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ON THE THEORY OF SURFACES IN THE FOUR-DIMENSIONAL EUCLIDEAN SPACE

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Abstract

For a two-dimensional surface M^2 in the four-dimensional Euclidean space $\mathbf{E}^4$ we introduce an invariant linear map of Weingarten type in the tangent space of the surface, which generates two invariants k and $\varkappa$.

The condition $k = \varkappa = 0$ characterizes the surfaces consisting of flat points. The minimal surfaces are characterized by the equality $\varkappa^2 - k = 0$. The class of the surfaces with flat normal connection is characterized by the condition $\varkappa = 0$. For the surfaces of general type we obtain a geometrically determined orthonormal frame field at each point and derive Frenet-type derivative formulas.

We apply our theory to the class of the rotational surfaces in $\mathbf{E}^4$, which prove to be surfaces with flat normal connection, and describe the rotational surfaces with constant invariants.

1. Introduction

In [4] T. Ōtsuki introduced curvatures $\lambda_1, \lambda_2, \dots, \lambda_n$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$) for a surface M^2 in a $(2+n)$-dimensional Euclidean space $\mathbf{E}^{2+n}$, defining a quadratic form in the normal space of the surface. In a suitable local frame of the normal space this quadratic form can be written in a diagonal form and the functions λ_α, $\alpha = 1, \dots, n$ are the coefficients in the diagonalized form (λ_α is called the *α-th curvature* of M^2). These curvatures are closely related to the Gauss curvature K of M^2:

$$K = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

The local cross-section, which diagonalizes the quadratic form is called a *Frenet cross-section* (*Frenet-frame*) of the surface.

For a surface M^2 in the four-dimensional Euclidean space $\mathbf{E}^4$ the curvatures λ_1 and λ_2 are the maximum and minimum, respectively of the Lipschitz-Killing curvature of the surface [5]. The function λ_1 is called the *principal curvature* and the function λ_2—the *secondary curvature* of M^2 in $\mathbf{E}^4$.

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Using the idea of the Frenet-frames, Shiohama [6] proved that a complete connected orientable surface M^2 in $\mathbf{E}^4$ with curvatures $\lambda_1 = \lambda_2 = 0$ is a cylinder. The same result is proved in [7] for a surface in a higher dimensional space $\mathbf{E}^{2+n}$.

Our aim is to find invariants of a surface M^2 in $\mathbf{E}^4$, considering a geometrically determined linear map (of Weingarten type) in the tangent space of the surface, as well as to obtain a geometric Frenet-type frame field of M^2.

In Section 2 we define a geometrical linear map in the tangent space of a surface M^2 in $\mathbf{E}^4$ and determine a second fundamental form II of the surface. We find invariants k and $\varkappa$ of M^2 (which are analogous to the Gauss curvature and the mean curvature of a surface in $\mathbf{E}^3$). These invariants divide the points of M^2 into four types: flat, elliptic, parabolic and hyperbolic.

In Section 3 we give a local geometric description of the surfaces consisting of flat points, proving that they are either planar surfaces (Proposition 3.1) or developable ruled surfaces (Proposition 3.2).

In Section 4 we characterize the minimal surfaces in $\mathbf{E}^4$ in terms of the invariants k and $\varkappa$ (Proposition 4.1).

For the surfaces of general type (which are not minimal and which have no flat points) in Section 5 we obtain a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point of the surface and derive Frenet-type derivative formulas. The tangent frame field $\{x, y\}$ is determined by the defined second fundamental form II, while the normal frame field $\{b, l\}$ is determined by the mean curvature vector field of the surface.

We also characterize the surfaces with flat normal connection in terms of the invariant $\varkappa$ (Theorem 5.1).

In the last section we apply our theory to the class of the rotational surfaces in $\mathbf{E}^4$, which prove to be surfaces with flat normal connection, and describe the rotational surfaces with $k = \text{const}$.

2. The Weingarten map

We denote by g the standard metric in the four-dimensional Euclidean space $\mathbf{E}^4$ and by ∇' its flat Levi-Civita connection. All considerations in the present paper are local and all functions, curves, surfaces, tensor fields etc. are assumed to be of the class $\mathscr{C}^\infty$.

Let $M^2 : z = z(u, v),\ (u, v) \in \mathscr{D}\ (\mathscr{D} \subset \mathbf{R}^2)$ be a 2-dimensional surface in $\mathbf{E}^4$. The tangent space to M^2 at an arbitrary point $p = z(u, v)$ of M^2 is $\operatorname{span}\{z_u, z_v\}$.

For an arbitrary orthonormal normal frame field $\{e_1, e_2\}$ of M^2 we have the standard derivative formulas:

$$
\begin{aligned}
\nabla'_{z_u} z_u &= z_{uu} = \Gamma_{11}^1 z_u + \Gamma_{11}^2 z_v + c_{11}^1 e_1 + c_{11}^2 e_2;\\
\nabla'_{z_u} z_v &= z_{uv} = \Gamma_{12}^1 z_u + \Gamma_{12}^2 z_v + c_{12}^1 e_1 + c_{12}^2 e_2;\\
\nabla'_{z_v} z_v &= z_{vv} = \Gamma_{22}^1 z_u + \Gamma_{22}^2 z_v + c_{22}^1 e_1 + c_{22}^2 e_2,
\end{aligned}
\tag{2.1}
$$

where Γ_{ij}^k are the Christoffel's symbols and $c_{ij}^k,\ i, j, k = 1, 2$ are functions on M^2.

We use the standard denotations $E(u,v) = g(z_u, z_u)$, $F(u,v) = g(z_u, z_v)$, $G(u,v) = g(z_v, z_v)$ for the coefficients of the first fundamental form and set $W = \sqrt{EG - F^2}$. If σ denotes the second fundamental tensor of M^2, then we have

$$\sigma(z_u, z_u) = c_{11}^1 e_1 + c_{11}^2 e_2,$$
$$\sigma(z_u, z_v) = c_{12}^1 e_1 + c_{12}^2 e_2,$$
$$\sigma(z_v, z_v) = c_{22}^1 e_1 + c_{22}^2 e_2.$$

We introduce the following functions:

$$\Delta_1 = \begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix};$$
$$L(u,v) = \frac{2\Delta_1}{W}, \quad M(u,v) = \frac{\Delta_2}{W}, \quad N(u,v) = \frac{2\Delta_3}{W}.$$

If

$$\begin{aligned} u &= u(\bar{u}, \bar{v}); \\ v &= v(\bar{u}, \bar{v}), \end{aligned} \quad (\bar{u}, \bar{v}) \in \overline{\mathscr{D}},\ \overline{\mathscr{D}} \subset \mathbf{R}^2 \tag{2.2}$$

is a smooth change of the parameters $\{u, v\}$ on M^2 with $J = u_{\bar{u}} v_{\bar{v}} - u_{\bar{v}} v_{\bar{u}} \neq 0$, then

$$z_{\bar{u}} = z_u u_{\bar{u}} + z_v v_{\bar{u}},$$
$$z_{\bar{v}} = z_u u_{\bar{v}} + z_v v_{\bar{v}}.$$

Let

$$\sigma(z_{\bar{u}}, z_{\bar{u}}) = \bar{c}_{11}^1 e_1 + \bar{c}_{11}^2 e_2,$$
$$\sigma(z_{\bar{u}}, z_{\bar{v}}) = \bar{c}_{12}^1 e_1 + \bar{c}_{12}^2 e_2,$$
$$\sigma(z_{\bar{v}}, z_{\bar{v}}) = \bar{c}_{22}^1 e_1 + \bar{c}_{22}^2 e_2.$$

Differentiating (2.2) and taking into account (2.1) we find

$$\begin{aligned} \bar{c}_{11}^k &= u_{\bar{u}}^2 c_{11}^k + 2u_{\bar{u}} v_{\bar{u}} c_{12}^k + v_{\bar{u}}^2 c_{22}^k, \\ \bar{c}_{12}^k &= u_{\bar{u}} u_{\bar{v}} c_{11}^k + (u_{\bar{u}} v_{\bar{v}} + u_{\bar{v}} v_{\bar{u}}) c_{12}^k + v_{\bar{u}} v_{\bar{v}} c_{22}^k, \quad (k = 1, 2) \\ \bar{c}_{22}^k &= u_{\bar{v}}^2 c_{11}^k + 2u_{\bar{v}} v_{\bar{v}} c_{12}^k + v_{\bar{v}}^2 c_{22}^k. \end{aligned} \tag{2.3}$$

Using (2.3), we obtain

$$\begin{aligned} \overline{\Delta}_1 &= J(u_{\bar{u}}^2 \Delta_1 + u_{\bar{u}} v_{\bar{u}} \Delta_2 + v_{\bar{u}}^2 \Delta_3); \\ \overline{\Delta}_2 &= J(2u_{\bar{u}} u_{\bar{v}} \Delta_1 + (u_{\bar{u}} v_{\bar{v}} + u_{\bar{v}} v_{\bar{u}}) \Delta_2 + 2v_{\bar{u}} v_{\bar{v}} \Delta_3); \\ \overline{\Delta}_3 &= J(u_{\bar{v}}^2 \Delta_1 + u_{\bar{v}} v_{\bar{v}} \Delta_2 + v_{\bar{v}}^2 \Delta_3). \end{aligned} \tag{2.4}$$

If $\bar{E} = g(z_{\bar{u}}, z_{\bar{u}})$, $\bar{F} = g(z_{\bar{u}}, z_{\bar{v}})$ and $\bar{G} = g(z_{\bar{v}}, z_{\bar{v}})$, then we have

$$\begin{aligned} \bar{E} &= u_{\bar{u}}^2 E + 2u_{\bar{u}} v_{\bar{u}} F + v_{\bar{u}}^2 G, \\ \bar{F} &= u_{\bar{u}} u_{\bar{v}} E + (u_{\bar{u}} v_{\bar{v}} + v_{\bar{u}} u_{\bar{v}}) F + v_{\bar{u}} v_{\bar{v}} G, \\ \bar{G} &= u_{\bar{v}}^2 E + 2u_{\bar{v}} v_{\bar{v}} F + v_{\bar{v}}^2 G \end{aligned} \tag{2.5}$$

and

$$\bar{E}\bar{G} - \bar{F}^2 = J^2(EG - F^2)$$

or

$$\overline{W} = \varepsilon J W, \quad \varepsilon = \operatorname{sign} J. \tag{2.6}$$

Taking into account (2.4) and (2.6), we find

$$\begin{aligned} \bar{L} &= \varepsilon(u_{\bar{u}}^2 L + 2u_{\bar{u}} v_{\bar{u}} M + v_{\bar{u}}^2 N), \\ \overline{M} &= \varepsilon(u_{\bar{u}} u_{\bar{v}} L + (u_{\bar{u}} v_{\bar{v}} + v_{\bar{u}} u_{\bar{v}}) M + v_{\bar{u}} v_{\bar{v}} N), \\ \overline{N} &= \varepsilon(u_{\bar{v}}^2 L + 2u_{\bar{v}} v_{\bar{v}} M + v_{\bar{v}}^2 N). \end{aligned} \tag{2.7}$$

Further we denote

$$\begin{aligned} \gamma_1^1 &= \frac{FM - GL}{EG - F^2}, \quad \gamma_1^2 = \frac{FL - EM}{EG - F^2}, \\ \gamma_2^1 &= \frac{FN - GM}{EG - F^2}, \quad \gamma_2^2 = \frac{FM - EN}{EG - F^2} \end{aligned} \tag{2.8}$$

and consider the linear map

$$\gamma : T_p M^2 \to T_p M^2$$

determined by the conditions

$$\begin{aligned} \gamma(z_u) &= \gamma_1^1 z_u + \gamma_1^2 z_v, \\ \gamma(z_v) &= \gamma_2^1 z_u + \gamma_2^2 z_v, \end{aligned} \qquad \gamma = \begin{pmatrix} \gamma_1^1 & \gamma_1^2 \\ \gamma_2^1 & \gamma_2^2 \end{pmatrix}. \tag{2.9}$$

Then a tangent vector $X = \lambda z_u + \mu z_v$ is transformed into the vector $X' = \gamma(X) = \lambda' z_u + \mu' z_v$ so that

$$\begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} = \gamma^t \begin{pmatrix} \lambda \\ \mu \end{pmatrix}.$$

We have

LEMMA 2.1. *The linear map γ given by* (2.9) *is geometrically determined.*

Proof. Let the change of the parameters be given by (2.2). Then we have

$$\begin{pmatrix} z_{\bar{u}} \\ z_{\bar{v}} \end{pmatrix} = T \begin{pmatrix} z_u \\ z_v \end{pmatrix}, \quad T = \begin{pmatrix} u_{\bar{u}} & v_{\bar{u}} \\ u_{\bar{v}} & v_{\bar{v}} \end{pmatrix}.$$

If we denote

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad h = \begin{pmatrix} L & M \\ M & N \end{pmatrix},$$

then the defining conditions (2.8) imply $\gamma = -hg^{-1}$.

With respect to the new coordinates $(\bar{u}, \bar{v})$ the linear map $\bar{\gamma}$ is determined by the equality $\bar{\gamma} = -\bar{h}\bar{g}^{-1}$.

On the other hand, the equalities (2.5) and (2.7) express that

$$\bar{g} = TgT^t, \quad \bar{h} = \varepsilon ThT^t.$$

Thus we obtain $\bar{\gamma} = -\bar{h}\bar{g}^{-1} = \varepsilon T\gamma T^{-1}$, which implies that $\bar{\gamma} = \varepsilon\gamma$.

Further, let $\{\tilde{e}_1, \tilde{e}_2\}$ be another orthonormal normal frame field of M^2. Then

$$\begin{aligned} e_1 &= \cos\theta \tilde{e}_1 + \varepsilon' \sin\theta \tilde{e}_2; \\ e_2 &= -\sin\theta \tilde{e}_1 + \varepsilon' \cos\theta \tilde{e}_2; \end{aligned} \qquad \theta = \angle(\tilde{e}_1, e_1),$$

and $\varepsilon' = 1$ $(\varepsilon' = -1)$ if the normal frame fields $\{e_1, e_2\}$ and $\{\tilde{e}_1, \tilde{e}_2\}$ have the same (opposite) orientation. The relation between the corresponding functions c_{ij}^k and $\tilde{c}_{ij}^k$, $i, j, k = 1, 2$ is given by the equalities

$$\begin{aligned} \tilde{c}_{ij}^1 &= \cos\theta c_{ij}^1 - \sin\theta c_{ij}^2; \\ \tilde{c}_{ij}^2 &= \varepsilon'(\sin\theta c_{ij}^1 + \cos\theta c_{ij}^2); \end{aligned} \qquad i, j = 1, 2.$$

Thus, $\tilde{\Delta}_i = \varepsilon'\Delta_i$, $i = 1, 2, 3$, and $\tilde{L} = \varepsilon' L$, $\tilde{M} = \varepsilon' M$, $\tilde{N} = \varepsilon' N$, which imply that $\tilde{\gamma} = \varepsilon'\gamma$. □

The linear map $\gamma : T_pM^2 \to T_pM^2$ is said to be the *Weingarten map* at the point $p \in M^2$. The following statement follows immediately from Lemma 2.1.

LEMMA 2.2. *The functions*

$$k := \det\gamma = \frac{LN - M^2}{EG - F^2}, \quad \varkappa := -\frac{1}{2}\operatorname{tr}\gamma = \frac{EN + GL - 2FM}{2(EG - F^2)} \tag{2.10}$$

are invariants of the surface M^2.

It is clear that the sign of $\varkappa$ depends on the orientations of the tangent plane and the normal space of M^2, while k is an absolute invariant.

The characteristic equation of the Weingarten map γ in view of Lemma 2.2 is

$$\nu^2 + 2\varkappa\nu + k = 0. \tag{2.11}$$

If X_1 and X_2 are two tangent vectors at a point $p \in M^2$, then $g(\gamma(X_1), X_2) = g(\gamma(X_2), X_1)$, i.e. γ is a symmetric linear operator and hence

$$\varkappa^2 - k \geq 0. \tag{2.12}$$

Using the defining equalities (2.10), it follows that

$$4(\varkappa^2 - k) = \left(\gamma_1^1 - \gamma_2^2 + 2\frac{F}{E}\gamma_1^2\right)^2 + 4\frac{EG - F^2}{E^2}(\gamma_1^2)^2.$$

This equality implies that the condition $\varkappa^2 - k = 0$ is equivalent to the equalities $\gamma_1^1 = \gamma_2^2$, $\gamma_1^2 = 0$, i.e. to the conditions

$$L = \rho E, \quad M = \rho F, \quad N = \rho G, \quad \rho \in \mathbf{R}. \tag{2.13}$$

Thus we get the following equivalence at a point $p \in M^2$:

$$L = M = N = 0 \quad \Leftrightarrow \quad k = \varkappa = 0. \tag{2.14}$$

As in the classical case (for a surface M^2 in $\mathbf{E}^3$), the invariants k and $\varkappa$ divide the points of M^2 into four types. A point $p \in M^2$ is said to be:

flat, if $k = \varkappa = 0$;
elliptic, if $k > 0$;
parabolic, if $k = 0$, $\varkappa \neq 0$;
hyperbolic, if $k < 0$.

Let $X = \lambda z_u + \mu z_v$, $(\lambda, \mu) \neq (0, 0)$ be a tangent vector at a point $p \in M^2$. The Weingarten map γ determines a second fundamental form of the surface M^2 at $p \in M^2$ as follows:

$$II(\lambda, \mu) = -g(\gamma(X), X) = L\lambda^2 + 2M\lambda\mu + N\mu^2, \quad \lambda, \mu \in \mathbf{R}.$$

First we study the class of surfaces whose points are flat.

3. Surfaces consisting of flat points

In this section we consider surfaces $M^2 : z = z(u, v)$, $(u, v) \in \mathcal{D}$ consisting of flat points, i.e. surfaces satisfying the conditions

$$k(u, v) = 0, \quad \varkappa(u, v) = 0, \quad (u, v) \in \mathcal{D}. \tag{3.1}$$

We give a local geometric description of these surfaces.

For the sake of simplicity, we shall assume that the parametrization of M^2 is orthogonal, i.e. $F = 0$. Denote the unit vector fields $x = \dfrac{z_u}{\sqrt{E}}$, $y = \dfrac{z_v}{\sqrt{G}}$. Then we write (2.1) in the form

$$\begin{array}{rlll}
\nabla'_x x = & \gamma_1 y & + \dfrac{c_{11}^1}{E} e_1 & + \dfrac{c_{11}^2}{E} e_2, \\[2ex]
\nabla'_x y = & -\gamma_1 x & + \dfrac{c_{12}^1}{\sqrt{EG}} e_1 & + \dfrac{c_{12}^2}{\sqrt{EG}} e_2, \\[2ex]
\nabla'_y x = & -\gamma_2 y & + \dfrac{c_{12}^1}{\sqrt{EG}} e_1 & + \dfrac{c_{12}^2}{\sqrt{EG}} e_2, \\[2ex]
\nabla'_y y = & \gamma_2 x & + \dfrac{c_{22}^1}{G} e_1 & + \dfrac{c_{22}^2}{G} e_2
\end{array} \tag{3.2}$$

Obviously, the surface M^2 lies in a 2-plane if and only if M^2 is totally geodesic, i.e. $c_{ij}^k = 0,\ i,j,k = 1,2$.

Now, let at least one of the coefficients c_{ij}^k not be zero. Then

$$\operatorname{rank}\begin{pmatrix} c_{11}^1 & c_{12}^1 & c_{22}^1 \\ c_{11}^2 & c_{12}^2 & c_{22}^2 \end{pmatrix} = 1$$

and the vectors $\sigma(x,x)$, $\sigma(x,y)$, $\sigma(y,y)$ are collinear. Let $\{b,l\}$ be a normal frame field of M^2, consisting of orthonormal vector fields, such that b is collinear with $\sigma(x,x)$, $\sigma(x,y)$, and $\sigma(y,y)$. It is clear that the normal frame field $\{b,l\}$ is invariant. Then the derivative formulas of M^2 can be written as follows:

$$\begin{array}{llll} & \nabla'_x x = \gamma_1 y + \nu_1 b, & \nabla'_x b = -\nu_1 x - \lambda y + \beta_1 l, \\ & \nabla'_x y = -\gamma_1 x + \lambda b, & \nabla'_y b = -\lambda x - \nu_2 y + \beta_2 l, \\ (3.3) & \nabla'_y x = -\gamma_2 y + \lambda b, & \nabla'_x l = -\beta_1 b, \\ & \nabla'_y y = \gamma_2 x + \nu_2 b, & \nabla'_y l = -\beta_2 b, \end{array}$$

for some functions ν_1, ν_2, λ, β_1, β_2, γ_1, γ_2 on M^2.

The Gauss curvature K of M^2 is expressed by

$$K = \nu_1 \nu_2 - \lambda^2. \tag{3.4}$$

Further we denote $\beta = \beta_1^2 + \beta_2^2$. It follows immediately that β does not depend on the change (2.2) of the parameters.

Since the curvature tensor R' of the connection ∇' is zero, then the equalities $R'(x,y,b) = 0$ and $R'(x,y,l) = 0$ together with (3.3) imply that either $K = 0$ or $\beta = 0$.

A surface M^2 is said to be *planar* if there exists a hyperplane $\mathbf{E}^3 \subset \mathbf{E}^4$ containing M^2. First we shall characterize the planar surfaces.

Proposition 3.1. *A surface M^2 is planar if and only if*

$$k = 0, \quad \varkappa = 0, \quad \beta = 0.$$

Proof. I. Let $M^2 \subset \mathbf{E}^3$ and b be the usual normal to M^2 in $\mathbf{E}^3$. Choosing l to be the normal to the hyperplane $\mathbf{E}^3$, from (3.3) we get $L = M = N = 0$ and $\beta = 0$.

II. Under the conditions $k = \varkappa = \beta = 0$, from (3.3) it follows that $l = \text{const}$ and M^2 lies in a hyperplane $\mathbf{E}^3$ orthogonal to l. □

A *ruled surface* M^2 is a one-parameter system $\{g(v)\}$, $v \in J$ of straight lines $g(v)$, defined in an interval $J \subset \mathbf{R}$. The straight lines $g(v)$ are called generators of M^2. A ruled surface $M^2 = \{g(v)\}$, $v \in J$ is said to be *developable*, if the tangent space $T_p M^2$ at all regular points p of an arbitrary fixed generator $g(v)$ is one and the same.

Each ruled surface M^2 can be parameterized as follows:

$$z(u,v) = x(v) + ue(v), \quad u \in \mathbf{R},\ v \in J, \tag{3.5}$$

where $x(v)$ and $e(v)$ are vector-valued functions, defined in J, such that the vectors $e(v)$ and $x'(v) + ue'(v)$ are linearly independent for all $v \in J$. The tangent space of M^2 is spanned by the vectors

$$\begin{aligned} z_u &= e(v); \\ z_v &= x'(v) + ue'(v). \end{aligned}$$

The ruled surface M^2 determined by (3.5) is developable if and only if the vectors $e(v)$, $e'(v)$ and $x'(v)$ are linearly dependent.

We shall characterize the developable ruled surfaces in terms of the invariants k, $\varkappa$ and the Gauss curvature K.

Proposition 3.2. *A surface M^2 is locally a developable ruled surface if and only if*

$$k = 0, \quad \varkappa = 0, \quad K = 0.$$

Proof. I. Let M^2 be a developable ruled surface, defined by the equality (3.5), where $e(v)$, $e'(v)$ and $x'(v)$ are linearly dependent. Without loss of generality we assume that $\|e(v)\| = 1$. Then, the vector fields $e(v)$ and $e'(v)$ are orthogonal and the tangent space of M^2 is $\operatorname{span}\{e(v), e'(v)\}$. Since $x'(v) \in \operatorname{span}\{e(v), e'(v)\}$, then $x'(v)$ is decomposed in the form $x'(v) = p(v)e(v) + q(v)e'(v)$ for some functions $p(v)$ and $q(v)$. Hence, the tangent space of M^2 is spanned by

$$\begin{aligned} z_u &= e; \\ z_v &= pe + (u+q)e'. \end{aligned}$$

Considering only the regular points of M^2 (where $u \neq -q$), we choose an orthonormal tangent frame field $\{x, y\}$ of M^2 in the following way:

$$\begin{aligned} x &= e = z_u; \\ y &= \frac{e'}{\|e'\|} = -\frac{p}{(u+q)\|e'\|} z_u + \frac{1}{(u+q)\|e'\|} z_v. \end{aligned} \tag{3.6}$$

Since the tangent space of M^2 does not depend on the parameter u, then the normal space of M^2 is spanned by vector fields $b_1(v)$, $b_2(v)$. With respect to the basis $\{e(v), e'(v), b_1(v), b_2(v)\}$ the derivatives of $b_1(v)$ and $b_2(v)$ are decomposed in the form

$$\begin{aligned} b_1' &= -c_1 e' + c_0 b_2, \\ b_2' &= -c_2 e' - c_0 b_1, \end{aligned} \tag{3.7}$$

where c_0, c_1, c_2 are functions of v.

Then the equalities (3.6) and (3.7) imply

$$\nabla'_x b_1 = 0,$$

$$\nabla'_y b_1 = -\frac{c_1}{u+q}\, y + \frac{c_0}{(u+q)\|e'\|}\, b_2,$$

$$\nabla'_x b_2 = 0,$$

$$\nabla'_y b_2 = -\frac{c_2}{u+q}\, y - \frac{c_0}{(u+q)\|e'\|}\, b_1.$$

Consequently, $L = M = N = 0$ and $K = 0$.

II. Let M^2 be a surface for which $L = M = N = 0$ and $K = 0$. We consider an orthonormal frame field $\{x, y, b, l\}$ of M^2, satisfying the equalities (3.3). Since $K = 0$, then $\nu_1 \nu_2 - \lambda^2 = 0$. If $\nu_1 = \nu_2 = 0$, then M^2 lies in a plane $\mathbf{E}^2$. So we assume that there exists a neighborhood $\tilde{\mathscr{D}} \subset \mathscr{D}$ such that $\nu_{2|\tilde{\mathscr{D}}} \neq 0$ (or $\nu_{1|\tilde{\mathscr{D}}} \neq 0$) and we consider the surface $\tilde{M}^2 = M^2_{|\tilde{\mathscr{D}}}$.

Let $\{\bar{x}, \bar{y}\}$ be the orthonormal tangent frame field of $\tilde{M}^2$, defined by

$$\bar{x} = \cos\varphi x + \sin\varphi y;$$

$$\bar{y} = -\sin\varphi x + \cos\varphi y,$$

where $\tan\varphi = -\dfrac{\lambda}{\nu_2}$. Then $\sigma(\bar{x}, \bar{x}) = 0$, $\sigma(\bar{x}, \bar{y}) = 0$. So the formulas (3.3) take the form

$$\begin{array}{llll}
\nabla'_{\bar{x}}\bar{x} = \bar{\gamma}_1 \bar{y}, & & \nabla'_{\bar{x}} b = & \bar{\beta}_1 l,\\
\nabla'_{\bar{x}}\bar{y} = -\bar{\gamma}_1 \bar{x}, & & \nabla'_{\bar{y}} b = -\bar{\nu}_2 \bar{y} & +\bar{\beta}_2 l,\\
\nabla'_{\bar{y}}\bar{x} = -\bar{\gamma}_2 \bar{y}, & & \nabla'_{\bar{x}} l = & -\bar{\beta}_1 b,\\
\nabla'_{\bar{y}}\bar{y} = \bar{\gamma}_2 \bar{x} & +\bar{\nu}_2 b, & \nabla'_{\bar{y}} l = & -\bar{\beta}_2 b,
\end{array}$$

where $\bar{\nu}_2 \neq 0$.

Since the curvature tensor R' is zero, then the equalities $R'(\bar{x}, \bar{y}, b) = 0$ and $R'(\bar{x}, \bar{y}, l) = 0$ imply that

$$\bar{\gamma}_1 = 0, \quad \bar{\beta}_1 = 0.$$

Hence,

$$\nabla'_{\bar{x}}\bar{x} = 0, \quad \nabla'_{\bar{x}} b = 0,$$

$$\nabla'_{\bar{x}}\bar{y} = 0, \quad \nabla'_{\bar{x}} l = 0.$$

Let $p = z(\bar{u}_0, \bar{v}_0)$, $(\bar{u}_0, \bar{v}_0) \in \tilde{\mathscr{D}}$ be an arbitrary point of $\tilde{M}^2$ and $c_1 : z(\bar{u}) = z(\bar{u}, \bar{v}_0)$ be the integral curve of the vector field $\bar{x}$, passing through p. It follows from $\nabla'_{\bar{x}}\bar{x} = 0$ that c_1 is contained in a straight line. Hence, $\tilde{M}^2$ lies on a ruled surface. Moreover, since $\nabla'_{\bar{x}} b = 0$ and $\nabla'_{\bar{x}} l = 0$ then the normal space $\operatorname{span}\{b, l\}$ of $\tilde{M}^2$ is constant at the points of c_1 and hence, the tangent space $\operatorname{span}\{\bar{x}, \bar{y}\}$ of $\tilde{M}^2$ at the points of c_1 is one and the same. Consequently, $\tilde{M}^2$ is part of a developable surface. □

From now on we exclude the flat points from our considerations.

4. Minimal surfaces

We recall that a surface M^2 is said to be *minimal* if the mean curvature vector $H = 0$. In this section we characterize the minimal surfaces in terms of the invariants k and $\varkappa$.

PROPOSITION 4.1. *Let M^2 be a surface in $\mathbf{E}^4$ without flat points. Then M^2 is minimal if and only if*

$$\varkappa^2 - k = 0.$$

Proof. Without loss of generality we assume that $F = 0$ and denote the unit vector fields $x = \frac{z_u}{\sqrt{E}}$, $y = \frac{z_v}{\sqrt{G}}$. Then we have

$$\begin{aligned}
\nabla'_x x &= \qquad \gamma_1 y + \frac{c_{11}^1}{E} e_1 \quad + \frac{c_{11}^2}{E} e_2,\\
\nabla'_x y &= -\gamma_1 x \qquad + \frac{c_{12}^1}{\sqrt{EG}} e_1 + \frac{c_{12}^2}{\sqrt{EG}} e_2,\\
\nabla'_y x &= \qquad -\gamma_2 y + \frac{c_{12}^1}{\sqrt{EG}} e_1 + \frac{c_{12}^2}{\sqrt{EG}} e_2,\\
\nabla'_y y &= \quad \gamma_2 x \qquad + \frac{c_{22}^1}{G} e_1 \quad + \frac{c_{22}^2}{G} e_2.
\end{aligned}$$

I. Let $H = \frac{1}{2}(\sigma(x, x) + \sigma(y, y)) = 0$. Then

$$\Delta_2 = \begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix} = 0, \quad \frac{\Delta_3}{G} = \frac{\Delta_1}{E}.$$

Therefore

$$L = \rho E, \quad M = \rho F, \quad N = \rho G,$$

where ρ is a function on M^2. Hence $\varkappa^2 - k = 0$.

II. Let $\varkappa^2 - k = 0$. Then

$$L = \rho E, \quad M = \rho F, \quad N = \rho G; \quad \rho \neq 0.$$

The condition $F = 0$ implies that $M = 0$. Then $\begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix} = 0$ and $c_{22}^1 = \tilde{\rho} c_{11}^1$, $c_{22}^2 = \tilde{\rho} c_{11}^2$. Further, the equality $\frac{L}{E} = \frac{N}{G}$ implies that $\tilde{\rho} = -\frac{G}{E}$. Hence $\operatorname{tr} \sigma = 0$, i.e. $H = 0$. □

5. Surfaces of general type

From now on we consider surfaces, satisfying the condition

$$\varkappa^2 - k \neq 0$$

and call them surfaces *of general type*.

As in the classical differential geometry of surfaces in $\mathbf{E}^3$ the second fundamental form determines conjugate tangents at a point p of M^2. A tangent $g : X = \lambda z_u + \mu z_v$ is said to be *principal* if it is perpendicular to its conjugate. The equation for the principal tangents at a point $p \in M^2$ is

$$\begin{vmatrix} E & F \\ L & M \end{vmatrix} \lambda^2 + \begin{vmatrix} E & G \\ L & N \end{vmatrix} \lambda\mu + \begin{vmatrix} F & G \\ M & N \end{vmatrix} \mu^2 = 0.$$

A line $c : u = u(q),\ v = v(q);\ q \in J$ on M^2 is said to be a *principal* curve if its tangent at any point is principal.

The surface M^2 is parameterized with respect to the principal lines if and only if

$$F = 0, \quad M = 0.$$

Let M^2 be parameterized with respect to the principal lines and denote the unit vector fields $x = \dfrac{z_u}{\sqrt{E}},\ y = \dfrac{z_v}{\sqrt{G}}$.

Since the mean curvature vector field $H \neq 0$, we determine the unit normal vector field b by the equality $b = \dfrac{H}{\|H\|}$. Further we denote by l the unit normal vector field such that $\{x, y, b, l\}$ is a positive oriented orthonormal frame field of M^2. Thus we obtain a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point $p \in M^2$. With respect to the frame field $\{x, y, b, l\}$ we have the following Frenet-type derivative formulas:

$$(5.1) \qquad \begin{array}{llll} \nabla'_x x = & \gamma_1 y + \nu_1 b; & \nabla'_x b = -\nu_1 x - \lambda y & + \beta_1 l; \\ \nabla'_x y = -\gamma_1 x & + \lambda b + \mu l; & \nabla'_y b = -\lambda x - \nu_2 y & + \beta_2 l; \\ \nabla'_y x = & -\gamma_2 y + \lambda b + \mu l; & \nabla'_x l = & -\mu y - \beta_1 b; \\ \nabla'_y y = \gamma_2 x & + \nu_2 b; & \nabla'_y l = -\mu x & - \beta_2 b, \end{array}$$

where $\gamma_1 = -y(\ln \sqrt{E}),\ \gamma_2 = -x(\ln \sqrt{G})$ and $\mu \neq 0$.

Hence we have

$$(5.2) \qquad k = -4\nu_1\nu_2\mu^2, \quad \varkappa = (\nu_1 - \nu_2)\mu, \quad K = \nu_1\nu_2 - (\lambda^2 + \mu^2).$$

Remark 1. We note that we determine the tangent frame field $\{x, y\}$ by the Weingarten map (the second fundamental form II) and the normal frame field $\{b, l\}$—by the mean curvature vector field, while the Frenet-cross section in the sense of Ōtsuki diagonalizes a quadratic form in the normal space. In general the geometric frame field $\{b, l\}$ is not a Frenet-cross section. Finding the relation between $\{b, l\}$ and the Frenet-cross section of Ōtsuki we derive the following relation between the invariant k and the curvatures λ_1 and λ_2 of Ōtsuki:

$$k = 4\lambda_1\lambda_2.$$

The same formula is valid in the cases of minimal surfaces and surfaces consisting of flat points.

Using (5.1) we find the length $\|H\|$ of the mean curvature vector field and taking into account (5.2) we obtain the formula

$$\|H\| = \frac{\sqrt{\varkappa^2 - k}}{2|\mu|},$$

which shows that $|\mu|$ is expressed by the invariants k, $\varkappa$ and the mean curvature function.

Let $z = g(z,x)x + g(z,y)y$ be an arbitrary tangent vector field of M^2. We define the one-form θ by the equality

$$\theta(z) = g(\nabla'_z b, l).$$

Then the formulas (5.1) imply that

$$\theta(z) = g(\beta_1 x + \beta_2 y, z),$$

which shows that the one-form θ corresponds to the tangent vector field $\beta_1 x + \beta_2 y$ and

$$\|\theta\| = \sqrt{\beta_1^2 + \beta_2^2}.$$

Using that $R'(x,y,x) = 0$, $R'(x,y,y) = 0$, $R'(x,y,b) = 0$ and $R'(x,y,l) = 0$, we get the following integrability conditions:

$$(5.3)\qquad \begin{aligned}
&\nu_1\nu_2 - (\lambda^2 + \mu^2) = x(\gamma_2) + y(\gamma_1) - ((\gamma_1)^2 + (\gamma_2)^2);\\
&2\mu\gamma_2 + \nu_1\beta_2 - \lambda\beta_1 = x(\mu);\\
&2\mu\gamma_1 - \lambda\beta_2 + \nu_2\beta_1 = y(\mu);\\
&2\lambda\gamma_2 + \mu\beta_1 - (\nu_1 - \nu_2)\gamma_1 = x(\lambda) - y(\nu_1);\\
&2\lambda\gamma_1 + \mu\beta_2 + (\nu_1 - \nu_2)\gamma_2 = -x(\nu_2) + y(\lambda);\\
&\gamma_1\beta_1 - \gamma_2\beta_2 + (\nu_1 - \nu_2)\mu = -x(\beta_2) + y(\beta_1).
\end{aligned}$$

At the end of this section we shall characterize the surfaces with flat normal connection in terms of the invariant $\varkappa$.

A surface M^2 is said to be of *flat normal connection* [3] if the normal curvature $R^\perp$ of M^2 is zero. The equalities (5.1) imply that the normal curvature $R^\perp$ of M^2 is expressed as follows:

$$(5.4)\qquad \begin{aligned}
R_b^\perp(x,y) &= D_xD_yb - D_yD_xb - D_{[x,y]}b = (x(\beta_2) - y(\beta_1) + \gamma_1\beta_1 - \gamma_2\beta_2)l,\\
R_l^\perp(x,y) &= D_xD_yl - D_yD_xl - D_{[x,y]}l = -(x(\beta_2) - y(\beta_1) + \gamma_1\beta_1 - \gamma_2\beta_2)b.
\end{aligned}$$

Taking in mind (5.4) and the last equality of (5.3) we get:

$$R_b^{\perp}(x, y) = -\varkappa l,$$
$$R_l^{\perp}(x, y) = \varkappa b,$$

i.e.

$$\varkappa = g(R_l^{\perp}(x, y), b) = g(R^{\perp}(x, y)l, b).$$

The function $g(R^{\perp}(x, y)l, b)$ is the curvature of the normal connection D of M^2. Hence, the invariant $\varkappa$ is the curvature of the normal connection.

Thus the surfaces with flat normal connection are characterized by the following

Proposition 5.1. *A surface M^2 in $\mathbf{E}^4$ is of flat normal connection if and only if*

$$\varkappa = 0.$$

Obviously, M^2 is a surface with flat normal connection if and only if $\nu_1 = \nu_2 =: \nu$. So, the Frenet-type formulas (5.1) of a surface M^2 with flat normal connection take the form:

$$\begin{array}{llll} \nabla'_x x = & \gamma_1 y + \nu b; & \nabla'_x b = -\nu x - \lambda y & + \beta_1 l; \\ \nabla'_x y = -\gamma_1 x & + \lambda b + \mu l; & \nabla'_y b = -\lambda x - \nu y & + \beta_2 l; \\ \nabla'_y x = & -\gamma_2 y + \lambda b + \mu l; & \nabla'_x l = & -\mu y - \beta_1 b; \\ \nabla'_y y = \gamma_2 x & + \nu b; & \nabla'_y l = -\mu x & - \beta_2 b. \end{array} \tag{5.5}$$

Hence the invariants k and K are expressed by

$$k = -4\nu^2\mu^2, \quad K = \nu^2 - (\lambda^2 + \mu^2).$$

Remark 2. The curvature of the normal connection of a surface M^2 in $\mathbf{E}^4$ is the Gauss torsion $\varkappa_G$ of M^2 [1]. The notion of the Gauss torsion is introduced by É. Cartan [2] for a p-dimensional submanifold of an n-dimensional Riemannian manifold and is given by the Euler curvatures. In case of a 2-dimensional surface M^2 in $\mathbf{E}^4$ the Gauss torsion can be expressed in terms of the ellipse of normal curvature at a point $p \in M^2$.

According to the theorem of Rodrigues, a curve c on a surface M^2 in $\mathbf{E}^3$ is a line of curvature if and only if the tangential component of the derivative of the normal vector field to M^2 along c is collinear with the tangent of c. Using this geometric characterization of the lines of curvature for surfaces in $\mathbf{E}^3$, É. Cartan generalized in [2] the notion of lines of curvature for a surface M^2 in $\mathbf{E}^4$. However, the lines of curvature in the sense of Cartan exist only in the class of the surfaces with zero Gauss torsion ($\varkappa_G = 0$), i.e. in the class of the surfaces with flat normal connection.

6. Rotational surfaces

Now we shall apply our theory to the class of the rotational surfaces in $\mathbf{E}^4$.

We denote by $Oe_1e_2e_3$ a fixed orthonormal base of $\mathbf{E}^3$. Let $\boldsymbol{c} : \tilde{z} = \tilde{z}(u)$, $u \in J$ be a smooth curve in $\mathbf{E}^3$, parameterized by

$$\tilde{z}(u) = (x_1(u), x_2(u), r(u)); \quad u \in J.$$

We denote by $\boldsymbol{c}_1$ the projection of $\boldsymbol{c}$ on the 2-dimensional plane Oe_1e_2.

Without loss of generality we can assume that $\boldsymbol{c}$ is parameterized with respect to the arc-length, i.e. $(x_1')^2 + (x_2')^2 + (r')^2 = 1$. We assume also that $r(u) > 0$, $u \in J$. Let us consider the rotational surface M^2 in $\mathbf{E}^4$ given by

$$z(u,v) = (x_1(u), x_2(u), r(u)\cos v, r(u)\sin v); \quad u \in J,\ v \in [0; 2\pi). \tag{6.1}$$

The tangent space of M^2 is spanned by the vector fields

$$z_u = (x_1', x_2', r'\cos v, r'\sin v);$$
$$z_v = (0, 0, -r\sin v, r\cos v).$$

Hence,

$$E = 1; \quad F = 0; \quad G = r^2(u); \quad W = r(u).$$

We consider the following orthonormal tangent vector fields

$$\bar{x} = (x_1', x_2', r'\cos v, r'\sin v);$$
$$\bar{y} = (0, 0, -\sin v, \cos v),$$

i.e. $z_u = \bar{x}$; $z_v = r\bar{y}$. The second partial derivatives of $z(u,v)$ are expressed as follows

$$z_{uu} = (x_1'', x_2'', r''\cos v, r''\sin v);$$
$$z_{uv} = (0, 0, -r'\sin v, r'\cos v);$$
$$z_{vv} = (0, 0, -r\cos v, -r\sin v).$$

Let κ and τ be the curvature and the torsion of the curve $\boldsymbol{c}$ (considered as a curve in $\mathbf{E}^3$). We consider the normal vector fields e_1 and e_2, defined by

$$e_1 = \frac{1}{\kappa}(x_1'', x_2'', r''\cos v, r''\sin v);$$
$$e_2 = \frac{1}{\kappa}(x_2'r'' - x_2''r', x_1''r' - x_1'r'', (x_1'x_2'' - x_1''x_2')\cos v, (x_1'x_2'' - x_1''x_2')\sin v).$$

Now it is easy to calculate that

$$L = 0; \quad M = -(x_1'x_2'' - x_1''x_2'); \quad N = 0.$$

Hence,

$$k = -\frac{(x_1'x_2'' - x_1''x_2')^2}{r^2}; \quad \varkappa = 0.$$

Applying Proposition 5.1 we get

COROLLARY 6.1. *Any rotational surface M^2 in $\mathbf{E}^4$, defined by* (6.1), *is a surface with flat normal connection.*

Let us denote the curvature of the plane curve $\boldsymbol{c}_1$ by $\kappa_1 = x_1' x_2'' - x_1'' x_2'$. Then with respect to the frame field $\{\bar{x}, \bar{y}, e_1, e_2\}$ the derivative formulas of M^2 look like:

$$\begin{array}{llll}
\nabla'_{\bar{x}}\bar{x} = & \kappa e_1; & \nabla'_{\bar{x}}e_1 = -\kappa\bar{x} & + \tau e_2; \\
\nabla'_{\bar{x}}\bar{y} = 0; & & \nabla'_{\bar{y}}e_1 = & \dfrac{r''}{\kappa r}\,\bar{y}; \\
\nabla'_{\bar{y}}\bar{x} = & \dfrac{r'}{r}\,\bar{y}; & \nabla'_{\bar{x}}e_2 = & -\tau e_1; \\
\nabla'_{\bar{y}}\bar{y} = -\dfrac{r'}{r}\,\bar{x} & -\dfrac{r''}{\kappa r}e_1 - \dfrac{\kappa_1}{\kappa r}e_2; & \nabla'_{\bar{y}}e_2 = & \dfrac{\kappa_1}{\kappa r}\,\bar{y}.
\end{array}$$

So, the Gauss curvature of M^2 is:

$$K = -\frac{r''}{r}.$$

Obviously M^2 is not parameterized with respect to the principal lines. The principal tangents of M^2 are:

$$\begin{array}{l}
x = \dfrac{\sqrt{2}}{2}\,\bar{x} + \dfrac{\sqrt{2}}{2}\,\bar{y}; \\
y = \dfrac{\sqrt{2}}{2}\,\bar{x} - \dfrac{\sqrt{2}}{2}\,\bar{y}
\end{array}.$$

With respect to the geometric frame field $\{x, y, b, l\}$ the Frenet-type formulas (5.5) hold good, where

$$\begin{array}{ll}
\gamma_1 = \gamma_2 = -\dfrac{\sqrt{2}}{2}\dfrac{r'}{r}; & \nu = \dfrac{\sqrt{(\kappa^2 r - r'')^2 + (\kappa_1)^2}}{2\kappa r}; \\
\lambda = \dfrac{\kappa^4 r^2 - (r'')^2 - (\kappa_1)^2}{2\kappa r}\sqrt{(\kappa^2 r - r'')^2 + (\kappa_1)^2}; & \mu = \dfrac{\kappa\kappa_1}{\sqrt{(\kappa^2 r - r'')^2 + (\kappa_1)^2}}.
\end{array}$$

Consequently, the invariants k, $\varkappa$ and K of the rotational surface M^2 are:

$$k = -\frac{(\varkappa_1)^2}{r^2}; \quad \varkappa = 0; \quad K = -\frac{r''}{r}.$$

At the end of the section we shall describe all rotational surfaces, for which the invariant k is constant.

1. The invariant $k = 0$ if and only if $\kappa_1 = 0$, which means that the projection of the curve $\boldsymbol{c}$ on the plane Oe_1e_2 lies on a straight line. There are two subcases:

1.1. If $K = 0$, i.e. $r'' = 0$, then M^2 is a developable ruled surface.

1.2. If $K \neq 0$, i.e. $r'' \neq 0$, then M^2 is a planar surface.

2. The invariant $k = \text{const}$ $(k \neq 0)$ if and only if $r(u) = a(x_1' x_2'' - x_1'' x_2')$, $a = \text{const}$. Moreover, if $r(u)$ satisfies $r''(u) = cr(u)$, then the Gauss curvature K is also a constant.

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