ON THE THEORY OF QUEUES WITH MANY SERVERS(1)

BY

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1. Introduction. The physical original of the mathematical problem to which this paper is devoted is a system of s "servers," who can be machines in a factory, ticket windows at a railroad station, salespeople in a store, or the like. Individuals (clients) who are to be served by these servers arrive at random and the duration of anyone's service (e.g., stay at the ticket window) is a chance variable whose distribution function may be arbitrary. The phrase "at random" used above is not to be interpreted to mean that the interval between successive arrivals is to have an exponential distribution. The assumption of an exponential or other special distribution for either the interval between arrivals or the service time of an individual or both usually makes the problem much easier. We also allow the distribution of the interval between arrivals to be arbitrary. The queue discipline is "first come, first served." The system is described precisely in §2.

In this system the waiting time of the individual who is ith in order of arrival, i.e., the time which elapses between his arrival and the beginning of his service, is a chance variable whose distribution function depends upon i. In §3 we prove that this distribution function approaches a limit as $i \to \infty$. This limit may not be a distribution function because its variation may be less than one. We assume that the expected value of the time interval between the arrivals of successive clients and the expected value of the service time of an individual both exist. In terms of these one defines a quantity ρ in §6. The situation may then be classified according as $\rho < 1$ or $\rho \geqq 1$. In the former and interesting case the limiting function is a distribution function (§6), in the latter case it is not a distribution function (§7). The limiting function is (a marginal function) obtained from a function which satisfies an integral equation derived in §3. This integral equation is satisfied by a unique distribution function on s-space when $\rho < 1$, and by no distribution function when $\rho \geqq 1$ (§8). These results for the case of one server were obtained by Lindley [1]. The problem when there are many servers offers many difficulties not present when there is only one server. The methods of the present paper are different from those of [1]. The proof of the result of §7, that the limit is not a distribution function when $\rho \geqq 1$, is obtained by reducing the problem to the case $s = 1$ by using our lemma of §4, and then employing the corresponding result of [1]; except for this argument our paper is self-contained. For special distributions of the time between successive arrivals

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and of the service time the results of the present paper have been obtained by various authors (we refer the reader to [5] and [6] which contain extensive bibliographies). The methods of these authors make use of their special assumptions in an essential way. The novelty of the results of the present paper lies in the fact that no restrictions are imposed on the distributions, with the exception of the assumption of finite first moment(2). Thus the results of the present paper include the corresponding ones of previous papers as special cases(3).

Mathematically speaking, our study is one of the ergodic character of the waiting time in our system, and the conditions under which the distribution of the latter approaches stability. Our problem can be reduced, and actually is so reduced by us, to studying a random walk in s-space with certain impassable but not absorbing barriers. We actually show that, when $\rho<1$, the distribution function of the particle engaged in the random walk approaches a limiting distribution which is the same no matter what the original starting point of the particle (§8).

Perhaps our principal device is to dominate the stochastic process to be studied by a lattice process to which we then apply available theorems from the theory of Markoff processes with discrete time parameter and denumerably many states. This device makes possible the argument of §6 and is also employed in §8. We are of the opinion that this device could be applied to other ergodic problems connected with random walks.

When the original process is a lattice process, i.e., when the chance variables R_1 and g_1 (defined in §2) take, with probability one, only values which are integral multiples of some positive number c, and when $\rho<1$, the limiting probabilities (which are shown to exist in §6) are reciprocals of certain mean recurrence times (this follows from the application of Theorem 2 of Chapter 15 of [3] to the argument of §6). Monte Carlo methods (see, e.g., [4]) may perhaps then be profitably employed to solve the integral equation (3.8).

It would be very desirable and interesting to solve the integral equation (3.8), at least for interesting or important functions G and H (see §2). This, however, is likely to be very difficult. Even in the simplest case, when $s=1$, the equation becomes the Wiener-Hopf equation, which has been of considerable interest to physicists but has been solved only in special cases. Some special cases of the equation (3.8) are discussed in [5], [6], and [1]. It may also interest the pure analyst that one can, by probabilistic methods

(2) Under stronger assumptions (e.g., existence of all moments), F. Pollaczek in recent notes (C. R. Acad. Sci. Paris vol. 236 (1953) pp. 578–580, 1469–1470) gives formally an integral equation for the Laplace transform of F (to be defined below), but does not consider the questions of the present paper.

(3) In a paper to be published elsewhere which makes extensive use of the present paper, the authors obtain, under minimal conditions, theorems on convergence of the mean of various variables connected with the queueing process.

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like ours, prove the existence or non-existence of distribution function solutions of (3.8).

Finally, in §9 we discuss the limiting distribution (as $i\to\infty$) of the queue size, i.e., of the number waiting to be served when the service of the ith customer begins.

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2. **Description of the system.** The system consists of s ($\geqq 1$) machines, $M_1, \cdots, M_s$. The ith individual arrives at time t_i ($\geqq 0$), with, of course, $t_i \leqq t_{i+1}$. If all machines are in service at his arrival he takes his place in the queue. His service begins as soon as at least one machine is unoccupied, and all individuals with smaller indices have been or are being served. If more than one machine becomes unoccupied at the time when it is the ith individual's turn to be served, we shall assume, for definiteness, that he takes his place at the unoccupied machine with smallest index.

Let $t_0=0$, $g_i=t_i-t_{i-1}$ for all $i\geqq 1$. We assume that the g_i are independently and identically distributed chance variables; let $G(z)=P\{g_1\leqq z\}$, where $P\{\ \}$ is the probability of the relation in braces. Throughout the paper we assume that $G(0)<1$; the case $G(0)=1$ is too trivial to discuss. We assume $Eg_1<\infty$.

Let R_i be the length of time the ith person spends being serviced by a machine. We assume that the R_i are independently and identically distributed chance variables, distributed independently of the g_i; let $H(z) =P\{R_1\leqq z\}$. We assume $ER_1<\infty$. We also assume $H(0)<1$, the case $H(0)=1$ being trivial.

Let $w_{i1}+t_i$ be the time at which service of the ith individual begins; w_{i1} is his waiting time. Then the ith individual leaves his machine at the time $w_{i1}+t_i+R_i$.

Let $u_{ij}+t_i$ be the time at which the jth machine finishes serving the last of those among the first $(i-1)$ individuals which it serves. Let $u'_{ij} =\max(0, u_{ij})$. Let $w_{i1}, \cdots, w_{is}$ be the quantities $u'_{i1}, \cdots, u'_{is}$ arranged in order of increasing size. It is easy to see that this definition of w_{i1} coincides with the former.

Let

$$F_i(x_1, \cdots, x_s) = P\{w_{i1} \leqq x_1, \cdots, w_{is} \leqq x_s\}. \tag{2.1}$$

If ever $x_j>x_{j+1}$ we may, since $w_{ij}\leqq w_{i(j+1)}$, replace x_j by x_{j+1} in both members of (2.1) without changing the value of either.

Write $w_i=(w_{i1}, \cdots, w_{is})$. The earliest times at which the various machines could attend to the $(i+1)$st individual are $t_i+w_{i1}+R_i$, $t_i+w_{i2}, \cdots,$ t_i+w_{is}. If t_{i+1} is greater than or equal to any of these quantities the $(i+1)$st individual finds at least one machine unoccupied at his arrival and does not have to wait at all. If t_{i+1} is less than all these quantities the $(i+1)$st individual has to wait for the first machine to be unoccupied. Since $t_{i+1}=t_i +g_{i+1}$, w_{i+1} is obtained from w_i as follows: Subtract g_{i+1} from *every* component

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of $(w_{i1}+R_i, w_{i2}, w_{i3}, \cdots, w_{is})$. Rearrange the resulting quantities in ascending order and replace all negative quantities by zero. The ensuing result is w_{i+1}.

3. **Recursion formula for F_i. Existence of the limit of F_i as $i\to\infty$.** Let $\phi_j(a, b, c)$, $j=1, \cdots, s$, be the value of $w_{(i+1),j}$ when $w_i=a$, $R_i=b$, $g_{i+1}=c$. If d is a point in s-space we shall say that $a\leqq d$ if every coordinate of a is not greater than the corresponding coordinate of d. If now $a\leqq d$ then obviously

$$\phi_j(a, b, c) \leqq \phi_j(d, b, c)$$

for $1\leqq j\leqq s$. Applying this argument k times we obtain the following result: Let $R_{i+j-1}=b_{i+j-1}$, $g_{i+j}=c_{i+j}$, $j=1, \cdots, k$. Let $w_{i+k}=e_1$ when $w_i=a_1$, and let $w_{i+k}=e_2$ when $w_i=a_2$. Then $a_1\leqq a_2$ implies $e_1\leqq e_2$.

Let S be the totality of points $(x_1, x_2, \cdots, x_s)$ of Euclidean s-space such that $0\leqq x_1\leqq x_2\leqq \cdots \leqq x_s$. Let x and y be generic points of S. For $i\geqq 1$, let

$$F_i(x \mid y) = P\{w_i \leqq x \mid w_1 = y\}.$$

Let 0 be the origin in s-space. Then

$$F_i(x \mid 0) = F_i(x)$$

and

$$F_{i+1}(x) = \int F_i(x \mid y) dF_2(y). \tag{3.1}$$

The conclusion of the preceding paragraph enables us to conclude that $y_1\in S$, $y_2\in S$, $y_1\leqq y_2$, imply

$$F_i(x \mid y_1) \geqq F_i(x \mid y_2)$$

for every x and every i. Now

$$F_{i+1}(x) - F_i(x) = \int [F_i(x \mid y) - F_i(x \mid 0)] dF_2(y). \tag{3.2}$$

Since the integrand is never positive we have that

$$F_{i+1}(x) \leqq F_i(x) \tag{3.3}$$

for all x and i. From (3.3) it follows that $F_i(x)$ approaches a limit, say $F(x)$, which is nondecreasing in every component of x, continuous to the right, and assigns non-negative measure to all rectangles. It need not, however, be a distribution function, i.e., its variation over S (hence over all of s-space) may be less than one.

Write

$$\phi(a, b, c) = (\phi_1(a, b, c), \cdots, \phi_s(a, b, c)). \tag{3.4}$$

For given $x\in S$, b, c, let $\psi(x, b, c)$ be the totality of points $y\in S$ such that

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$\phi(y, b, c) \leqq x$ and $\in S$. Obviously

$$F_{(i+1)}(x) = \int P_i\{\psi(x, b, c)\} dH(b) dG(c) \tag{3.5}$$

where P_i is the measure according to F_i. This equation determines each F_i uniquely by recursion, since of course $F_1(0)=1$. For $s=1$ and $x \geqq 0$, $\psi(x, b, c)$ is $\{y \mid 0 \leqq y \leqq x-b+c\}$. Hence (3.5) becomes, for $x \geqq 0$,

$$F_{(i+1)}(x) = \int F_i(x - b + c) dH(b) dG(c), \tag{3.6}$$

an equation due to Lindley [1]. (In (3.6) it is understood that $F_i(x-b+c)=0$ whenever $x-b+c<0$.) For $s=2$ and $x=(x_1, x_2) \in S$ we have that $\psi(x, b, c)$ is the set of points $y \in S$ such that $y \leqq (x_1-b+c, x_2+c)$, together with the set of points $y \in S$ such that $y \leqq ([\min (x_2-b+c, x_1+c)], x_1+c)$. We extend the definition of $F_i(y_1, y_2)$ to all of s-space in the natural way as follows: $F_i(y_1, y_2)=0$ if either y_1 or $y_2<0$, $F_i(y_1, y_2)=F_i(y_2, y_2)$ if $y_1>y_2$. Then for all (x_1, x_2) in S and $s=2$, (3.5) becomes

$$\begin{aligned} F_{(i+1)}(x_1, x_2) = \int & [F_i(x_1 - b + c, x_2 + c) + F_i(x_2 - b + c, x_1 + c) \\ & - F_i(x_1 - b + c, x_1 + c)] dH(b) dG(c). \end{aligned} \tag{3.7}$$

In general, when (3.5) is written in the form of (3.6) and (3.7) the integrand contains (2^s-1) terms. With the integrand in this form let $i \to \infty$ in (3.5). By Lebesgue's bounded convergence theorem we obtain for $x \in S$,

$$F(x) = \int P_\infty\{\psi(x, b, c)\} dH(b) dG(c) \tag{3.8}$$

where P_∞ is the measure according to $F(x)$. (When $s=1$ or 2 equation (3.8) becomes (3.6) and (3.7) with the subscripts of F deleted.) This is an integral equation satisfied by $F(x)$. We shall later prove that, when $\rho<1$, $F(x)$ is a distribution function (d.f.), and the only d.f. over S which satisfies (3.8). Moreover, we shall prove that, when $\rho \geqq 1$, $F(x)$ is not a d.f., and (3.8) has no solution which is a d.f. over S.

We remark that (3.8) implies that if $F(x)$ is a d.f., the latter defines a stationary absolute probability distribution for our (Markoff) stochastic process, i.e., if w_1 is distributed according to $F(x)$ then w_i has this distribution for every value of i.

Write $\bar{x}_1=(x_1, \infty, \cdots, \infty)$, $F_i^*(x_1)=F_i(\bar{x}_1)$. Then, from (3.5), the Lebesgue bounded convergence theorem, and the structure of ψ, it follows that

$$F^*_{(i+1)}(x_1) = \int P_i\{\psi(\bar{x}_1, b, c)\} dH(b) dG(c). \tag{3.9}$$

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We proved earlier that

$$F_i(x) \geqq F_{(i+1)}(x) \tag{3.10}$$

for every i and x. In (3.10) let the last $(s-1)$ coordinates of x approach infinity. We obtain

$$F_i^*(x_1) \geqq F_{(i+1)}^*(x_1). \tag{3.11}$$

We conclude that $\lim_{i\to\infty} F_i^*(x_1)$ exists; call it $F^*(x_1)$, say. Clearly we have

$$F^*(x_1) \geqq F(\bar{x}_1). \tag{3.12}$$

We shall prove in §5 that equality holds in (3.12). It will then follow from (3.8) that

$$F^*(x_1) = \int P_\infty\{\psi(\bar{x}_1, b, c)\} dH(b) dG(c). \tag{3.13}$$

4. **An essential lemma.** In this section we shall prove the following

LEMMA.

$$\lim_{y'\to\infty} \liminf_{i\to\infty} P\{w_{i,s} - w_{i,1} \leqq y'\} = 1$$

for $s>1$.

Proof. Let

$$B_i = (s-1)w_{i,s} - \sum_{j=1}^{s-1} w_{i,j} \tag{4.1}$$

for $i \geqq 1$. It follows easily from the way in which $w_{(i+1)}$ is obtained from w_i that

$$B_{(i+1)} \leqq \begin{cases} B_i - R_i & \text{when } R_i \leqq w_{is} - w_{i1}, \\ B_i - s(w_{is} - w_{i1}) + (s-1)R_i \leqq (s-1)R_i & \\ & \text{when } R_i \geqq w_{is} - w_{i1}. \end{cases} \tag{4.2}$$

In either case we have

$$B_{(i+1)} \leqq \max\,(B_i - R_i,\ (s-1)R_i). \tag{4.3}$$

Applying (4.3) to B_i we obtain

$$B_{(i+1)} \leqq \max\,(B_{i-1} - R_{i-1} - R_i,\ (s-1)R_{i-1} - R_i,\ (s-1)R_i). \tag{4.4}$$

Continuing in this manner and noting that $B_1 = 0$ we obtain

$$\begin{aligned} B_{i+1} \leqq \max\,[&(s-1)R_i,\ (s-1)R_{i-1} - R_i,\ (s-1)R_{i-2} - R_{i-1} - R_i, \\ &\cdots,\ (s-1)R_1 - R_2 - \cdots - R_i] = Y_i \text{ (say)}. \end{aligned} \tag{4.5}$$

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Since the R_j are independently and identically distributed, we may interchange indices j and $i-j+1$, $j=1, \cdots, i$, in the middle member of (4.5) without altering its distribution. Hence, setting $h=(s-1)^{-1}$, we have

$$P\{Y_n \leqq y'\} = P\{R_1 \leqq hy', R_2 \leqq h(R_1 + y'), \cdots, \\ R_n \leqq h(R_1 + \cdots + R_{n-1} + y')\} \tag{4.6}$$

(where, for $n=1$, we replace $R_1 + \cdots + R_{n-1}$ by 0). Since $B_i \geqq w_{is} - w_{i1}$, our proof will be complete if we show that

$$\lim_{y'\to\infty} \liminf_{n\to\infty} P\{Y_n \leqq y'\} = 1. \tag{4.7}$$

From the strong law of large numbers we have

$$P\left\{\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} R_i = ER_1,\ \lim_{n\to\infty} \frac{R_n}{n} = 0\right\} = 1. \tag{4.8}$$

Now, for $y' \geqq 0$,

$$\begin{aligned} &P\{Y_n \leqq y'\} \\ &\geqq P\{R_n \leqq h(R_1 + \cdots + R_{n-1} + y') \text{ for } n = 1, 2, \cdots, \text{ad inf.}\} \\ &= P\left\{\frac{R_n}{hn} \leqq \frac{R_1 + \cdots + R_{n-1}}{n} + \frac{y'}{n} \text{ for } n = 1, 2, \cdots, \text{ad inf.}\right\}. \end{aligned} \tag{4.9}$$

Because of (4.8), for any $\epsilon > 0$ there exists an integer N such that

$$P\left\{\frac{R_n}{hn} \leqq \frac{R_1 + \cdots + R_{n-1}}{n} \text{ for } n > N\right\} > 1 - \epsilon.$$

Clearly, there is a value y_0' such that for $y' > y_0'$

$$P\left\{\frac{R_n}{hn} \leqq \frac{R_1 + \cdots + R_{n-1}}{n} + \frac{y'}{n} \text{ for } n \leqq N\right\} > 1 - \epsilon.$$

Equation (4.7) is an immediate consequence.

5. Certain immediate consequences of the lemma.

(A) $F^*(x_1) = F(\bar{x}_1)$.

Proof. Let $x(x_1, y')$ be the point $x_1, y', \cdots, y'$. From the lemma it follows at once that for any $\epsilon > 0$ and i and y' sufficiently large we have

$$\left| P\{w_{i1} \leqq x_1\} - P\{w_{i1} \leqq x_1, w_{i2} \leqq y', \cdots, w_{is} \leqq y'\} \right| < \epsilon. \tag{5.1}$$

Let $i \to \infty$. We obtain

$$\left| F^*(x_1) - F(x(x_1, y')) \right| \leqq \epsilon. \tag{5.2}$$

Let $y' \to \infty$. We obtain

$$\left| F^*(x_1) - F(\bar{x}_1) \right| \leqq \epsilon. \tag{5.3}$$

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Since ϵ was arbitrary the desired result follows.

(B) Either F and F^* are both distribution functions or neither is a distribution function.

This follows from the fact that (A) above implies that $\lim F(x)$ as all coordinates of x approach infinity is the same as $\lim F^*(x_1)$ as x_1 approaches infinity.

6. **Proof that F is a distribution function when $\rho<1$.** We define

$$\rho = (ER_1)(sEg_1)^{-1}. \tag{6.1}$$

We shall now prove that, if $\rho<1$, $F(x)\to 1$ as all coordinates of x approach infinity. Then, by (3.12), we have $\lim F^*(x_1)=1$ as $x_1\to\infty$.

I. We show that it is sufficient to prove this result in a "dominating" case where, for some $c>0$,

$$\sum_{i=0}^{\infty} P\{R_1 = ci\} = \sum_{i=0}^{\infty} P\{g_1 = ci\} = 1. \tag{6.2}$$

Let $[a]$ be the largest integer $\leqq a$ and for some one $c>0$ define, for all i,

$$g_i' = c\left[\frac{g_i}{c}\right], \qquad R_i' = c\left[\frac{R_i}{c}\right] + c.$$

Then $g_i' \leqq g_i$ and $R_i' \geqq R_i$. Let w_i' be the same function of $\{g_j'\}$ and $\{R_j'\}$ that w_i is of $\{g_j\}$ and $\{R_j\}$. It follows from an argument like that of §3 that $w_i' \geqq w_i$ for all i. Hence if we can show that

$$\lim_{y'\to\infty} \liminf_{i\to\infty} P\{w_{i1}' \leqq y', \cdots, w_{is}' \leqq y'\} = 1 \tag{6.3}$$

it will follow that

$$\lim_{y'\to\infty} F(y', y', \cdots, y') = 1 \tag{6.4}$$

which is the desired result.

We have

$$Eg_j' \geqq Eg_j - c, \qquad ER_j' \leqq ER_j + c.$$

Hence, if c is sufficiently small, $(ER_1')(sEg_1')^{-1}<1$.

In the remainder of this section we assume that (6.2) is satisfied with $\rho<1$, and we shall prove (6.3) for this process.

II. We show that (6.3) is valid if $P\{R_1=0\}>0$. We recall that $P\{g_1=0\}<1$. Hence, for any i and integral $a_1, \cdots, a_s$, with $0\leqq a_1\leqq a_2\leqq \cdots \leqq a_s$,

$$P\{w_{(i+j)} = 0 \text{ for some } j \mid w_{i,1} = a_1c,\ w_{i,2} = a_2c, \cdots, w_{i,s} = a_sc\}$$

is positive. Let Z be the totality of all points $(a_1c, a_2c, \cdots, a_sc)$ with integral

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a's such that $0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_s$. Let z be a generic point of Z. The points z are the states of our Markoff process $\{w_i\}$. The preceding argument shows that the origin 0 and all the points z which can be reached by the process $\{w_i\}$ with positive probability form a chain C which is irreducible. C is aperiodic, since

$$(6.5) \qquad P\{w_{i+1} = 0 \mid w_i = 0\} \geqq P\{R_1 = 0\} > 0.$$

The desired result then follows by III below.

III. We shall show that, if C is aperiodic and irreducible, (6.3) holds. From §3 or [3, Chap. 15, Theorems 1 and 2], it follows that

$$\lim_{i\to\infty} P\{w_i = z\}$$

exists for all z in C; call it $f(z)$. From the theorems of [3] cited above it follows that

$$(6.6) \qquad \sum_{z\in C} f(z) = 0 \quad \text{or} \quad 1.$$

Our result is proved if we show that the sum in (6.6) is 1. Suppose it were 0; every $f(z)$ is then zero. We would then have

$$(6.7) \qquad F(y', y', \cdots, y') = 0$$

for every y'. Hence from (A), §5, we obtain, using (6.7),

$$(6.8) \qquad F^*(x_1) = 0$$

for every x_1. From the definition of w_i and the fact that $\rho < 1$ we obtain that there exists an $M > 0$ such that, whenever

$$(6.9) \qquad M \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_s,$$

we have

$$(6.10) \qquad E\left\{\sum_{j=1}^{s} w_{(i+1),j} \,\middle|\, w_{ij} = a_j,\ j = 1, \cdots, s\right\} < \sum_{j=1}^{s} a_j - \delta$$

for some $\delta > 0$. It is to be noted that, whether (6.9) holds or not, the left member of (6.10) is never greater than

$$(6.11) \qquad \sum_{j=1}^{s} a_j + ER_1.$$

Since $F^*(x_1) \equiv 0$ we can find an $N > 0$ such that for $i \geqq N$ we have

$$(6.12) \qquad P\{w_{i,1} < M\} < \frac{\delta}{\delta + ER_1}\cdot$$

Then, for $i \geqq N$, we have

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$$
\begin{aligned}
&E\left\{\sum_{j=1}^{s} w_{i+1,j} - \sum_{j=1}^{s} w_{i,j}\right\} \\
(6.13)\qquad &\quad = E\left\{E\left\{\sum_{j=1}^{s} w_{i+1,j} - \sum_{j=1}^{s} w_{i,j} \,\Big|\, w_{ij}, j = 1, \cdots, s\right\}\right\} \\
&\quad < \frac{\delta}{\delta + ER_1}(ER_1) + \frac{ER_1}{\delta + ER_1}(-\delta) = 0.
\end{aligned}
$$

Hence, for $i>0$,

$$
(6.14)\qquad E\sum_{j=1}^{s} w_{i+N,j} < E\sum_{j=1}^{s} w_{N,j}
$$

so that $E\sum_{j=1}^{s} w_{i,j}$ is bounded uniformly in i. This contradicts (6.7) and proves III.

IV. We now suppose $P\{R_1=0\}=0$, and we construct a suitable "dominating" process for which we can prove results analogous to II and III.

Let k be a positive integer such that $(sk-1)>0$,

$$
(6.15)\qquad P\{R_1 \geqq (sk-1)c\} = 1,
$$

and

$$
(6.16)\qquad P\{R_1 = (sk-1)c\} > 0.
$$

If necessary, we can decrease the c of (6.2) so that such a k can always be found. If now

$$
(6.17)\qquad P\{g_1 \geqq skc\} > 0,
$$

then it is clear that

$$
(6.18)\qquad P\{w_{i+j} = 0 \text{ for some } j > 0 \mid w_i = z\} > 0
$$

for every z in C. Hence C is irreducible. It is also aperiodic, because

$$
(6.19)\qquad P\{w_{i+1} = 0 \mid w_i = 0\} > 0.
$$

Hence the desired result follows by III. We therefore assume that (6.17) does not hold, i.e., that

$$
(6.20)\qquad P\{g_1 \leqq (sk-1)c\} = 1.
$$

Let m be the largest integer for which

$$
P\{g_1 = mc\} > 0.
$$

Let A_1 be the set of α (say) non-negative integers $j<m$ such that

$$
P\{g_1 = jc\} = 0, \qquad j \in A_1.
$$

Let $\{g_i'\}$ be independently and identically distributed chance variables with

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the following distribution:

$$P\{g_1' = jc\} = \lambda, \qquad j \in A_1,$$
$$P\{g_1' = mc\} = P\{g_1 = mc\} - \alpha\lambda,$$
$$P\{g_1' = jc\} = P\{g_1 = jc\}, \qquad j \neq m, j \notin A_1.$$

Here λ is a small positive number, whose choice will be more fully described shortly, but which should in any case be such that $P\{g_1 = mc\} - \alpha\lambda > 0$ and $\lambda < P\{R_1 = (sk-1)c\}$.

Let A_2 be the totality of integers $j > (sk-1)$ such that

$$P\{R_1 = jc\} = 0, \qquad j \in A_2.$$

Let $\{R_i'\}$ be independently and identically distributed chance variables, independent of $\{g_i'\}$ and with the following distribution:

$$P\{R_1' = jc\} = \frac{\lambda}{2^j}, \qquad j \in A_2,$$
$$P\{R_1' = jc\} = P\{R_1 = jc\}, \qquad j \neq (sk-1), j \notin A_2,$$
$$P\{R_1' \geqq (sk-1)c\} = 1.$$

We choose $\lambda > 0$ so small that

$$(ER_1')(sEg_1')^{-1} < 1.$$

Any such λ will suffice.

Let $\{w_i'\}$ be the same functions of $\{R_i', g_i'\}$ as w_i are of $\{R_i, g_i\}$. Let F_i' and F' be the corresponding functions for the primed w_i. Comparing corresponding sequences in the manner of §3 we obtain that

$$F_i'(x) \leqq F_i(x) \qquad \text{for every } x.$$

Hence

$$F'(x) = \lim F_i'(x) \leqq F(x) \qquad \text{for every } x.$$

If, therefore, we prove the desired result for the system $\{w_i'\}$ we have a fortiori proved the desired result for the system $\{w_i\}$. We may therefore drop the accents and henceforth assume that

$$P\{g_1 = jc\} > 0, \qquad j = 0, \cdots, (sk-1), \tag{6.21}$$

$$P\{g_1 \leqq (sk-1)c\} = 1, \tag{6.22}$$

$$P\{R_1 = jc\} > 0, j = (sk-1), (sk), (sk+1), \cdots, \tag{6.23}$$

$$P\{R_1 \geqq (sk-1)c\} = 1. \tag{6.24}$$

(We note that these imply that we are in the case $s > 1$, for $s = 1$ would

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violate the requirement that $\rho<1$.)

Let

$$z^* = (skc, skc, \cdots, skc).$$

Let

$$z = (a_1c, a_2c, \cdots, a_sc)$$

with $0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_s$ be any point in Z. Let

$$L = w_{is} + skc.$$

If $R_{i+j-1} = L - w_{ij}$ and $g_{i+j} = 0, j = 1, \cdots, s$, an event of positive probability, we have

$$w_{i+s} = (L, L, \cdots, L).$$

If $R_{i+s+j-1} = (sk-1)c$ and $g_{i+s+j} = kc, j = 1, \cdots, s$, again an event of positive probability, we have, since $L \geqq skc$, that

$$w_{i+2s} = (L - c, L - c, \cdots, L - c).$$

Applying the above argument a_s times we conclude that, for any z and i, $P\{w_{i+j} = z^* \text{ for some } j \geqq 0 \mid w_i = z\} > 0$. Let D be the set of all points in Z which can be reached from z^* with positive probability. The above argument shows that the states of D form an irreducible Markoff chain. This chain is aperiodic, because a modification of the above argument shows (using (6.23)) that there exists a number N such that, whatever be $n \geqq N$, there is a positive probability of moving from z^* back to z^* in exactly n steps.

If now, with probability one, $w_i \in D$ for some i, an argument similar to that of III applies and the desired result is proved.

V. We now prove that, with probability one, $w_i \in D$ for some i.

From (6.23) and the fact that $P\{g_1 = 0\} > 0$ it follows that any point $z = (a_1c, \cdots, a_sc) \in Z$ such that $(2sk-1) \leqq a_1$ is a member of D. We now note that the probability of entering D in at most s steps from any point z not in D is bounded below by a number (say) $\mu > 0$, independently of z (not in D). To see this, we note that this can be accomplished in at most s steps where each $R = 2skc$ and each $g = 0$. From this it follows that the probability of entering D for some i is one.

The proof of the result of this section is now complete.

7. **Proof that F is not a distribution function when** $\rho \geqq 1$. To prove this result we must in addition assume that, when $\rho = 1$,

$$P\{R_i - sg_i = 0\} < 1. \tag{7.1}$$

For if (7.1) does not hold we have, for some $e > 0$,

$$P\{g_1 = e\} = P\{R_1 = se\} = 1. \tag{7.2}$$

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(Hence $\rho=1$ here.) Therefore, with probability one,

$$\begin{aligned} w_1 &= (0, 0, \cdots, 0), \\ w_2 &= (0, 0, \cdots, 0, (s-1)e), \\ w_3 &= (0, 0, \cdots, 0, (s-2)e, (s-1)e), \\ &\ \ \vdots \\ w_s &= (0, e, 2e, \cdots, (s-1)e), \\ w_i &= w_s, \qquad\qquad i > s. \end{aligned}$$

Hence a limiting distribution function F does exist.

We therefore assume that $\rho \geqq 1$ and (7.1) holds. We shall show that $F(x) \equiv 0$, and hence (see §5, A) that $F^*(x_1) \equiv 0$.

Let $\{L_i\}$ be a sequence of chance variables defined as follows: $L_1=0$ with probability one. For $i \geqq 1$

$$L_{i+1} = \max (0, L_i + R_i - sg_{i+1}).$$

Thus L_i is the waiting time of the ith individual in a system such as described in §2 where $s=1$, the service time of the ith individual is R_i, and the interval between the ith arrival and $(i+1)$st arrival is sg_{i+1}. In this system $\rho \geqq 1$, so that the theorem of Lindley (which treats the case $s=1$) is applicable, i.e.,

$$\lim_{x' \to \infty} \lim_{i \to \infty} P\{L_i \leqq x'\} = 0. \tag{7.3}$$

Now, if $0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_s$, and b, c, and d are non-negative numbers with $b \leqq \sum_{j=1}^{s} a_j$, we clearly have $\max(0, a_1+c-d) + \sum_{j=2}^{s} \max(0, a_j - d) \geqq \max(0, b+c-sd)$. We conclude, using induction, that for all $i \geqq 1$,

$$L_i \leqq \sum_{j=1}^{s} w_{i,j} \tag{7.4}$$

with probability one. It follows from (7.3) and (7.4) that

$$\begin{aligned} \lim_{x' \to \infty} F(x', x', \cdots, x') &= \lim_{x' \to \infty} \lim_{i \to \infty} P\{w_{i,j} \leqq x', j = 1, \cdots, s\} \\ &\leqq \lim_{x' \to \infty} \lim_{i \to \infty} P\left\{ \sum_{j=1}^{s} w_{i,j} \leqq sx' \right\} \qquad (7.5) \\ &\leqq \lim_{x' \to \infty} \lim_{i \to \infty} P\{L_i \leqq sx'\} = 0, \end{aligned}$$

which proves the desired result.

8. **Proof that $\lim_{i\to\infty} F_i(x|y)$ exists and is independent of y. Uniqueness of the solution of the integral equation (3.8).** Suppose first that $\rho \geqq 1$. Since $F_i(x|y) \leqq F_i(x)$ for every i and every x and $y \in S$, it follows from the results

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of §7 that

$$\lim_{i\to\infty} F_i(x \mid y) = 0$$

when $\rho \geqq 1$. We shall shortly show that, when $\rho \geqq 1$, (3.8) has no solution which is a distribution function over S.

Assume, therefore, that $\rho < 1$, which is the interesting case. We shall show that, for all x and y in S, the ergodic property

$$\lim_{i\to\infty} F_i(x \mid y) = F(x) \tag{8.1}$$

holds. From this it follows easily that (3.8) has at most one solution which is a distribution function over S (thus, by §6, it has exactly one such solution). For, suppose, to the contrary, that there were another such distribution function, say $V(x)$. It is clear then that, if w_1 is distributed according to $V(x)$, so are $w_2, w_3, \cdots$, so that $V(x)$ is the limiting distribution. On the other hand, it follows from (8.1) and the Lebesgue bounded convergence theorem applied to

$$V(x) = \int F_i(x \mid y) dV(y)$$

that

$$V(x) = \int F(x) dV(y) = F(x)$$

which is the desired result. (Thus we have proved that, when $\rho < 1$, $F(x)$ is the unique stationary absolute probability distribution; see the paragraph following equation (3.8).)

Conversely, if (3.8) has a solution V which is a distribution function over S, then

$$F(x) = \lim_{i\to\infty} F_i(x) \geqq \lim_{i\to\infty} \int F_i(x \mid y) dV(y) = V(x),$$

so that (from the result of §7) $\rho < 1$ and hence $V(x) = \lim_{i\to\infty} F_i(x)$ is the unique solution of (3.8).

Denote by $[a, b]$ and $[c, d]$ the smallest closed intervals for which

$$P\{a \leqq R_1 \leqq b\} = P\{c \leqq g_1 \leqq d\} = 1.$$

We shall conduct the proof separately for several cases.

Case 1: $b > sc$. Let $b - sc = 2\nu > 0$. Then, for any positive integer n,

$$P\{w_{sn,1} > \nu n\} = q_n > 0.$$

Fix n. For any x and $\delta > 0$ there exists an integer M such that, for all $j \geqq M$

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and $k>0$, we have

$$|F_j(x) - F_{j+k}(x)| < q_n\delta.$$

We recall that, if $y_1 \leqq y_2$, $y_1 \in S$, $y_2 \in S$, we have, for all i,

$$F_i(x \mid y_1) \geqq F_i(x \mid y_2).$$

Hence, for $j \geqq M$, we have

$$\begin{aligned}
0 &\leqq q_n[F_j(x) - F_j(x \mid (nv, nv, \cdots, nv))] \\
&\leqq \int [F_j(x) - F_j(x \mid y)]dF_{sn}(y) \qquad (8.2)\\
&< q_n\delta + \int [F_{j+sn-1}(x) - F_j(x \mid y)]dF_{sn}(y) = q_n\delta.
\end{aligned}$$

Therefore

$$0 \leqq [F_j(x) - F_j(x \mid y)] < \delta \tag{8.3}$$

for all $y \leqq (nv, nv, \cdots, nv)$, $y \in S$, and all $j \geqq M$. Since x, n, and δ were arbitrary this proves (8.1) for Case 1.

Case 2: $a<d$. Let y be any point in S, and

$$p_n(y) = P\{w_n = 0;\ w_i \neq 0,\ i < n \mid w_1 = y\}.$$

We shall show that, for all y in S,

$$\sum_{n=1}^{\infty} p_n(y) = 1. \tag{8.4}$$

This is sufficient to prove the desired result, because

$$F_i(x \mid y) - \sum_{n \leqq i} p_n(y)F_{i-n}(x) \tag{8.5}$$

then approaches zero as $i \to \infty$.

To prove (8.4) we proceed as in §6 to construct a "dominating" random walk on a lattice. The walk begins at a point on the lattice all of whose coordinates are no less than the corresponding ones of y. As in §6 one proves that with probability one the walk enters an irreducible aperiodic chain. Since $a<d$ this chain contains the origin. Since $\rho<1$ and $F(x)$ is a distribution function this chain constitutes a positive recurrent class. For an irreducible, recurrent class (8.4) must hold for all y in the class. Since the walk enters the class with probability one, (8.4) holds for all y in S.

Case 3: $c=d\leqq a=b\leqq sc$. In this case we have $P\{R_1=b\}=P\{g_1=c\}=1$. Since $\rho<1$, we also have $b<sc$.

Let $y^*=(y_1, \cdots, y_s)$ be any point in S, and let $y''=(y_s, \cdots, y_s)$. Given the process $\{R_i, g_i\}$, let w_i^* be the position of w_i if $w_1=y^*$, let w_i' be the posi-

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tion of w_i if $w_1=0$, and let w_i'' be the position of w_i if $w_1=y''$. Clearly, $w_i \leqq w_i^* \leqq w_i''$ for all i with probability one, and for $i \geqq s$ we have $w_i' = \bar{w}$, where $\bar{w}$ is defined by

$$\bar{w} = (0, u_{s-1}, \cdots, u_2, u_1), \tag{8.6}$$

where

$$u_j = \max (b - jc, 0).$$

We shall show that $w_i'' = \bar{w}$ with probability one for i sufficiently large, which implies that for sufficiently large i with probability one, $w_i^* = \bar{w}$, and proves the desired result.

It is clear that, for all i,

$$\begin{aligned} w''_{i+1,j} &= \max (0, w''_{i,j+1} - c) \qquad \text{for } 1 \leqq j < s, \\ w''_{i+1,s} &= w''_{i,1} + b - c. \end{aligned} \tag{8.7}$$

For $n \geqq 0$ and $0 \leqq i \leqq s-1$, we evidently have

$$w''_{1+ns+i,1} \leqq w''_{ns+1,1}. \tag{8.8}$$

Let N be a positive integer such that $y_s - N(sc-b) \leqq 0$. Then $w''_{ns+1,1}=0$ for $n \geqq N$, and hence from (8.8) we have $w''_{i,1}=0$ for $i \geqq Ns+1$. It follows from (8.7) that $w_i''=\bar{w}$ for $i \geqq (N+1)s$.

Case 4: $d \leqq a$, $b \leqq sc$, *and either* $a<b$ *or* $c<d$. Let u_j be as in (8.6) and for $\epsilon > 0$ define $\bar{w}^\epsilon = (0, u^\epsilon_{s-1}, \cdots, u^\epsilon_2, u^\epsilon_1)$, where $u^\epsilon_j = \max (0, u_j - \epsilon)$. From the definition of b and c we have that, for every $\epsilon > 0$,

$$P\{ w_{s,j} \geqq u^\epsilon_{s-j+1}, j = 1, \cdots, s\} = \gamma > 0.$$

An argument like that of Case 1 (with γ for q_n and F_s for F_{sn}) then shows that

$$\lim_{i \to \infty} F_i(x \mid y) = F(x)$$

for all $y \in \Gamma^\epsilon = \{ y \mid y \in S, y \leqq \bar{w}^\epsilon\}$ and all x.

Let y be any point in S, and

$$p^\epsilon_n(y) = P\{ w_n \in \Gamma^\epsilon, w_i \notin \Gamma^\epsilon, i < n \mid w_1 = y\}.$$

We shall show that, for some $\epsilon > 0$ and all $y \in S$, we have

$$\sum_{n=1}^{\infty} p^\epsilon_n(y) = 1; \tag{8.9}$$

from this and the result of the previous paragraph, the desired result is proved in the manner of the first paragraph of Case 2.

Let

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$$(8.10)\qquad E = \{y \mid y = (y_1, \cdots, y_s) \in S;\ y_1 = 0;\ y_s \leqq (s-1)b\}.$$

In order to prove (8.9) for some $\epsilon > 0$ and for all $y \in S$, it clearly suffices to show that

$$(8.11)\qquad P\{w_i \in E \text{ for infinitely many } i \mid w_1 = y\} = 1$$

for all $y \in S$, and that there exists a positive integer M and positive numbers α and ϵ such that

$$(8.12)\qquad P\{w_M \in \Gamma^\epsilon \mid w_1 = y\} > \alpha$$

for all $y \in E$.

We first prove (8.11). To this end, let $y = (y_1, \cdots, y_s)$ be any fixed point in S. Since we have always assumed $d > 0$, we have in Case 4 that $a > 0$. It follows from equation (4.3) that for $n > (s-1)y_s/a$ we have

$$(8.13)\qquad P\{B_n \leqq (s-1)b \mid w_1 = y\} = 1.$$

Let $\{e_i\}$, $\{f_i\}$ be any sequences of non-negative numbers, and let $\{v_i\}$ be the corresponding sequence of values of $\{w_i\}$ when $w_1 = y^*$, $R_i = e_i$, and $g_i = f_i$. Then, if $v_{i1} = 0$ for only finitely many i, it would follow that $\liminf_{n\to\infty} (1/n) \sum_{i=1}^n (e_i - sf_{i+1}) \geqq 0$. However, since $\rho < 1$, the strong law of large numbers implies that

$$P\left\{\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n (R_i - sg_{i+1}) = ER_1 - sEg_1 < 0\right\} = 1.$$

Hence

$$(8.14)\qquad P\{w_{i,1} = 0 \text{ for infinitely many } i \mid w_1 = y^*\} = 1$$

for all $y^* \in S$. Equation (8.14) is a fortiori true for the original process, and (8.11) is an immediate consequence of (4.1), (8.13), and (8.14).

It remains to prove (8.12). We recall that in Case 4 we have $c < b \leqq sc$ and that there are numbers b', c' such that $P\{R_1 \leqq b'\} = p > 0$, $P\{g_1 \geqq c'\} = q > 0$, and $b' - c' = b - c - \epsilon$ for some $\epsilon > 0$. An obvious modification of the argument of Case 3 (put b', c' for b, c) shows that if $w_1 = y \in E$ and if $R_j = b'$ and $g_{j+1} = c'$ for $1 \leqq j < M$ where M/s is the greatest integer contained in $2 + (s-1)b/(sc' - b')$, then

$$(8.15)\qquad w_{M,i} \leqq \max(0, b' - (s+1-i)c') \leqq u^\epsilon_{s-i+1}.$$

Equation (8.15) is a fortiori true if $R_j \leqq b'$ and $g_{j+1} \geqq c'$ for $1 \leqq j < M$ (the argument being similar to that of §3). We conclude that (8.12) is satisfied for ϵ and M as defined here and for $\alpha = (pq)^{M-1}$.

9. **Distribution of the number of individuals waiting in the queue.** In order to avoid trivial cases and the circumlocutions required to dispose of them, we shall assume in this section that $G(0) = 0$. This means that the prob-

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ability is zero that two or more individuals arrive simultaneously.

Let Q_i be the number of arrivals in the open time interval (t_i, t_i+w_{i1}); i.e., Q_i is the number of individuals in the queue waiting to be served, just before the service of the ith individual begins.

Since $g_{i+1}, g_{i+2}, \cdots$ are independent of t_i, we have

$$\begin{aligned} P\{Q_i \geqq n\} &= \int P\{g_1 + g_2 + \cdots + g_n < a\} dF_i^*(a) \\ &= \int G^{n*}(a-) dF_i^*(a), \end{aligned} \tag{9.1}$$

where $G^{n*}(a)$ denotes the n-fold convolution of $G(a)$ with itself. Since $F_i^*(a)$ tends nonincreasingly to $F^*(a)$ as $i \to \infty$ for all a, and since $G^{n*}(a-)$ is continuous from the left, we obtain, in the case $\rho < 1$,

$$\lim_{i \to \infty} P\{Q_i \geqq n\} = \int G^{n*}(a-) dF^*(a). \tag{9.2}$$

If $\rho \geqq 1$, equation (9.1) shows that $\lim_{i \to \infty} P\{Q_i \geqq n\} = 1$ for all n, except in the trivial case where $P\{R_1 - sg_1 = 0\} = 1$.

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