ordinate of the upper surface, we have

$$C_{,} + g\rho_{,} y + c\rho_{,}\frac{d\phi_{,}}{dx} = 0,$$

where y is to be replaced by $-h_{,}$ in the last term. Let us consider the motion corresponding to the value of ϕ given by (35). We must evidently have

$$\phi_{,} = (A_{,}\epsilon^{my} + B_{,}\epsilon^{-my})\sin mx,$$

where $A_{,}$ and $B_{,}$ have to be determined. The conditions (32), (33) and (45) give

$$DA + A_{,} - B_{,} = 0,$$

$$\rho (gD - mc^2 S) A + \rho_{,} (g + mc^2) A_{,} - \rho_{,} (g - mc^2) B_{,} = 0,$$

$$(g + mc^2) \epsilon^{-mh_{,}} A_{,} - (g - mc^2) \epsilon^{mh_{,}} B_{,} = 0.$$

Eliminating A, $A_{,}$ and $B_{,}$ from these equations, and putting

$$c^2 = \frac{g\zeta}{m},$$

we find

$$(\rho SS_{,} + \rho_{,} DD_{,}) \zeta^2 - \rho (SD_{,} + S_{,}D) \zeta + (\rho - \rho_{,}) DD_{,} = 0. \ldots (46).$$

The equilibrium of the fluid being supposed to be stable, we must have $\rho_{,} < \rho$. This being the case, it is easy to prove that the two roots of (46) are real and positive. These two roots correspond to two systems of waves of the same length, which are propagated with the same velocity.

In the limiting case in which $\frac{\rho}{\rho_{,}} = \infty$, (46) becomes

$$SS_{,} \zeta^2 - (SD_{,} + S_{,}D) \zeta + DD_{,} = 0,$$

the roots of which are $\frac{D}{S}$ and $\frac{D_{,}}{S_{,}}$, as they evidently ought to be, since in this case the motion of the under fluid will not be affected by that of the upper, and the upper fluid can be in motion by itself.

When $\rho_{,} = \rho$ one root of (46) vanishes, and the other becomes $\frac{SD_{,} + S_{,}D}{SS_{,} + DD_{,}}$ or $\frac{\epsilon^{m(h+h_{,})} - \epsilon^{-m(h+h_{,})}}{\epsilon^{m(h+h_{,})} + \epsilon^{-m(h+h_{,})}}$. The former of these roots corresponds to the waves propagated at the common surface of the fluids, while the latter gives the velocity of propagation belonging to a single fluid having a depth equal to the sum of the depths of the two considered.

When the depth of the upper fluid is considered infinite, we must put $\frac{D_{,}}{S_{,}} = 1$ in (46). The two roots of the equation so transformed are 1 and $\frac{(\rho - \rho_{,}) D}{\rho S + \rho_{,} D}$, the former corresponding to waves propagated at the upper surface of the upper fluid, and the latter agreeing with Art. 15.

When the depth of the under fluid is considered infinite, and that of the upper finite, we must put $\frac{D}{S} = 1$ in (46). The two roots will then become 1 and $\frac{(\rho - \rho_{,}) D_{,}}{\rho S_{,} + \rho_{,} D_{,}}$. The value of the former root shows that whatever be the depth of the upper fluid, one of the two systems of waves will always be propagated with the same velocity as waves of the same length at the surface of a single fluid of infinite depth. This result is true even when the motion is in three dimensions, and the form of the waves changes with the time, the waves being still supposed to be such as could be excited in the fluids, supposed to have been previously at rest, by means of forces applied at the upper surface. For the most general small motion of the fluids in this case may be regarded as the resultant of an infinite number of systems of waves of the kind considered in this paper. It is remarkable that when the depth of the upper fluid is very great, the root $\zeta = 1$ is that which corresponds to the waves for which the upper fluid is disturbed, while the under is sensibly at rest; whereas, when the depth of the upper fluid is very small, it is the other root which corresponds to those waves which are analogous to the waves which would be propagated in the upper fluid if it rested on a rigid plane.

When the depth of the upper fluid is very small compared with the length of a wave, one of the roots of (46) will be very small; and if we neglect square and products of $mh_{\prime}$ and ζ, the equation becomes $2\rho D\zeta - 2(\rho - \rho_{\prime}) m h_{\prime} D = 0$, whence

$$\zeta = \frac{\rho - \rho_{\prime}}{\rho} m h_{\prime}, \quad c^2 = \frac{\rho - \rho_{\prime}}{\rho} g h_{\prime} \qquad (47).$$

These formulæ will not hold good if mh be very small as well as $mh_{\prime}$, and comparable with it, since in that case all the terms of (46) will be small quantities of the second order, $mh_{\prime}$ being regarded as a small quantity of the first order. In this case, if we neglect small quantities of the third order in (46), it becomes

$$4\rho\zeta^2 - 4m\rho(h + h_{\prime})\zeta + 4(\rho - \rho_{\prime}) m^2 h h_{\prime} = 0,$$

whence
$$c^2 = \frac{g}{2}\left\{h + h_{\prime} \pm \sqrt{(h - h_{\prime})^2 + \frac{4\rho_{\prime}}{\rho} h h_{\prime}}\right\} \qquad (48).$$

Of these values of c^2, that in which the radical has the negative sign belongs to that system of waves to which the formulæ (47) apply when $h_{\prime}$ is very small compared with h.

If the two fluids are water and mercury, $\frac{\rho}{\rho_{\prime}}$ is equal to about 13.57. If the depth of the water be very small compared both with the length of the waves and with the depth of the mercury, it appears from (47) that the velocity of propagation will be less than it would have been, if the water had rested on a rigid plane, in the ratio of .9624 to 1, or 26 to 27 nearly.

G. G. STOKES.