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# ON THE THEORY OF RAPIDLY ROTATING WHITE DWARFS

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SUMMARY

The structure of rotating white dwarfs is investigated using a second-order perturbation analysis. It is shown that the second-order corrections are very small and that the first-order perturbation theory gives good results.

## 1. INTRODUCTION

Many previous papers (**1**)–(**10**) have considered the theory of the rotation of stellar configurations, using Newton's theory of gravitation as well as the theory of General Relativity for slowly rotating stars only. It is assumed that the parameter of rotation $\beta = \Omega^2/8\pi\kappa\rho_c$ is small and only the theory of rotation to the first order in $\beta$ is considered. The maximum value of the parameter, $\beta_{max}$ is chosen with the condition that centrifugal force balances the gravitational force at the equator. This condition occurs at the critical value of $\beta$ for which equatorial instability arises. Now the question arises whether we can use the first-order theory for values up to $\beta = \beta_{max}$. This approximation is only valid if the results of the theory of rotation calculated to the second order in $\beta$ agree with the first-order theory. Evidently, this comparison may depend on the equation of state of the matter in the stellar configuration. A similar problem has been considered by James (**12**) and Anand (**13**) for rotating polytropes and simple models of white dwarfs. Comparison of the results of other papers (**1**)–(**6**) with the work of James (**12**) shows that the theory to the first order in $\beta$ is quite accurate. Similar results have been obtained by Anand (**13**), but only for larger values of the polytropic index.

In the present paper we consider the second-order theory of rotation which is valid for any equation of state in the form $P = P(\rho)$. We then use this theory to consider problems of rapidly rotating white dwarfs, neutron stars and hyperon stars and give results for two models of white dwarfs. Calculations for neutron and hyperon stars will be considered in a subsequent paper. It will be shown that the corrections which are given by second-order theory are small, both for a simple model of white dwarfs as well as for a more realistic model.

The present approach is entirely Newtonian. In Einstein's theory of General Relativity, which is important at higher densities, the problem of rotation has been considered using only the theory of rotation to the first order in $\beta$ (**8**)–(**11**). In the latter case it is very difficult to consider the next approximation. However, if we show that the first-order Newtonian theory of rotation is adequate for $\beta \leqslant \beta_{max}$, we can also expect that it will be sufficiently correct in General Relativity.

It will be shown that the order of magnitude of the corrections of the next approximation for different models of white dwarfs and for various values of the central density is the same. The corrections depend on the equation of state and the central density, but it turns out that the dependence is very weak. We expect that the corrections in the case of neutron and hyperon stars will be similarly small.

## 2. BASIC EQUATIONS

The structure of a configuration which rotates like a solid body can be obtained from the system of differential equations.

$$\nabla^2\phi = 4\pi\kappa\rho \tag{1}$$

$$\mathbf{\nabla} P = -\rho\mathbf{\nabla}\phi + \rho\Omega^2\mathbf{r} \tag{2}$$

$$P = P(\rho) \tag{3}$$

where $\kappa$ is the gravitational constant, and $\mathbf{r}$ the distance from the axis of rotation. The equation of state takes the form (3), if the matter of the configuration is degenerate or if the temperature is a function of the density only.

Let us introduce the spherical coordinates $R$, $\mu = \cos\theta$, $\phi$. The equation of hydrostatic equilibrium can be integrated and equations (1) and (2) now take the form (4)

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right] = G \tag{4}$$

$$F = -\phi + G_c\beta R^2(1-\mu^2) + c \tag{5}$$

where

$$F = \int\frac{dP}{\rho}, \quad G = 4\pi\kappa\rho, \quad G_c = 4\pi\kappa\rho_c \tag{6}$$

$$\beta = \frac{\Omega^2}{8\pi\kappa\rho_c}, \quad c = c_0 + \beta c_1 + \beta^2 c_2 \tag{7}$$

$\Omega$ is the constant angular velocity of rotation and $c$ is the constant of integration. We define $\rho$ and $\rho_c$ as the density of the matter and its value in the centre of the star respectively. In order to find solutions of equations (4) and (5) we expand the functions $F$ and $G$ in the form:

$$\begin{aligned} F(R,\mu) = {} & f(R) + \beta\left[f_0(R) + \sum_{l=2}^{\infty} A_l f_l(R) P_l(\mu)\right] \\ & + \beta^2\Big[f_0^{(1)}R + f_2^{(1)}(R)P_2(\mu) + f_4^{(1)}(R)P_4(\mu) \\ & \qquad + \sum_{l=2}^{\infty} A_l^{(1)}\phi_l(R)P_l(\mu)\Big], \end{aligned} \tag{8}$$

$$\begin{aligned} G(R,\mu) = {} & g(R) + \beta\left[g_0(R) + \sum_{l=2}^{\infty} B_l g_l(R) P_l(\mu)\right] \\ & + \beta^2\Big[g_0^{(1)}(R) + g_2^{(1)}(R)P_2(\mu) + g_4^{(1)}(R)P_4(\mu) \\ & \qquad + \sum_{l=2}^{\infty} B_l^{(1)}\Psi_l(R)P_l(\mu)\Big] \end{aligned} \tag{9}$$

where $\beta$ is a small expansion parameter and $P_l(\mu)$ is the Legendre polynomial of order $l$. $A_l$, $B_l$ and $A_l^{(1)}$, $B_l^{(1)}$ are constants. Solutions of equations (4) and (5) in the spherical case are denoted by $f$ and $g$.

The relation between $A_l$, $A_l^{(1)}$ and $B_l$, $B_l^{(1)}$ can be obtained by considering $G$ as a function of $F$ and expanding it in Taylor's series in a region close to the value $F = f$. Then we have

$$\begin{aligned} G(f) &= g(R), & \gamma f_0(R) &= g_0(R) \\ \gamma A_l f_l(R) &= B_l g_l(R), & \gamma A_l^{(1)}\phi_l(R) &= B_l^{(1)}\Psi_l(R) \end{aligned} \tag{10}$$

where

$$\gamma = \left.\frac{dG}{dF}\right|_{F=f} = \frac{dg}{df}.$$

Substituting $F$ and $G$ from equations (8) and (9) in the equations (4) and (5), and using the relations (10) we derive the system of equations:

$$\begin{aligned} \Delta f(R) &= -g(R) \\ \Delta_0 f_0(R) + \gamma f_0(R) &= 4G_c \\ \Delta_l f_l(R) + \gamma f_l(R) &= 0, \quad l>0 \\ \Delta_0 f_0^{(1)}(R) + \gamma f_0^{(1)}(R) + \frac{\gamma_1}{2}\,[f_0^2(R) + \tfrac{1}{5}A_2^2 f_2^2(R)] &= 0 \\ \Delta_2 f_2^{(1)}(R) + \gamma f_2^{(1)}(R) + \gamma_1 A_2 f_2(R)[f_0(R) + \tfrac{1}{7}A_2 f_2(R)] &= 0 \\ \Delta_4 f_4^{(1)}(R) + \gamma f_4^{(1)}(R) + \frac{9}{35}\gamma_1 A_2^2 f_2^2(R) &= 0 \\ \Delta_l \phi_l(R) + \gamma \phi_l(R) &= 0, \quad l>0 \end{aligned} \tag{11}$$

with

$$\Delta_l = \frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d}{dR}\right) - \frac{l(l+1)}{R^2}$$

$$\gamma_1 = \left.\frac{d^2G}{dF^2}\right|_{F=f} = \frac{d^2g}{df^2}.$$

Equations (11) can be used to obtain the radial functions $f$, $g$, $f_0$, $f_l$, $f_0^{(1)}$, $f_2^{(1)}$, $f_4^{(1)}$, $\phi_l$. However, the solution is only determined if we can find the values of the coefficients which appear in the solutions (5), (8) and (9). The shape of the star must also be obtained at the same time. The boundary of the star can be expressed in the form

$$Q(\mu) = R_0 + \beta\sum_{l=0}^{\infty} q_l P_l(\mu) + \beta^2 \sum_{l=0}^{\infty} q_l^{(1)} P_l(\mu) \tag{12}$$

where $R_0$ is the radius of a corresponding non-rotating star and $q_l$ and $q_l^{(1)}$ are constants.

We impose the usual boundary condition of zero pressure and density at the surface of the star. It is easy to show that the function $F$ becomes equal to zero at the surface as well (4). From the condition $F = 0$ we get:

$$\begin{aligned} f(R_0) = 0, \quad f'(R_0)q_0 + f_0(R_0) &= 0, \\ f'(R_0)q_l + A_l f_l(R_0) &= 0, \quad l>0 \\ f'(R_0)q_0^{(1)} + f_0^{(1)}(R_0) &= -q_0T_0 - \tfrac{1}{5}q_2T_2 \\ f'(R_0)q_2^{(1)} + f_2^{(1)}(R_0) + A_2^{(1)}\phi_2(R_0) &= -q_0T_2 - q_2T_0 - \tfrac{2}{7}q_2T_2 \\ f'(R_0)q_4^{(1)} + f_4^{(1)}(R_0) + A_4^{(1)}\phi_4(R_0) &= -\tfrac{18}{35}q_2T_2 \\ f'(R_0)q_l^{(1)} + A_l^{(1)}\phi_l(R_0) &= 0, \quad l>4 \end{aligned} \tag{13}$$

where

$$f'(R_0) = \left.\frac{df}{dR}\right|_{R=R_0}, \quad T_0 = \left.\left(\frac{df_0(R)}{dR}+\frac{f_0(R)}{R}\right)\right|_{R=R_0}, \quad T_2 = A_2\left.\left(\frac{df_2(R)}{dR}+\frac{f_2(R)}{R}\right)\right|_{R=R_0}.$$

Relations (13) connect the constants in equation (12) with the constants in the solutions (5), (8) and (9). These constants are determined from the boundary conditions on the surface of the star.

$$\phi(Q\mu) = \phi^{(e)}(Q\mu)$$
$$\left.\frac{d\phi}{dR}\right|_{Q(\mu)} = \left.\frac{d\phi^{(e)}}{dR}\right|_{Q(\mu)} \tag{14}$$

which implies that the potential and its gradient must be continuous on the boundary. $\phi^{(e)}$ is the external solution of equation (4) of the form

$$\phi^{(e)} = \frac{\kappa_0}{R}+\beta\sum_{l=0}^{\infty}\frac{\kappa_{1l}}{R^{l+1}}P_l(\mu)+\beta^2\sum_{l=0}^{\infty}\frac{\kappa_{2l}}{R^{l+1}}P_l(\mu)$$

where $k_0$, $k_{1l}$, $k_{2l}$ are integration constants and are defined from the conditions. Solving the algebraic equations given by the boundary conditions and by equations (13), we get

$$\kappa_0 = R_0^2 f'(R_0), \quad \kappa_{10} = R_0^2 f_0'(R_0)-\frac{4G_c}{3}R_0^3, \quad c_0 = R_0 f'(R_0), \quad q_0 = -\frac{f_0(R_0)}{f'(R_0)},$$

$$c_1 = R_0 f_0'(R_0)+f_0(R_0)+2G_cR_0^2, \qquad A_2 = -\frac{10G_cR_0^2}{3}\,\frac{1}{R_0 f_2'(R_0)+3f_2(R_0)}$$

$$\kappa_{21} = -\frac{2G_cR_0^5}{3}\cdot\frac{R_0f_2'(R_0)-2f_2(R_0)}{R_0f_2'(R_0)+3f_2(R_0)}, \quad q_2 = -\frac{A_2f_2(R_0)}{f'(R_0)}, \tag{15}$$

$$A_l = q_l = \kappa_{1l} = 0 \quad \text{for} \quad l>2$$

$$\kappa_{20} = R_0^2 f_0'^{(1)}(R_0), \quad c_2 = f_0^{(1)}(R_0)+R_0f_0'^{(1)}(R_0),$$

$$q_2^{(1)} = -\frac{R_0^2}{\kappa_0}\left[q_0T_0+\tfrac{1}{5}q_2T_2+f_0^{(1)}(R_0)\right]$$

$$A_2^{(1)} = -\frac{R_0f_2'^{(1)}(R_0)+3f_2^{(1)}(R_0)}{R_0\phi_2'(R_0)+3\phi_2(R_0)}, \quad \kappa_{22} = \frac{R_0^4[\phi_2(R_0)f_2'^{(1)}(R_0)-\phi_2\,(R_0)f_2^{(1)}(R_0)]}{R_0\phi_2'(R_0)+3\phi_2(R_0)}$$

$$q_2^{(1)} = -\frac{R_0^2}{k_0}\left[q_{02}+Tq_2T_0+\tfrac{2}{7}q_2T_2-\frac{\kappa_{22}}{R_0^3}\right],$$

$$A_4^{(1)} = \frac{R_0f_4'^{(1)}(R_0)+5f_4^{(1)}(R_0)}{R_0\phi_4'(R_0)+5\phi_4(R_0)}, \quad \kappa_{24} = \frac{R_0^6[\phi_4(R_0)f_4'^{(1)}(R_0)-\phi_4'(R_0)f_4^{(1)}(R_0)]}{R_0\phi_4'(R_0)+5\phi_4(R_0)},$$

$$q_4^{(1)} = -\frac{R_0^2}{\kappa_0}\left[\tfrac{18}{35}q_2T_2-\frac{\kappa_{24}}{R_0^5}\right],$$

$$A_l^{(1)} = q_l^{(1)} = k_{2l} = 0 \quad \text{for} \quad l>4$$

where primes denote differentional with respect to $R$. From the knowledge of the constants (15) the mass of the star is given by

$$M = M_0-\beta\kappa_{10}-\beta^2R_0^2f_0'^{(1)}(R_0) \tag{16}$$

where $M_0$ is the mass of the non-rotating star. The equatorial radius $Q_l$ and the polar radius $Q_p$ are then derived from the expressions

$$Q_l = R_0 + \beta L_1 + \beta^2 L_2 \tag{17}$$

$$Q_p = R_0 + \beta(q_0 + q_2) + \beta^2(q_0^{(1)} + q_2^{(1)} + q_4^{(1)}) \tag{18}$$

where

$$L_1 = q_0 - 0{\cdot}5\, q_2, \quad L_2 = q_0^{(1)} - 0{\cdot}5\, q_2^{(1)} + 0{\cdot}375\, q_4^{(1)}.$$

The expansion parameter $\beta$ runs from zero to $\beta = \beta_{\max}$ and the value of $\beta_{\max}$ corresponding to equatorial break-up is

$$\beta_{\max} = -\frac{M_0}{PR_0^2} \cdot \frac{P^2R_0^2 - M_0 N}{P^2 R_0^2 - \frac{1}{5} M_0 N} \tag{19}$$

where

$$P = -2G_c R_0 - \frac{2M_0 L_1}{R_0^3} - \frac{\kappa_{10}}{R_0^2} + \frac{3\kappa_{12}}{2R_0^4},$$

$$N = -2G_c L_1 - \frac{M_0(2L_2 R_0 - 3L_1^2)}{R_0^4} - \frac{6\kappa_{12} L_1}{R_0^5} + \frac{2\kappa_{10} L_1}{R_0^3} - \frac{\kappa_{20}}{R_0^2} + \frac{3\kappa_{22}}{2R_0^4} - \frac{15\kappa_{24}}{8R_0^6}.$$

We again emphasize that this method can be used for any equation of state of the form $P = P(\rho)$.

## 3. EQUATIONS FOR ROTATING WHITE DWARFS

We restrict our investigation to the problem of white dwarfs, that is, to the case where the density of matter is greater than $10^6$ g cm$^{-3}$. It is evident that under these conditions, the density is contributed by the nuclei while the electrons determine the pressure. We introduce the parameter $x = P_e/m_e c$ and write the equation of state for white dwarfs in the form

$$\rho = \frac{32}{3}\left(\frac{m_e}{m_n}\right)^3 \kappa_n \left(\frac{A}{Z}\right) x^3$$
$$P = \frac{4}{3}\left(\frac{m_e}{m_n}\right)^4 \kappa_n [x(2x^2-3)\sqrt{1+x^2} + 3\ln(x + \sqrt{1+x^2})] \tag{20}$$

where

$$\kappa_n = \frac{m_n^4 c^5}{32\pi^2 h^3}$$

$P_e$ and $m_e$ are maximal momentum and the electron mass, respectively, and $m_n$ is the neutron mass. $c$ and $h$ are the velocity of light and Planck's constant, respectively.

In the case of completely degenerate electrons, for a given barion number $A$, the nuclear charge $eZ$ depends on Fermi's energy level of electrons $Ee$ and decreases as the density becomes smaller. The dependence of $A/Z$ on $E_e$ describes step-wise curves, which varies for different nuclei. For investigating white dwarfs with a complicated chemical composition it will be more convenient to replace these curves by a continuous function for which the polynomial

$$\frac{A}{Z} = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

with

$$a_0 = 2, \quad a_1 = 1.255\ 10^{-2}, \quad a_2 = 1.755\ 10^{-5}, \quad a_3 = 1.376\ 10^{-6}$$

gives us a sufficiently correct approximation (14). We also consider the case of simple white dwarfs, i.e. $A/Z = 2$. For the integration of the system of equations (11) we need the functions $\gamma$ and $\gamma_1$, for white dwarfs. From the determination of these functions, using (20), we obtain

$$\gamma = \frac{32}{3}\left(\frac{m_e}{m_n}\right)^2\left(\frac{A}{Z}\right) M_1 L x\sqrt{1+x^2}$$

$$\gamma_1 = \frac{32}{3}\left(\frac{m_e}{m_n}\right)\frac{L}{x}\left(\frac{A}{Z}\right) \tag{21}$$

where

$$M_1 = 3a_0+4a_1x+5a_2x^2+6a_3x^3$$

$$L = A_1+2A_2x+(2A_1+3A_3)x^2+(3A_2+4A_4)x^3+(4A_3+5A_5)x^4 + (5A_4+6A_6)x^5+(6A_5+7A_7)x^6+7A_6x^7+8A_7x^8$$

$$A_1 = 3a_0^2, \quad A_2 = 7a_1a_0, \quad A_3 = 8a_2a_0+4a_1^2, \quad A_4 = 9(a_3a_0+a_1a_2),$$

$$A_5 = 10a_1a_3+5a_2^2, \quad A_6 = 11a_2a_3, \quad A_7 = 6a_3^2.$$

We now write the structural equations for white dwarfs in a form suitable for numerical integration.

$$\frac{du}{dR} = \frac{32}{3}\left(\frac{m_e}{m_n}\right)^3\left(\frac{A}{Z}\right)x^3R^2, \quad \frac{dx}{dR} = -\left(\frac{m_n}{m_e}\right)\left(\frac{A}{Z}\right)\frac{\sqrt{1+x^2}}{x}\cdot\frac{U}{R^2}$$

$$\frac{dv}{dR} = (4G_c-\gamma f_0(R))R^2, \quad \frac{df_0}{dR} = \frac{v}{R^2},$$

$$\frac{dv^{(1)}}{dR} = -\gamma f_0^{(1)}R^2-\tfrac{1}{2}\gamma_1R^2[f_0^2+\tfrac{1}{5}A_2^2f_2^2], \quad \frac{df_0^{(1)}}{dR} = \frac{v^{(1)}}{R^2},$$

$$\frac{dz}{dR} = (6-\gamma R^2)f_2, \quad \frac{df_2}{dR} = \frac{Z}{R^2}, \tag{22}$$

$$\frac{dZ_2^{(1)}}{dR} = (6-\gamma R^2)f_0^{(1)}-\gamma_1A_2R^2f_2[f_0+\tfrac{1}{7}A_2f_2], \quad \frac{df_2^{(1)}}{dR} = \frac{Z_2^{(1)}}{R^2},$$

$$\frac{dZ_4^{(1)}}{dR} = (20-\gamma R^2)f_4^{(1)}-\frac{9}{35}\gamma_1A_2^2R^2f_2^2, \quad \frac{df_4^{(1)}}{dR} = \frac{Z_4^{(1)}}{R^2}$$

$$\frac{dw}{dR} = (20-\gamma R^2)\phi_4, \quad \frac{d\phi_4}{dR} = \frac{w}{R^2}.$$

We note the relation

$$f_2(R) \equiv \phi_2(R).$$

In order to integrate these equations we need the initial conditions at the centre of the star. The integration is started from the point $R = R_1$ where $R_1$ is very close to the centre, and we chose the following values for the unknown functions at this point.

$$x = x_c, \quad u = 0, \quad f_0 = \frac{4G_c}{6}R_1^2, \quad v = 0, \quad f_2 = R_1^2,$$

$$z = 0, \quad f_0^{(1)} = 0, \quad v^{(1)} = 0, \quad f_2^{(1)} = R_1^2, \quad Z_2^{(1)} = 0,$$

$$f_4^{(1)} = R_1^4, \quad Z_4^{(1)} = 0, \quad \phi_4 = R_1^4, \quad w = 0.$$

Now to integrate equations (22) we need only the value of $x_c$ in the centre of the star.

TABLE I

*Integral parameters for rotating White Dwarfs; $A/Z = a_0+a_1x+a_2x^2+a_3x^3$*

| | Spherical solutions | | The first-order approximation | | | | The second-order approximation | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho_c$ g cm³ | $M_0/M_\odot$ | $R_0$ km | $\beta_{max}$ | $M/M_\odot$ | $Q_e$ km | $Q_p$ km | $\beta_{max}$ | $M/M_\odot$ | $Q_e$ km | $Q_p$ km |
| $1\cdot0145.10^6$ | 0·3921 | $9\cdot7188.10^3$ | $1\cdot738.10^{-2}$ | 0·475 | $1\cdot385.10^4$ | $9\cdot083.10^3$ | $1\cdot286.10^{-2}$ | 0·4754 | $1\cdot447.10^4$ | $9\cdot498.10^3$ |
| $1\cdot9847.10^6$ | 0·4991 | $9\cdot574.10^3$ | $1\cdot622.10^{-2}$ | 0·6005 | $1\cdot233.10^4$ | $8\cdot075.10^3$ | $1\cdot204.10^{-2}$ | 0·6015 | $1\cdot276.10^4$ | $8\cdot426.10^3$ |
| $1\cdot5972.10^7$ | 0·8714 | $6\cdot369.10^3$ | $1\cdot145.10^{-2}$ | 1·0139 | $8\cdot113.10^3$ | $5\cdot506.10^3$ | $8\cdot626.10^{-3}$ | 1·013 | $8\cdot421.10^3$ | $5\cdot706.10^3$ |
| $5\cdot425.10^7$ | 1·052 | $4\cdot907.10^3$ | $8\cdot649.10^{-3}$ | 1·202 | $6\cdot205.10^3$ | $4\cdot318.10^3$ | $6\cdot596.10^{-3}$ | 1·2 | $6\cdot458.10^3$ | $4\cdot45.10^3$ |
| $1\cdot294.10^8$ | 1·148 | $4\cdot032.10^3$ | $6\cdot974.10^{-3}$ | 1·292 | $5\cdot075.10^3$ | $3\cdot592.10^3$ | $5\cdot358.10^{-3}$ | 1·289 | $5\cdot291.10^3$ | $3\cdot688.10^3$ |
| $7\cdot0643.10^8$ | 1·25 | $2\cdot68.10^3$ | $4\cdot54.10^{-3}$ | 1·387 | $3\cdot35.10^3$ | $2\cdot439.10^3$ | $3\cdot529.10^{-3}$ | 1·364 | $3\cdot505.10^3$ | $2\cdot491.10^3$ |
| $2\cdot099.10^9$ | 1·264 | $2\cdot03.10^3$ | $3\cdot478.10^{-3}$ | 1·365 | $2\cdot53.10^3$ | $1\cdot867.10^3$ | $2\cdot722.10^{-3}$ | 1·359 | $2\cdot648.10^3$ | $1\cdot9.10^3$ |
| $7\cdot3108.10^9$ | 1·24 | $1\cdot458.10^3$ | $2\cdot582.10^{-3}$ | 1·321 | $1\cdot813.10^3$ | $1\cdot356.10^3$ | $2\cdot033.10^{-3}$ | 1·317 | $1\cdot902.10^3$ | $1\cdot377.10^3$ |
| $3\cdot614.10^{10}$ | 1·158 | $9\cdot478.10^2$ | $1\cdot726.10^{-3}$ | 1·216 | $1\cdot177.10^3$ | $8\cdot917.10^2$ | $1\cdot366.10^{-3}$ | 1·212 | $1\cdot237.10^3$ | $9\cdot032.10^2$ |
| $6\cdot465.10^{10}$ | 1·114 | $8\cdot126.10^2$ | $1\cdot458.10^{-3}$ | 1·164 | $1\cdot008.10^3$ | $7\cdot675.10^2$ | $1\cdot156.10^{-3}$ | 1·161 | $1\cdot060.10^3$ | $7\cdot768.10^2$ |
| $1\cdot652.10^{11}$ | 1·026 | $6\cdot425.10^2$ | $1\cdot048.10^{-3}$ | 1·066 | $7\cdot974.10^2$ | $6\cdot11.10^2$ | $8\cdot328.10^{-4}$ | 1·062 | $8\cdot391.10^2$ | $6\cdot173.10^2$ |
| $2\cdot449.10^{11}$ | 0·9834 | $5\cdot866.10^2$ | $8\cdot842.10^{-4}$ | 1·018 | $7\cdot277.10^2$ | $5\cdot596.10^2$ | $7\cdot04.10^{-4}$ | 1·016 | $7\cdot664.10^2$ | $5\cdot649.10^2$ |

TABLE II

*Integral parameters for rotating white dwarfs; $A/Z = 2$*

| | Spherical solutions | | The first-order approximation | | | | The second-order approximation | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho_c$ g cm³ | $M_0/M_\odot$ | $R_0$ km | $\beta_{max}$ | $M/M_\odot$ | $Q_e$ km | $Q_p$ km | $\beta_{max}$ | $M/M_\odot$ | $Q_e$ km | $Q_p$ km |
| $1\cdot0096.10^6$ | 0·3953 | $1\cdot0857.10^4$ | $1\cdot7478.10^{-2}$ | 0·4791 | $1\cdot4029.10^4$ | $9\cdot1023.10^3$ | $1\cdot292.10^{-2}$ | 0·4782 | $1\cdot4481.10^4$ | $9\cdot5325.10^3$ |
| $1\cdot9719.10^6$ | 0·5041 | $9\cdot605.10^3$ | $1\cdot637.10^{-2}$ | 0·607 | $1\cdot2374.10^4$ | $8\cdot0967.10^3$ | $1\cdot212.10^{-2}$ | 0·6062 | $1\cdot2796.10^4$ | $8\cdot4598.10^3$ |
| $1\cdot5775.10^7$ | 0·886 | $6\cdot405.10^3$ | $1\cdot164.10^{-2}$ | 1·0352 | $8\cdot16.10^3$ | $5\cdot3896.10^3$ | $8\cdot771.10^{-3}$ | 1·0344 | $8\cdot4707.10^3$ | $5\cdot7321.10^3$ |
| $5\cdot324.10^7$ | 1·0817 | $4\cdot94.10^3$ | $8\cdot897.10^{-3}$ | 1·239 | $6\cdot25.10^3$ | $4\cdot3415.10^3$ | $6\cdot783.10^{-3}$ | 1·2358 | $6\cdot5034.10^3$ | $4\cdot4758.10^3$ |
| $1\cdot262.10^8$ | 1·1903 | $4\cdot062.10^3$ | $7\cdot2696.10^{-3}$ | 1·342 | $5\cdot1156.10^3$ | $3\cdot6127.10^3$ | $5\cdot58.10^{-3}$ | 1·3389 | $5\cdot3334.10^3$ | $3\cdot7113.10^3$ |
| $6\cdot7636.10^8$ | 1·3278 | $2\cdot702.10^3$ | $4\cdot945.10^{-3}$ | 1·4587 | $3\cdot3798.10^3$ | $2\cdot4509.10^3$ | $3\cdot84.10^{-3}$ | 1·455 | $3\cdot5332.10^3$ | $2\cdot5057.10^3$ |
| $1\cdot9719.10^9$ | 1·376 | $2\cdot043.10^3$ | $3\cdot9825.10^{-3}$ | 1·4921 | $2\cdot5468.10^3$ | $1\cdot8714.10^3$ | $3\cdot111.10^{-3}$ | 1·4884 | $2\cdot6674.10^3$ | $1\cdot9084.10^3$ |
| $6\cdot6552.10^9$ | 1·4067 | $1\cdot46.10^3$ | $3\cdot2366.10^{-3}$ | 1·5079 | $1\cdot8166.10^3$ | $1\cdot3508.10^3$ | $2\cdot542.10^{-3}$ | 1·5032 | $1\cdot9043.10^3$ | $1\cdot3727.10^3$ |
| $3\cdot0811.10^{10}$ | 1·425 | $9\cdot3475.10^2$ | $2\cdot653.10^{-3}$ | 1·5097 | $1\cdot161.10^3$ | $8\cdot7105.10^2$ | $2\cdot091.10^{-3}$ | 1·5042 | $1\cdot2186.10^3$ | $8\cdot8761.10^2$ |
| $5\cdot324.10^{10}$ | 1·43 | $7\cdot9255.10^2$ | $2\cdot5046.10^{-3}$ | 1·51 | $9\cdot8311.10^2$ | $7\cdot4035.10^2$ | $1\cdot984.10^{-3}$ | 1·505 | $1\cdot0328.10^3$ | $7\cdot5117.10^2$ |
| $1\cdot262.10^{11}$ | 1·433 | $6\cdot08.10^2$ | $2\cdot339.10^{-3}$ | 1·511 | $7\cdot5432.10^2$ | $5\cdot6937.10^2$ | $1\cdot848.10^{-3}$ | 1·506 | $7\cdot9241.10^2$ | $5\cdot7759.10^2$ |
| $1\cdot7969.10^{11}$ | 1·433 | $5\cdot446.10^2$ | $2\cdot284.10^{-3}$ | 1·511 | $6\cdot756.10^2$ | $5\cdot1046.10^2$ | $1\cdot806.10^{-3}$ | 1·506 | $7\cdot0966.10^2$ | $5\cdot1759.10^2$ |

## 4. NUMERICAL RESULTS

Numerical solutions for white dwarfs have been obtained for different values of the central densities (or $x_0$), using two models of white dwarfs. The results of these calculations are given in Table I and II, respectively.

The tables show the total mass $M_0$ and radius $R_0$ of non-rotating stars, and also the total mass and the equatorial as well as polar radii, both for the first- and second-order approximations in $\beta$. A variety of central densities in the range $10^6$ g cm$^{-3}$ to $10^{11}$ g cm$^{-3}$ have been considered. Comparing the integral parameters given by the two approximations for $\beta_{max}$, we find that:

(i) In the second-order approximation $\beta_{max}$ is about 25 per cent smaller than in the first-order approximation for both models and for all densities considered. Thus, value of $\beta_{max}$ does not change significantly in the second-order approximation.

(ii) For small central densities, i.e. $\rho_c \cong 10^6$ g cm$^{-3}$ the mass of the rotating star in the first-order approximation is increased by 17 to 20 per cent compared to the non-rotating case. The correction to the mass given by the second-order approximation is only of the order of 0·1 per cent.

In the first-order approximation the equatorial radius of the rotating star is increased by 30 per cent and the polar radius is decreased by 15 per cent with respect to the non-rotating star. The respective second-order corrections are only 4 per cent and 3 per cent.

(iii) For the higher central densities, i.e. $\rho_c \cong 10^{11}$ g cm$^{-3}$, the mass of the rotating star in the first-order approximation is increased by 5 to 8 per cent compared to the non-rotating case. The correction to the mass given by the second-order approximation is only of the order of 0·4 per cent.

In the first-order approximation the equatorial radius of the rotating star is increased by 20 per cent and the polar radius is decreased by 8 per cent with respect to the non-rotating case. The respective second-order corrections are only 5 per cent and 1 per cent.

Since the above results apply to both models of white dwarfs we conclude that the correction given by the second-order approximation is not appreciably dependent on the equation of state and is therefore also small for values of $\beta_{max}$.

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