On the Theory of the Advance and Retreat of Glaciers

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Summary

The main effect on a glacier of a change of climate is to change the rate of supply (accumulation) and removal (ablation) of ice. As a result the end of the glacier changes in position. The differential equations governing this phenomenon, derived in a previous paper, show that the effects of changes of accumulation and ablation are propagated down a glacier as kinematic waves. The present paper examines the effect of diffusion of the waves, since diffusion can have a large effect on quantitative results.

The validity of the basic assumptions is re-examined and particular attention is given to the proper choice of boundary conditions. A special model which shows a realistic amount of diffusion allows an explicit solution for the response to any variation in rate of accumulation, in terms of certain averages over past history. The responses to a step-function and to a pulse are found, and also the response to a harmonic variation of rate of accumulation as a function of frequency. The inverse problem of calculating the climatic changes from the variation of the position of the end of the glacier has a simple general solution for this model. An interesting asymmetry is then revealed between two inference problems: to calculate the current glacier behaviour requires a record of the climate extending back for many hundreds of years; but to calculate the current climate only requires a record of the glacier behaviour over the very recent past (say 10 yr).

For an actual glacier the frequency response can be calculated by numerical methods. The usefulness of the special model considered in this paper is that it indicates the general nature of the results to be expected from numerical analysis.

1. Introduction

The response of a glacier to changes in climate may be studied theoretically by considering a glacier as a one-dimensional flow system. In the higher parts of a glacier the annual supply of new ice by snowfall exceeds the annual loss by melting and evaporation, while in the lower parts the reverse is true. On balance, therefore, new material enters the system at all points above the snow line; it flows downwards and is removed at all points below the snow line. When there is no more ice left to be removed, the glacier ends.

The main effect of a change in climate is to change the rates of supply and removal of ice. If the glacier is not at the melting point throughout, a change of climate may also change the temperature of the ice; this is an effect not taken into account in the present treatment, which is therefore only strictly applicable to temperate glaciers. Thus, the climate will here be specified by giving the rate of supply of ice to the glacier, the "rate of accumulation", as a function of x, the distance down the glacier, and t, the time. Specifically, $a(x, t)$ is the thickness of ice added to the upper surface of the glacier per unit time; it may be positive (accumulation) or negative (ablation). A change in climate causes a change in the position of the end of the glacier, the snout, and this will be our main concern. There are two problems: (1) to find out how to calculate the response of a given glacier to a given change of climate, and (2) to find out whether it is possible, by observing the fluctuations of a glacier snout, to calculate the changes in climate which have caused them.

In a previous paper (Nye 1960), which will be referred to as I, the basic differential equations which govern the response of a glacier to climate were derived. It was shown that the effects of changes in $a(x, t)$ are transmitted down the glacier as kinematic waves which travel at 2 to 5 times the surface speed of the ice itself. It was also shown that these waves are subject to diffusion, whereby they tend to decrease in sharpness as they travel. The emphasis in I was on solutions obtained by ignoring the diffusion effect, and a broad understanding of the response of glaciers to climate was reached by that approximation. The order of magnitude of the diffusion to be expected in practice was estimated, in certain cases, on the basis of these diffusionless solutions. A short non-mathematical account is given in (Nye 1961).

When diffusion is properly taken into account from the beginning, the mathematical problem is to solve a type of one-dimensional diffusion equation—the diffusivity being a function of position which goes to zero at each end of the range of x. In the present paper (II) this problem is approached by considering a special model which has a realistic diffusivity, and for which there exist simple solutions in closed form. The results for the chosen model can thus be studied in full detail—in particular the model gives simple explicit answers to problems (1) and (2) posed above.

The application of the present type of theory to an actual glacier, where the necessary parameters are given in numerical or graphical form, is taken up in a further paper (Nye 1963a) referred to as III. Whereas the present paper deals with an infinitely wide glacier, III takes account of the finite and non-uniform cross-section of the glacier valley. In this general case, where there is naturally no analytical solution in closed form, the result—in fact the frequency response—is ultimately expressed in purely numerical or graphical form for the particular glacier selected. The usefulness of the solutions for the special model in the present paper is that, being algebraic rather than numerical, they serve as a guide to the type of result to be expected from the numerical analysis. For instance, they throw light on the time scales likely to be encountered, and also on the nature of the general problems (1) and (2), by showing what length of historical record of glacier behaviour is likely to be needed before inferences can be drawn about climatic trends—and, conversely, what length of climatic record suffices to determine current glacier behaviour. A brief account of some of the results of this paper and of III has been given in (Nye 1963b).

2. The general differential equations

We consider the same general model as in I, with the same notation. The glacier is represented by a sheet of ice which flows down a slope of non-uniform inclination $\beta(x)$ in the x direction, the sheet being of unlimited extent in the horizontal direction perpendicular to x. $\beta(x)$ is supposed to be a slowly varying function of x.

Suppose first that there is a steady rate of accumulation, so that $a(x, t)$, as already defined, is equal to $a_0(x)$, say. $a_0(x)$ will be some function which is positive in the upper parts of the glacier and negative in the lower parts. Corresponding to $a_0(x)$ there will be a certain steady-state configuration of the glacier, and we shall take this as our datum state. We have not yet said precisely how x is to be measured, but we now specify that x is to be measured along the upper surface of this steady-state datum glacier.

In general, the glacier will not be in the datum state. We define $h(x, t)$ as the thickness of the glacier measured perpendicular to the x direction, and we define $\alpha(x, t)$ as the inclination of the upper surface, being positive for a downhill slope in the x direction. The volume of ice, for unit breadth, passing in unit time through a given section perpendicular to the x direction is denoted by $q(x, t)$ and is called the "flow" or "discharge". It is assumed that the ice is incompressible, which is a good approximation except in the upper layers of the firn region, and we therefore have a continuity equation

$$\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = a(x, t), \qquad (1)$$

where, to be strict, the accumulation rate $a(x, t)$ must be measured perpendicular to the x direction.

The basic assumption about the mechanism of flow, made throughout this treatment, is that, at any given point x, q is a definite function of h and α; thus

$$q = q(h, \alpha, x). \qquad (2)$$

This kinematic law is equivalent to putting q, at given x, a function of h and $\partial h/\partial x$, since the slope of the bed β is itself a function of x only. As pointed out in I, the dependence on h gives rise to the main phenomenon, namely the propagation of kinematic waves; the dependence on $\partial h/\partial x$ leads to diffusion of these waves; any dependence on $\partial^2 h/\partial x^2$ and higher derivatives would lead to higher-order forms of diffusion. By assuming (2) we are therefore confining attention to the effect of the first two terms of the Taylor expansion of h about the point x.

The validity of equation (2) is frequently assumed tacitly in glaciological writings, but there is very little direct observational evidence one way or the other. Many instances have been reported where a glacier has diminished in thickness and at the same time has decreased in velocity, but this of course is far from establishing a functional relation between q, h and x. Striking evidence in favour of (2) has recently come from Nisqually Glacier (Meier and Johnson 1962), where a number of bulges have recently been seen to travel down the glacier. The bulges have caused a roughly sinusoidal variation of h at given x, of large amplitude, and there has been a corresponding sinusoidal variation of velocity, with a slight discrepancy that can be explained as an effect of a dependence of q on α. However, one does not want to rely too heavily on one field example, and it is perhaps preferable to base the case for equation (2) on a combination of theory with the experimentally observed behaviour of ice under stress in the laboratory, as discussed in I. The experimental results give the law of flow, strain-rate $\propto$ (shear stress)^n, $n \simeq 3$ or 4. Combining this with a simple model of the geometry of flow (shear distortion in the ice on planes parallel to the bed, plus sliding at the ice-rock interface) gives equation (42) of I:

$$q = Fh^{n+2} \sin^n\alpha + Gh^{m+1} \sin^m\alpha, \tag{3}$$

where F and G are function only of x, and $m \simeq 2$. More complicated geometries of flow could be considered, but (3) is a plausible first approximation. In fact we are only at this stage using the feature of (3) that q is a function of x, h and α, and there is no need to be more specific than this. In further justification of (2) it may be remarked that it is found to be of wide validity in the closely analogous problem of flood waves on long rivers (Lighthill and Whitham 1955)*.

None of this discussion on the validity of (2) properly applies to the extreme end of the glacier, the snout, and here the whole picture is much less clear. Temperate glaciers commonly have wedge-shaped snouts as indicated in Figure 1,

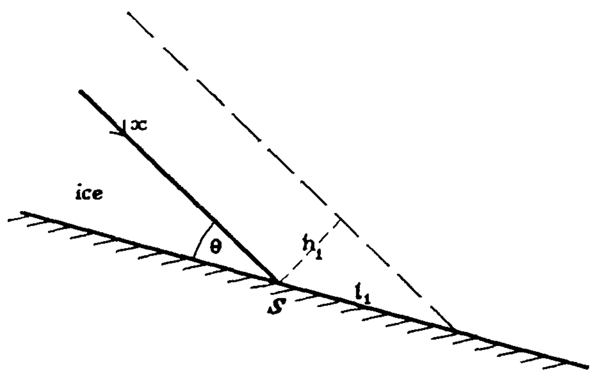

FIG. 1.—An increase of thickness h_1 at the snout S of the datum glacier results in an advance l_1.

with a wedge angle θ which is often from 5° to 15°, but which may be larger. The ice has a definite forward velocity right up to the extreme tip, and there is no obvious sign of any violent change in the flow pattern as the tip is approached. Melting occurs, mainly at the upper surface, and a rough balance is maintained between forward motion and backward melting. The difference between the two gives the advance or retreat that we are concerned with.

No adequate theory exists yet for the flow of the snout region of a glacier. The assumptions on which equation (3) is based certainly break down, and (3) predicts a forward velocity $u = q/h$ which is zero at the extreme tip, thus contradicting observation (by this definition, u is the component in the x direction of the average velocity through the thickness). It is plausible to suppose that q in the snout region, at given x, is determined mainly by the values of h and α over a certain range of x. Thus the kinematic law (2) is probably not strictly true unless h and α are reinterpreted to refer to some average values over a range of x, possibly a range of x extending over a distance of order h. (Note that, even in

* There seem to be only two basic differences between rivers and glaciers so far as the application of kinematic wave theory is concerned. One is that the Reynolds Number in glacier flow is so small that dynamical (inertia) effects can always be safely ignored—for example, there are no dynamical waves on glaciers. The other is that the feeding of a river by tributaries normally causes a monotonic increase in q from the source to the sea or other outlet. In glaciers, on the other hand, q first increases, and then decreases to zero at the snout, because of ablation. It is this decrease which leads to the interesting instability in the ablation area discussed in I. Analogous instability effects would be expected in those rivers which, by excessive evaporation and run-off, end on land in deserts, or, in general, whenever the flow velocity decreases with x.

(3), h and α should really be interpreted in this way). From this point of view the assumption of (2) at the snout is likely to go badly wrong if α changes very rapidly with x—say by a large amount in a distance $\sim h$. Since this does not happen in our solutions we shall assume from now on, as a reasonable simplification, that $q = q(h, \alpha, x)$ both at the snout and over the rest of the glacier.

An example of a plausible function q in the neighbourhood of the snout would be

$$q = Fh \sin^r \alpha, \tag{4}$$

where F is a function of x only, and $r \simeq 3$. Now $q \to 0$ at the snout proportionally to y, the distance from the snout. Hence (4) is satisfied if α remains finite and $h \to 0$ proportionally to y. The velocity $u = q/h = F \sin^r \alpha$. Thus the function (4) is consistent with the finite slope and finite velocity at the snout which is observed in the field. The main difference between (4) and (3) is in the power of h, and one can imagine a smooth transition between them. Equations (3) and (4) are only given as examples of the general statement (2); in fact neither of them will be used explicitly in what follows, all the analysis being based on (2).

Having discussed the validity of the kinematic law (2), we now continue by expressing the various quantities in terms of perturbations from the datum state, which was defined as the steady state resulting from the steady rate of accumulation $a_0(x)$. Thus for the actual rate of accumulation we write

$$a(x, t) = a_0(x) + a_1(x, t),$$

so that $a_1(x, t)$ is the perturbation. If the datum-state values of q, h, and α are $q_0(x)$, $h_0(x)$, $\alpha_0(x)$ and the perturbations are $q_1(x, t)$, $h_1(x, t)$, $\alpha_1(x, t)$, equation (1) gives

$$\frac{dq_0}{dx} = a_0 \tag{5}$$

and

$$\frac{\partial q_1}{\partial x} + \frac{\partial h_1}{\partial t} = a_1. \tag{6}$$

We now restrict attention to the linearized theory in which the perturbations h_1 and α_1 are small. Then, in view of assumption (2), we may write, for given x,

$$q_1 = c_0 h_1 + D_0 \alpha_1, \tag{7}$$

where $c_0(x) = (\partial q/\partial h)_0$ and $D_0(x) = (\partial q/\partial \alpha)_0$, the derivatives being evaluated at the datum values h_0, α_0. Thus $c_0(x)$ specifies the dependence of q on h in the neighbourhood of the datum state; in fact, as we shall see in a moment, $c_0(x)$ is the velocity of kinematic waves of small amplitude. Similarly $D_0(x)$ specifies the dependence of q on α in the neighbourhood of the datum state; we shall see that $D_0(x)$ plays the role of a diffusion coefficient for kinematic waves.

Now, since $\alpha_1 = -\partial h_1/\partial x$, equation (7) can also be written as

$$q_1 = c_0 h_1 - D_0 \partial h_1/\partial x. \tag{8}$$

D_0 thus relates a concentration gradient, $\partial h_1/\partial x$, to a flow, q_1. (Note that (8) is only true when x is measured along the upper surface of the datum glacier. It would be possible to measure x along the glacier bed, or horizontally, provided that, in order to keep equation (1) valid, h and a were always measured perpendicularly to x. But in these cases α_1 would not be simply $-\partial h_1/\partial x$, and so equation (8) would have a slightly different form.)

Equations (6) and (8) are the basic equations of the theory. A convenient, but not essential, simplification is to put a_1 a function of t only, and we shall do this, since it seems to represent a reasonable first approximation to the changes which actually occur in many glaciers. (In III the more general assumption $a(x, t) = \bar{a}(t)X(x)$ is made.) Then, differentiating (6) and (8) with respect to x and t respectively, we can eliminate h_1 to obtain the following equation for q_1:

$$\frac{\partial q_1}{\partial t} = c_0 a_1 - c_0 \frac{\partial q_1}{\partial x} + D_0 \frac{\partial^2 q_1}{\partial x^2}. \tag{9}$$

In a non-linearized theory the same equation for q_1 would result, except that the coefficients c_0 and D_0 would be functions of q_1 as well as of x [I, equation (24)].

If, on the other hand, equation (8) is differentiated with respect to x, we can eliminate q_1 between (6) and (8) to obtain the following equation for h_1:

$$\frac{\partial h_1}{\partial t} = a_1 - c_0' h_1 - (c_0 - D_0') \frac{\partial h_1}{\partial x} + D_0 \frac{\partial^2 h_1}{\partial x^2}, \tag{10}$$

where the primes denote differentiation with respect to x.

Equation (9) is slightly simpler than equation (10). The coefficient of $\partial q_1/\partial x$ and of $\partial h_1/\partial x$ may evidently be interpreted in each case as a kinematic wave velocity: c_0 for waves of q_1 and $(c_0 - D_0')$ for waves of h_1—although when the D_0 term is present the paths of the kinematic waves are not, of course, true characteristics of the equations. The last term in each equation represents diffusion, the diffusion coefficient being D_0 in both equations. (10) contains an extra term, $-c_0' h_1$, which has no counterpart in (9) and which leads to the instability effects discussed in I. We shall have occasion to use both equations (9) and (10) in what follows.

3. Form of the functions $c_0(x)$ and $D_0(x)$

In order to set up a model for calculation we must now discuss what form the functions $c_0(x)$ and $D_0(x)$, the wave velocity and the diffusion coefficient, might take in a typical glacier. It was shown in I that in most regions of a glacier $c_0 \sim 4u_0$, where u_0 is the datum-state velocity of the ice (defined as q_0/h_0). Observation shows that the ice velocity in a glacier tends to be greatest in the middle regions and small near the head and snout. We therefore expect $c_0(x)$ to show similar behaviour.

Simple models of the glacier flow mechanism similarly lead to the result (I) that, $D_0 \sim 3q_0 \cot \alpha_0$. α_0, the slope of the upper surface, will show some variation with x, which will be characteristic of the particular glacier, but, broadly speaking, we may expect $D_0(x)$ to be of similar form to $q_0(x)$. Since $a_0(x)$ is positive in the upper part of the glacier and negative in the lower part, $q_0(x)$, which equals

$$\int_0^x a_0 \, dx,$$

is a maximum in the middle part. $D_0(x)$ thus tends to be small at the two ends and a maximum in the middle, like $c_0(x)$.

Now it turns out that the solution of the differential equations (9) and (10), and indeed the whole problem of calculating the response of a glacier to climate, depends critically upon the behaviour of $c_0(x)$ and $D_0(x)$ at the extreme snout, and we must consider this problem in detail. In Figure 1 the full line shows the profile of the glacier in the datum state, the end S being the point $x = L$. The forward velocity at S is finite, as already explained; since we are here dealing with a steady state the forward velocity is exactly balanced by the ablation. If the thickness at S now increases by a small amount h_1 (non-steady state), the snout will advance a distance l_1, which to the first order equals h_1 cosec θ; we have to take account of the change in θ only in the next approximation. Thus, our main task will be to calculate h_1 at S, because this is directly proportional to the advance of the actual glacier beyond the datum length.

Let us now consider the value of c_0 at S, $c_0(L)$. By definition,

$$c_0(x) = \left(\frac{\partial q}{\partial h}\right)_0 = \left(\frac{\partial(uh)}{\partial h}\right)_0 = u_0 + h_0\left(\frac{\partial u}{\partial h}\right)_0,$$

and, since $h_0(L) = 0$, $c_0(L) = u_0(L)$.

On the other hand, for $D_0(L)$ we have

$$D_0(x) = \left(\frac{\partial q}{\partial \alpha}\right)_0 = \left(\frac{\partial(uh)}{\partial \alpha}\right)_0 = h_0\left(\frac{\partial u}{\partial \alpha}\right)_0,$$

and hence $D_0(L) = 0$*. Thus $c_0(x)$ and $D_0(x)$ are both comparatively small near the end of the glacier, but $c_0(L)$ is finite while $D_0(L)$ is zero. If we assume the form (4) for q it follows that, as $x \to L$, $D_0 \to 0$ proportionally to q_0, that is proportionally to $L-x$.

For the end point S, equation (8) becomes

$$q_1 = c_0(L)h_1. \tag{11}$$

The validity of (11) may be questioned on the grounds that h_1 is much larger than h_0 near the snout. However, the condition for (11) to be valid is not that h_1 is much smaller than h_0, but that h_1 is small enough for the q:h relation in the neighbourhood of $h_0(=0)$ to be considered as linear, with slope $c_0(L)$. h_1 can in fact be made as small as we like by choosing the perturbation a_1 small enough. The fact that $h_1 \gg h_0$ near the snout is thus no reason for disbelieving (11).

An increase in rate of accumulation will naturally cause non-zero values of q_1 at S. Equation (11) shows that it is essential to keep $c_0(L)$ non-zero, and not to approximate the small but finite velocity at the snout of the glacier by putting it zero. If we did, we should force h_1, and the advance of the glacier, which is what we want to calculate, to be infinite.

A similar distinction between the behaviour of c_0 and D_0 may occur sometimes at the head of a glacier—namely that D_0 is strictly zero, but c_0 is only approximately so—but it turns out that no difficulty results if both are taken to be zero. This accords with the intuitive notion that the *details* of what is done at the extreme head of the glacier can hardly affect the glacier as a whole—since the feeding takes place down the whole length.

4. Boundary conditions

To determine a solution to the differential equations, which are parabolic, boundary conditions at $x = 0$ and $x = L$ and an initial condition must be imposed. The proper boundary condition to put on at $x = L$ is not immediately obvious, since no particular physical restriction seems to be placed on the glacier at that place. This problem did not arise in I, since when diffusion is neglected the

* Even if q is assumed to depend on x, h, $\partial h/\partial x$, $\partial^2 h/\partial x^2$ and any number of higher derivatives, the argument that $D_0 = 0$ at S still holds. Moreover the higher-order diffusion coefficients which then appear also vanish at S in a similar way.

differential equation for h_1 or q_1 is of first order, and so an initial condition and a condition at $x = 0$ are sufficient to fix the solution.

To clarify the question consider first the simplest case $a_1 = 0$. If a_1 is zero for a sufficiently long time the system may be expected to reach a steady state. Assume it has done so. Then, putting $\partial h_1/\partial t = 0$ and $a_1 = 0$ in (6) gives $dq_1/dx = 0$. Hence $q_1 =$ constant. It is physically reasonable to require $q_1 = 0$ at $x = 0$. In this case $q_1 = 0$ for all x. Equation (8) is then

$$D_0 dh_1/dx - c_0 h_1 = 0. \qquad (12)$$

The equation has singularities at $x = 0$ and $x = L$, D_0 being zero at these end points. The solution that makes immediate physical sense is $h_1 = 0$ for all x, that is, the glacier is in the datum state. But there are also solutions with $h_1 \neq 0$, which must be looked at. Near $x = L$, $D_0 \to 0$ as $L - x$, while c_0 remains non-zero; hence the equation is sufficiently well represented near $x = L$ by

$$\gamma y \frac{dh_1}{dy} + c_0(L)\, h_1 \mathrm{o} = ,$$

where $y = L - x$ and γ is a positive constant, which has the general solution

$$h_1 = Py^{-B},$$

where $B = c_0(L)/\gamma$ and P is an arbitrary constant. Since B is positive, either $P = 0$, giving $h_1 = 0$, or the solution is unbounded at $y = 0$. Thus, there is a family of solutions of (12), all except one of them ($h_1 = 0$) being unbounded at $x = L$. In an unbounded solution, with h_1 positive, the positive q_1 produced by the increase in thickness ($c_0 h_1$) is just balanced by the negative q_1 produced by the decrease in surface slope ($D_0\, dh_1/dx$)—and similarly if h_1 is negative. Physically this is quite acceptable; the only objection arises at $x = L$. Here an unbounded solution cannot be taken quite literally because h_1 is restricted to be small, but it does suggest that some corresponding physical solution exists. This is in fact a "cliff" solution, which will now be described.

Consider afresh the form of the datum glacier itself, leaving aside for the moment any possible perturbations from it. At the lower end $q_0 = 0$. Now, since $q_0 = u_0 h_0$, there are two ways of achieving $q_0 = 0$; either (1) $h_0 = 0$, which gives the wedge-shaped snout of Figure 1, or (2) $u_0 = 0$ with non-zero h_0. This last represents a terminal cliff, which is a possibility not yet considered. The zero u_0 might be achieved by having the surface slope α zero. Case (2) corresponds to a whole family of datum profiles of different cliff height, while case (1) corresponds to a single datum profile only. The two types of snout profile can be obtained analytically (unpublished) but the possibility of their existence is fairly obvious without analysis. The important point for the present argument is that q_0 is the same at given x for all the members of the family—in other words, fixing $q_0(x)$ does not, in general, fix the datum profile. But the profile *is* fixed if the stipulation is added that the snout is wedge-shaped.

It is natural, therefore, that in the perturbation theory a restriction of this same type is needed at $x = L$ to fix the profile. We therefore *assume* that, if a glacier terminates as a wedge, it will normally continue to do so (as in Figure 1)—basing the assumption on general field observation. Singularities of h_1 at $x = L$ must then be ruled out and the solution to (12) becomes unique. The boundary conditions thus suggested are:

$$\left.\begin{array}{l} q_1 = 0 \text{ at } x = 0 \\ h_1 \text{ not singular at } x = L. \end{array}\right\} \qquad (13)$$

Consider next the case $a_1 = A$, a constant. Suppose, for example, that the glacier starts in the datum state under the steady rate of accumulation $a_0(x)$, and that the rate is suddenly increased by a uniform amount A. Thus, in the upper parts the rate of accumulation increases, and in the lower parts the rate of ablation decreases; there is a sudden permanent lowering of the snow-line. Eventually the glacier may be expected to reach a new steady state. To find this, put $\partial h_1/\partial t = 0$ in (6) to give $dq_1/dx = A$ and hence

$$q_1 = Ax, \tag{14}$$

if the boundary condition $q_1 = 0$ at $x = 0$ is applied. Equation (8) is now

$$D_0 dh_1/dx - c_0 h_1 = -Ax. \tag{15}$$

Following the same procedure as before we represent the equation near $x = L$ by

$$\gamma y \, dh_1/dy + c_0(L)h_1 = AL$$

and obtain the general solution

$$h_1 = \frac{AL}{c_0(L)} + Py^{-B}. \tag{16}$$

Once more, all solutions except the one with $P = 0$ give an infinite h_1 at $y = 0$. The required solution near $y = 0$ is therefore the particular integral $h_1 = AL/c_0(L)$.

Let us note two facts about this solution near $y = 0$. First, it may be obtained direct from equation (15) by setting the first term equal to zero. Second, the required solution is that one out of the family (16) which $\to 0$ as $A \to 0$. We can in fact advantageously replace the boundary conditions (13) by the physical requirement that the solution should be the one which approaches zero as $A \to 0$. The role of the boundary conditions in this problem, as we shall see more clearly when time-dependent solutions are considered, is to prevent any h_1 or q_1 flowing or diffusing in from either end of the system. The system is then completely controlled by the forcing term A.

The argument about boundary conditions has so far considered only the steady-state solutions, but identical reasoning can be applied to the time-dependent solutions by taking Laplace transforms. Consider, for example, the solutions of equation (9) for $t > 0$, with initial condition $q_1(x, 0) = 0$, $0 \leqslant x \leqslant L$. Taking the Laplace transform we have the ordinary differential equation

$$D_0 \frac{d^2\bar{q}_1}{dx^2} - c_0 \frac{d\bar{q}_1}{dx} - p\bar{q}_1 = -c_0 \bar{a}_1, \tag{17}$$

where

$$\bar{q}_1(x, p) = \int_0^\infty q_1(x, t)\, e^{-pt}\, dt,$$

and

$$\bar{a}_1(p) = \int_0^\infty a_1(t)\, e^{-pt}\, dt.$$

Suppose a particular integral of (17) has been found, namely $\bar{Q}(x, p)$, which is such that $\bar{Q} \to 0$ as $\bar{a}_1 \to 0$. Then the general solution will be of the form

$$\bar{q}(x, p) = A(p) f_1(x, p) + B(p) f_2(x, p) + \bar{Q}(x, p), \tag{18}$$

where A and B are arbitrary, and f_1 and f_2 are two linearly independent functions not involving $\bar{a}_1$. As $\bar{a}_1 \to 0$ the solution becomes the complementary function

$$\bar{q}(x, p) = A(p)f_1(x, p)+B(p)f_2(x, p).$$

Now, on physical grounds, the only acceptable solution of equation (9) with initial condition $q_1(x, 0) = 0$ and $a_1 = 0$ is $q_1(x, t) = 0$. The boundary conditions must be chosen so as to ensure this. Hence the only acceptable solution of (17) for $\bar{a}_1 = 0$ is $\bar{q}(x, p) = 0$, and so $A(p)$ and $B(p)$ must be taken zero. It follows from (18) that the solution of (17) is $\bar{Q}(x, p)$, and hence that the solution of (9) with the given initial condition is $Q(x, t)$, the inverse Laplace transform of $\bar{Q}(x, p)$. In other words, if a solution $Q(x, t)$ can be found which satisfies the given initial condition and which approaches zero as $a_1(t) \to 0$, we may be sure that it is the unique solution required*. A similar discussion can be given for equation (10) for h_1.

This argument shows clearly that, as noted above in relation to the steady state, the role of the boundary conditions is to prevent any h_1 or q_1 from flowing or diffusing in from either end. In this approach the only requirement we make of the boundary conditions is that they should enable us to find a solution with the required property—that it vanishes as a_1 vanishes. The boundary conditions found satisfactory from this point of view are the obvious one $q_1(0, t) = 0$, and the less obvious ones that h_1 and q_1 and their derivatives with respect to x should not approach infinity as $x \to L$.

5. The special model

To obtain an explicit solution of the equations a special model is now adopted in which the functions $c_0(x)$ and $D_0(x)$ have simple polynomial forms (Figure 2)

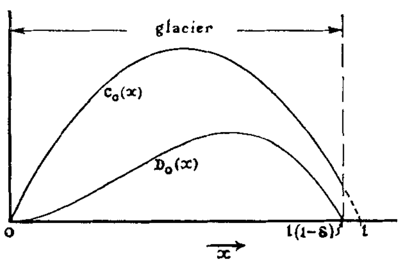

FIG. 2.—Wave velocity c_0 and diffusion coefficient D_0 as functions of x, in the special case chosen for detailed study. δ is exaggerated for clarity (0·50 instead of $\sim$ 0·006).

consistent with the considerations put forward in § 3:

$$c_0(x) = \frac{x}{\sigma}\left(1-\frac{x}{l}\right), \tag{19}$$

$$D_0(x) = \frac{Ex^2}{\sigma}\left(1-\delta-\frac{x}{l}\right), \tag{20}$$

* If the initial condition is not $q_1(x, 0) = 0$, some extension of the reasoning is needed. One way is first to note that the argument given also applies when a_1 is a function of x as well as of t (in which case an additional term in $\partial a_1/\partial x$ appears in (9)). Other initial conditions than $q_1(x, 0) = 0$ can then be dealt with by adding on to $a_1(x, t)$ at $t = 0$ a δ-function which is sharp with respect to t, but which has the appropriate x variation to give the required initial function $h_1(x, 0)$.

where the glacier in the datum state runs from $x = 0$ to $x = l(1-\delta) = L$. σ, which is a time, and E, which is dimensionless, are constants, and δ is a small dimensionless constant introduced to prevent c_0 being zero at the end of the glacier. c_0 in fact extrapolates to zero at a distance $l\delta$ beyond the end of the datum glacier. Equations (19) and (20) constitute the special model referred to in § 1.

It might be thought better to put $D_0 \sim x$ at the head of the glacier, rather than $D_0 \sim x^2$, but the latter is in fact a better representation when the glacier width is finite (III). The point is not important because the precise form taken for $D_0(x)$ at the head of the glacier is not expected to matter very much.

It is convenient now to choose σ as a natural unit of time and l as a natural unit of length, so that in these units

$$c_0(x) = x(1-x), \tag{21}$$

$$D_0(x) = Ex^2(1-\delta-x), \tag{22}$$

and the datum glacier runs from $x = 0$ to $x = 1-\delta$.

Numerical values of σ, E *and* δ. To assign a suitable numerical value to σ, and thus to fix a natural time scale for the problem, note that, from (19), $\sigma = l/4c_{0\max} \simeq l/16u_{0\max}$. Thus for example, if $l = 10$ km and $u_{0\max} = 100$ m/yr, $\sigma \simeq 6$ yr.

The small dimensionless number δ may be estimated as follows.

$$\delta \simeq c_0(L)/4c_{0\max} \simeq u_0(L)/16u_{0\max}.$$

Putting $u_0(L) = 10$ m/yr, for example, and $u_{0\max} = 100$ m/yr, as before, gives $\delta \simeq 0{\cdot}006$.

E may be estimated by considering the value of D_0 at $x = \frac{1}{2}l$, which is approximately $El^2/8\sigma$. If we use the approximation $D_0 \simeq 3q_0 \cot \alpha_0$, we have

$$E \simeq (24\sigma u_{0\max} h_0 \cot \alpha_0)/l^2 \simeq \frac{3}{2}\left(\frac{h_0}{l}\right)\cot \alpha_0,$$

where h_0 and α_0 are here the thickness and slope at $x = \frac{1}{2}l$. If, for example, $h_0 = 250$ m, $\alpha_0 = 4 \times 10^{-2}$, with $l = 10$ km as before, we find $E \simeq 0{\cdot}9$.

As a check that the numerical data used are consistent and reasonable note that $h_0\alpha_0$, which is proportional to the shear stress on the glacier bed, is 10 m, a common average figure. We can also note that to deal with the maximum discharge of $u_{0\max}h_0$ requires an average ablation rate over a distance $\frac{1}{2}l$ of 5 m/yr, which is reasonable. The above figures are simply illustrative. No one glacier can be taken as typical of all, but very roughly we may conclude that the natural time unit $\sigma \sim 10$ yr, that $\delta \sim 0{\cdot}01$ and $E \sim 1$.

6. Solution for a step function

We are now in a position to solve the problem set in § 4—the glacier response when $a_1(t)$ is a step function. As initial condition we have

$$q_1 = h_1 = 0 \qquad (t = 0,\ 0 \leqslant x \leqslant 1-\delta)$$

and thereafter

$$a_1 = A \qquad (t \geqslant 0).$$

(i) *The final steady state.* We have already found that eventually $q_1 = Ax$. To find the eventual change in profile $h_1(x)$ we have to solve (15), which now has the form

$$Ex^2(1-\delta-x)\frac{dh_1}{dx} - x(1-x)h_1 = -Ax. \tag{23}$$

From the discussion in § 4 the value of h_1 at $x = 1-\delta$ is found directly from the equation as A/δ, the term in dh_1/dx being zero.

If $E = 1$ the solution is readily found to be

$$h_1 = A(1+x/\delta). \tag{24}$$

If $E = 0$ the solution is $h_1 = A/(1-x)$. Both solutions vanish with A as required. The solutions for other values of E may be found by numerical integration of (23) starting at $x = 1-\delta$. To start the integration the value of dh_1/dx at $x = 1-\delta$ is needed, and this cannot be found directly from (23) since $D_0 = 0$ at this point. However, by differentiating (23) and assuming that d^2h_1/dx^2 remains finite it is found that at $x = 1-\delta$

$$\frac{dh_1}{dx} = \frac{A}{\delta\{\delta+E(1-\delta)\}}.$$

The step-by-step integration then proceeds without difficulty and the result is shown in Figure 3 for various values of E.

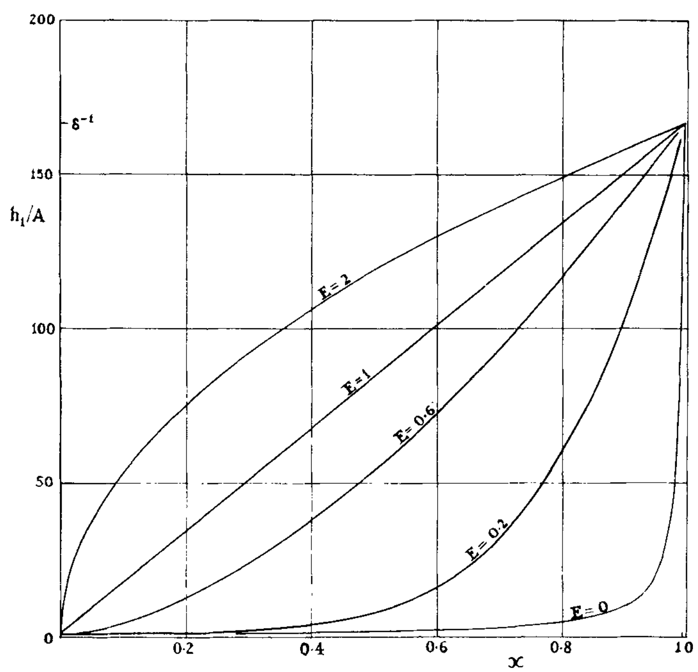

FIG. 3.—Steady-state profiles, h_1:x, after a sudden uniform increase A in rate of accumulation. Natural units, $l = 1$, $\sigma = 1$, are used, and δ is taken as 0·006.

Remembering that a realistic value of E is about 1, and comparing the curves for $E = 0$ and 1, we see that the effect of diffusion on the final steady-state profile is very marked. On the other hand, whatever the value of E the value of h_1 at

$x = 1-\delta$ is always A/δ, so that the ultimate advance of the glacier is always (A/δ) cosec θ, in natural units. (As a numerical illustration, if the change A is 0·1 m/yr, with $\delta = 0{\cdot}006$, $\theta = 20°$, $\sigma = 6$ yr, the ultimate advance is 300 m.) Thus a diffusionless model correctly predicts the final distribution of the flow $q(x)$ and the final advance of the glacier, but not the final profile of its surface. The previous paper (I) is misleading in implying that one can draw significant conclusions about the final profile, as distinct from the final advance of the snout, without taking account of diffusion.

(ii) *Time dependence.* If we now examine the manner in which the new steady state is reached it turns out that simple solutions in closed form can be given for the cases $E = 0$ and $E = 1$. Other cases are much more difficult.

Noticing that the ultimate steady-state solution for $E = 1$ is linear in x we look for a time-dependent solution for $E = 1$ of the form.

$$h_1 = b_0+b_1x+b_2x^2+\ldots, \tag{25}$$

where b_0, $b_1\ldots$ are functions of t. Substituting into equation (10) with the coefficients (21) and (22), equating coefficients of x, and using the condition that the b's are initially all zero, gives the closed solution

$$\frac{h_1}{A} = 1-e^{-t}+\left\{\frac{1}{\delta}+\frac{2}{1-2\delta}e^{-t}-\frac{1}{\delta(1-2\delta)}e^{-2\delta t}\right\}x. \tag{26}$$

We note that h_1 vanishes with A, and that therefore, from the discussion in § 4, this solution is the only one that is physically acceptable (the physically unacceptable solutions would presumably not be of the form (25) even near $x = 0$, and so were excluded from the beginning). As $t \to \infty$, (26) approaches (24) as expected.

The corresponding solution for q_1 can either be found by solving the original differential equation (9) in a similar way, or, more quickly, by substituting the solution for h_1 into equation (8):

$$\frac{q_1}{A} = (1-e^{-t})x+\frac{1}{1-2\delta}(e^{-t}-e^{-2\delta t})x^2. \tag{27}$$

We note that q_1 vanishes with A, as required.

The solution (26) for h_1 is remarkable in that it is linear in x at all times. At $x = 0$, h_1 approaches the new steady value $A(=A\sigma)$ exponentially with time constant $1(=\sigma)$. The value of h_1 at $x = 1-\delta$, which is proportional to the advance of the glacier, is given by

$$\frac{h_1}{A} = \frac{1}{\delta}+\frac{1}{1-2\delta}e^{-t}-\frac{1-\delta}{\delta(1-2\delta)}e^{-2\delta t}, \tag{28}$$

which is plotted in Figure 4a and 4b as the full curve labelled $t_0 \to \infty$. As $t \to \infty$, $h_1 \to A/\delta$, and thus the eventual thickening at $x = 1-\delta$ is $1/\delta$ times greater (say 160 times greater) than the thickening at $x = 0$, as could be seen already from the steady-state solution. But the most interesting feature of the equation for h_1 is the appearance of two time scales: in addition to the time constant 1, there is the longer time constant $1/2\delta$. The part played by this long time constant may be seen by writing down the expression for $\partial h_1/\partial t$ at $x = 1-\delta$, and then making the approximation $\delta \ll 1$:

$$\frac{\partial h_1}{\partial t} \simeq (2e^{-2\delta t}-e^{-t})A \qquad (x = 1-\delta). \tag{29}$$

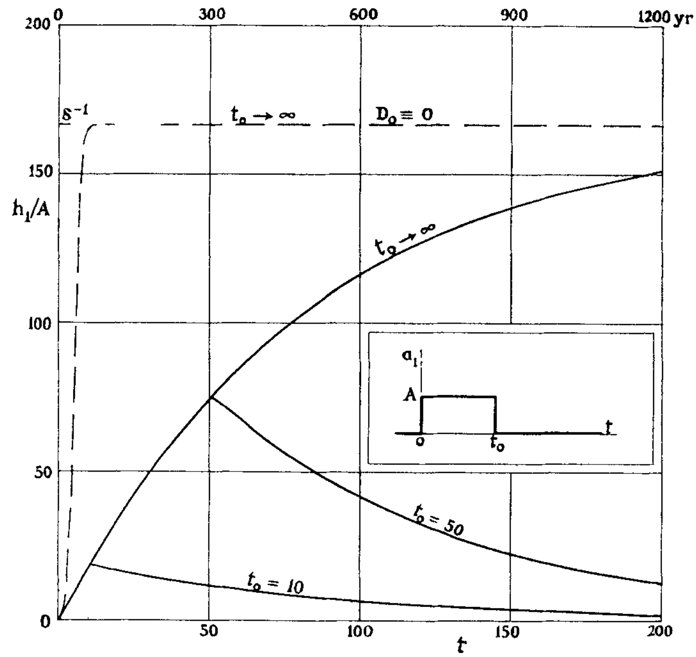

FIG. 4(a).—Response h_1 at the datum position of the snout to a rectangular pulse of increased rate of accumulation is plotted against time t for various pulse-lengths t_0 (see inset). The broken curve shows the response at the snout to a sudden increase in rate of accumulation when diffusion is ignored ($t_0 \to \infty$, $D_0 \equiv 0$). Natural units, $l = 1$, $\sigma = 1$, are used, and δ is taken as 0·006. A scale of years is inserted at the top for illustration, taking $\sigma = 6$ yr.

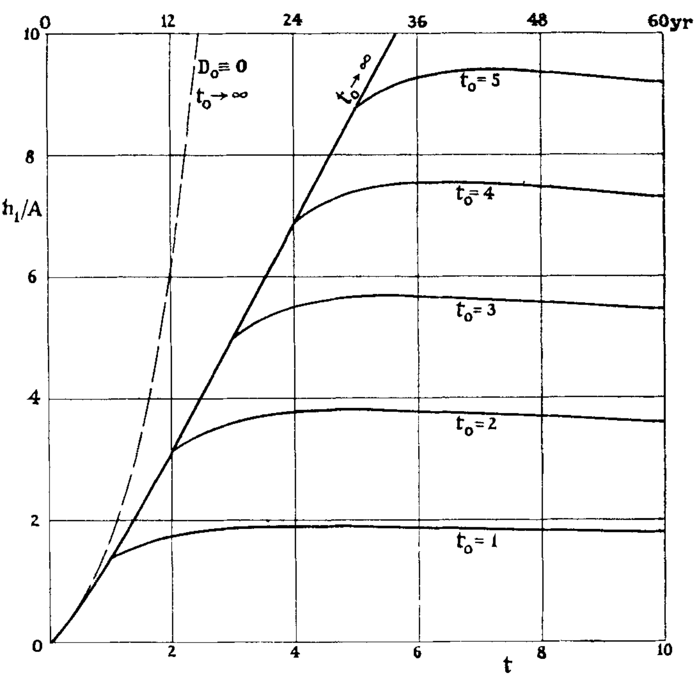

FIG. 4(b).—The lower left corner of Fig. 4(a) on a larger scale, showing curves for $t_0 = 1$ to 5.

Thus $\partial h_1/\partial t$ is initially A, it rapidly increases to nearly $2A$, and then slowly decreases to zero, almost exponentially with time constant $1/2\delta$. Thus h_1 itself initially accelerates during the period while t is of order 1 (Figure 4b), but it then decelerates, and for $t \gg 1$, h_1 approaches the value A/δ virtually exponentially with time constant $1/2\delta$. The accelerating period is precisely the instability which was treated in I and which arises from the term $-c_0'h_1$ in equation (10).

The long time constant $1/2\delta$ represents a new phenomenon which only appears when diffusion is introduced. To see this we may compare our solution for $E = 1$ with the corresponding solution for $E = 0$, which may be readily obtained by the method used in I, namely by integrating along the characteristics, which are the paths of the kinematic waves. The result is

$$\frac{h_1}{A} = \frac{1-e^{-t}}{1-x(1-e^{-t})}, \tag{30}$$

or, at $x = 1-\delta$,

$$\frac{h_1}{A} = \frac{1-e^{-t}}{\delta(1-e^{-t})+e^{-t}}. \tag{31}$$

This equation, which is shown graphically in Figures 4a and b as a broken line, only involves the short time constant 1. As $t \to \infty$, h_1 approaches the same final value as before, namely A/δ, and again there is an initial acceleration, followed by a deceleration. The difference is that, without diffusion ($E = 0$) the motion is substantially complete after a time of, say, 10 natural units, whereas, with a definite amount of diffusion present ($E = 1$) the motion is still continuing after times of order $1/2\delta$, which are very much longer; with the numbers we have been using $1/2\delta$ is 500 yr.

The difference introduced by diffusion is also brought out very clearly by examining the maximum rate of increase of h_1 at the snout. Without diffusion the maximum rate is approximately $A/4\delta$, and since δ is small the rate is large. On the other hand, with $E = 1$ the rate is never more than $2A$, as we have seen. Qualitatively it was to be expected that diffusion would both prolong the advance of the glacier and reduce its maximum rate. What was not clear beforehand, but is revealed by this example, is the very large size of the effect.

The origin of the long time constant can now be understood in the following way. When there is a sudden increase A in rate of accumulation, the eventual value of q_1 at the snout of the datum glacier is AL, by equation (14), where L is the length of the datum glacier; this is so whatever the detailed forms of $c_0(x)$ and $D_0(x)$. The resulting thickening at the datum snout is then $AL/c_0(L)$ by equation (11). The factor $L/c_0(L)$ is the long time constant we are concerned with; it is the time taken to travel the length of the datum glacier at the constant speed $c_0(L) = u_0(L)$, and in the special model it is σ/δ. Now, just after the sudden increase A, h_1 grows at the rate A. *If* h_1 continued to grow at this rate it would reach its final value only after this long time $L/c_0(L)$. In fact, when there is no diffusion h_1 grows much more quickly than this (Figures 4(a) and (b)) and the long time constant does not appear. But when diffusion is present it prevents such rapid growth, and so longer times are needed. In the special model with $E = 1$ the maximum rate of growth of h_1 is about $2A$, and the final value is therefore approached after a time of order $\frac{1}{2}L/c_0(L)$. This equals $\sigma/2\delta$, which is precisely the long time constant that appears in the exact solution.

This long time constant for $E = 1$ has the practical significance that the glacier continues to respond to a climatic change for many hundreds of years after it has happened—the glacier has a very long memory—in marked contrast to the result given by the diffusionless approximation ($E = 0$). We may therefore ask whether the behaviour found for the case $E = 1$ is exceptional. For values of E other than 0 and 1 the behaviour does not seem to be exactly describable in terms of one, two or any finite number of time constants. But the argument above suggests that, as diffusion increases, the response times will be longer, and that, since $E = 1$ is a realistic value, times $\sim \sigma/2\delta$ must be expected. The question would be further clarified if a solution could be found for a general value of E. Although no exact solution has been obtained, we can, following a suggestion by Dr. W. Chester, find a solution for a general value of E close to 1, which exactly satisfies the differential equation and which *approximately* satisfies the initial condition. This solution (see Appendix) continues to show a time scale $\sim \sigma/2\delta$. Thus there is good reason for thinking that long response times of several hundred or a thousand years are involved in the reaction of a glacier to climatic change. The numerical calculation of the frequency response of South Cascade Glacier in III fully supports this conclusion.

7. Solution for varying rate of accumulation

(i) *General solution*

We now derive a solution analogous to (26) and (27) for the case where a_1 is an arbitrary function of t. c_0 and D_0 are supposed given by equations (21) and (22) with $E = 1$. We choose the origin of t arbitrarily and assume that at $t = -T$, where T is as large as we like ($T > 0$), the glacier was in the datum state with $a_1 = 0$, $h_1 = 0$ and $\partial h_1/\partial t = 0$. The real initial condition would, of course, be different from this, in general, but we argue that the very remote past can have no effect on the current behaviour of the glacier, and that we are therefore entitled to assume any initial condition that is convenient.

We look for a solution of equation (10) of the form (25), and proceed exactly as for a_1 constant. Substituting (25) into the differential equation and equating coefficients of x gives for the constant term

$$\dot{b}_0 = a_1(t) - b_0,$$

and hence, using the initial condition,

$$b_0 = e^{-t}\int_{-T}^{t} a_1(\tau)e^{\tau}\,d\tau.$$

The coefficients of x give an equation for $\dot{b}_1$, b_1 and b_0 whose solution is

$$b_1 = 2\,e^{-2\delta t}\int_{-T}^{t}\left\{e^{-(1-2\delta)t'}\int_{-T}^{t'} a_1\,e^{\tau}\,d\tau\right\}dt',$$

again using the initial condition. Integrating by parts this becomes

$$b_1 = -\frac{2}{1-2\delta}\left(e^{-t}\int_{-T}^{t} a_1\,e^{\tau}\,d\tau - e^{-2\delta t}\int_{-T}^{t} a_1\,e^{2\delta\tau}\,d\tau\right).$$

The coefficients of x^2 give the equation

$$\dot{b}_2 = 3b_2(1-2\delta),$$

of which the solution satisfying the initial condition is $b_2 \equiv 0$. It then follows in a similar way that $b_3 \equiv b_4 \equiv \ldots \equiv 0$.

Now define two time averages of a_1 as follows:

$$\bar{a}_1 \int_{-T}^{t} e^{\tau}\, d\tau = \int_{-T}^{t} a_1 e^{\tau}\, d\tau \quad \text{and} \quad \bar{\bar{a}}_1 \int_{-T}^{t} e^{2\delta\tau}\, d\tau = \int_{-T}^{t} a_1 e^{2\delta\tau}\, d\tau.$$

Since T is as large as we please, we may put e^{-T} equal to zero and obtain

$$\bar{a}_1 = \int_{-T}^{t} a_1 e^{-(t-\tau)}\, d\tau \quad \text{and} \quad \bar{\bar{a}}_1 = 2\delta \int_{-T}^{t} a_1 e^{-2\delta(t-\tau)}\, d\tau. \tag{32}$$

Thus, if we imagine ourselves situated at time t, $\bar{a}_1$ is an average of a_1 over the recent part (time scale 1), while $\bar{\bar{a}}_1$ is an average of a_1 extending into the more distant past (time scale $1/2\delta$). They may be called the short-time and the long-time averages, respectively*.

With this notation the solution may be written

$$h_1(x, t) = \bar{a}_1 - \frac{2x}{1-2\delta}\left(\bar{a}_1 - \frac{\bar{\bar{a}}_1}{2\delta}\right). \tag{33}$$

The increase in thickness at the datum position of the snout ($x = 1-\delta$) is then

$$h_1(t) = -\frac{1}{1-2\delta}\bar{a}_1 + \frac{1-\delta}{\delta(1-2\delta)}\bar{\bar{a}}_1, \tag{34}$$

or, since $\delta \ll 1$, simply

$$h_1(t) \simeq -\bar{a}_1 + \delta^{-1}\bar{\bar{a}}_1. \tag{35}$$

Multiplied by cosec θ, these expressions give the advance of the glacier.

Differentiating equation (34) with respect to t gives for the rate of increase of thickness at the datum snout, which is proportional to the rate of advance of the glacier,

$$\dot{h}_1 = a_1 + \frac{1}{1-2\delta}\bar{a}_1 - \frac{2(1-\delta)}{1-2\delta}\bar{\bar{a}}_1 \tag{36}$$

or simply,

$$\dot{h}_1 \simeq a_1 + \bar{a}_1 - 2\bar{\bar{a}}_1. \tag{37}$$

The physical meaning of equation (35) is that, if $\bar{a}_1$ and $\bar{\bar{a}}_1$ are of comparable magnitude, the position of the snout is very largely determined by $\bar{\bar{a}}_1$ alone. This is a very reasonable result, for the position of the snout is the outcome of conditions extending over a long period of time. On the other hand, equation (37) tells us that the *rate* of advance is approximately proportional to $(a_1-\bar{\bar{a}}_1)+(\bar{a}_1-\bar{\bar{a}}_1)$; that is, proportional to the departures of a_1 and $\bar{a}_1$ from the long-time average $\bar{\bar{a}}_1$. Thus in this respect the long-time average sets the datum from which the changes should be measured.

The a_1 term in equation (37) represents the direct effect of decreased ablation in thickening the snout, while $\bar{a}_1$ and $\bar{\bar{a}}_1$ represent the effect of past changes in a_1

* An analogue would be a damped galvanometer, with zero inertia, subject to a force $a_1(t)$. The response would be $\bar{a}_1(t)$ if the relaxation time was 1, and $\bar{\bar{a}}_1(t)$ for relaxation time $1/2\delta$.

transmitted from higher up the glacier. The role of the three terms may be clearly seen in the example already treated, where a_1 was zero up to a certain time, and constant, say A, thereafter. Just after the discontinuity, $\bar{a}_1$ and $\bar{\bar{a}}_1$ are both zero, and $\dot{h}_1$, by (37), is simply equal to A. After a short time $\bar{\bar{a}}_1$ is still effectively zero, but $\bar{a}_1$ approaches the value A. Thus, from (37), $\dot{h}_1$ approaches $2A$. This is the accelerating period already discussed. After this time $\bar{\bar{a}}_1$ begins to increase significantly, and finally $\bar{\bar{a}}_1 \to A$. Thus, after long times a_1, $\bar{a}_1$ and $\bar{\bar{a}}_1$ all approach A, and $\dot{h}_1$ becomes zero. The glacier is then in its new steady state, and, from equation (35), the final h_1 is effectively A/δ.

(ii)† *Effect of a pulse of a_1*

Equations (34) and (36) may be applied to study the response of the glacier snout to a temporary deterioration or improvement in climate, as distinct from the permanent change discussed in § 6. Let the glacier be in a steady state and suppose that a rectangular pulse of a_1 is then applied: from $t = 0$ to $t = t_0$, $a_1 = A$, and a_1 is zero at all other times (Figure 4(a)). From $t = 0$ to $t = t_0$ the solution will be given simply by equation (28). To find the solution for $t > t_0$ we use equations (34) and (36). From equations (32), $\bar{a}_1$ and $\bar{\bar{a}}_1$ are calculated as

$$\bar{a}_1 = Ae^{-t}(e^{t_0}-1), \quad \bar{\bar{a}}_1 = Ae^{-2\delta t}(e^{2\delta t_0}-1) \quad (t \geqslant t_0). \tag{38}$$

Then equation (34) gives, for the datum snout,

$$h_1 = \frac{A}{1-2\delta}\left\{-e^{-t}(e^{t_0}-1)+\frac{1-\delta}{\delta}e^{-2\delta t}(e^{2\delta t_0}-1)\right\} \quad (t \geqslant t_0). \tag{39}$$

An examination of this equation, which is shown graphically in Figures 4(a) and (b), shows that when the pulse ends, at $t = t_0$, h_1 at the datum snout may either continue to rise or may fall immediately. For small t_0, h_1 continues to rise a little after $t = t_0$, it reaches a maximum, and eventually falls to zero exponentially with time constant $1/2\delta$; on the other hand, for large t_0, h_1 falls immediately the pulse ends. There is thus a range of t_0 for which a flood maximum occurs at the datum snout after the pulse is over. To find the time of arrival of this maximum, say $t = t^*$, put $\dot{h}_1 = 0$ in (36). Since $a_1 = 0$ when the maximum occurs,

$$\bar{a}_1 = 2(1-\delta)\bar{\bar{a}}_1. \tag{40}$$

Then, using equations (38), we find for t^*,

$$e^{-(1-2\delta)t^*} = 2(1-\delta)\frac{e^{2\delta t_0}-1}{e^{t_0}-1} \qquad (t \geqslant t_0). \tag{41}$$

When $t_0 \ll 1$, and $\delta \ll 1$, this gives

$$t^* \simeq -\ln 4\delta$$

as the delay time for the flood maximum after a short pulse. (This is also the time taken for a kinematic wave of velocity $c_0 - D_0'$ to travel from $x = \frac{1}{2}$ to the snout. Note that, since h_1 is linear in x at all times, the term $D_0\partial^2 h_1/\partial x^2$ in equation (10) vanishes for all x and t. The equation is then effectively of first order and the solution is propagated along the characteristics $dx/dt = c_0 - D_0'$.) The flood maximum after a short pulse thus occurs after a delay of a few multiples of the natural time unit σ. For larger values of t_0, $t^* - t_0$ decreases (that is, the maximum occurs sooner after the end of the pulse) until, when $t_0 \simeq -(1/2\delta)\ln 2$, $t^* = t_0$.

† This section may be omitted without loss of continuity.

For longer pulses than this the maximum h_1 is attained at the end of the pulse itself.

The height of the flood maximum at the datum snout is obtained by eliminating $\bar{\bar{a}}_1$ between (34) and (40) to give

$$\begin{aligned} h_{1\max} &= \bar{a}_1/2\delta \\ &= (A/2\delta)e^{-t^*}(e^{t_0}-1) \end{aligned} \tag{42}$$

from (38). For moderate values of t_0, that is, values small compared with δ^{-1}, equation (41) is essentially

$$e^{-t^*} = \frac{4\delta t_0}{e^{t_0}-1},$$

and hence $h_{1\max}$ is given by

$$h_{1\max} = 2At_0 = 2w \qquad (t_0 \ll \delta^{-1}), \tag{43}$$

where w is the total thickness of additional accumulation due to the pulse. As a numerical illustration of this very simple result, suppose $\delta = 0{\cdot}006$ as before, and that the natural time unit $\sigma = 6$ yr; then the duration δ^{-1} is 1000 yr. Thus, if the rate of ablation at the snout changes from 10 m./yr to 9 m./yr for 20 yr, $w = 20$ m and $h_{1\max} = 40$ m. If θ, the angle at the snout, is 20°, the advance of the glacier is $h_{1\max} \operatorname{cosec} \theta = 120$ m.

If t_0 is comparable with δ^{-1} the flood maximum occurs relatively very soon after the end of the pulse, for we have seen that the interval is always shorter than $-\ln 4\delta$. Hence the flood maximum in these cases is only slightly greater than the height attained at the end of the pulse itself.

In the absence of diffusion ($E = 0$) it may be shown that the corresponding formula to (43) for very short pulses ($t_0 \ll 1$) is $h_{1\max} = w/4\delta$. Thus diffusion, with $E = 1$, has the effect of reducing the flood height by a factor of order $1/8\delta$, say 20, for very short pulses. For very long pulses, however, diffusion makes no difference to the thickness attained, for $h_1 \to A/\delta$ with or without diffusion.

The pulse does not have to be rectangular for equation (43) to hold, for, by superposition of rectangular pulses, one can see that the equation will be true for any pulse whose duration is small compared with δ^{-1}. Another way of seeing this is to write (42) in terms of $\bar{\bar{a}}_1$ by using (40):

$$h_{1\max} = \bar{\bar{a}}_1(1-\delta)/\delta.$$

Now we know from the analysis of rectangular pulses that the flood maximum will occur fairly soon after the pulse itself, that is, the factor $\exp -2\delta(t-\tau)$ in the definition (32) of $\bar{\bar{a}}_1$ is essentially 1. Hence

$$h_{1\max} = 2(1-\delta)\int_{-T}^{t} a_1\, d\tau \simeq 2w$$

where $$w = \int_{-T}^{t} a_1\, d\tau.$$

An interesting result follows if we make the duration of the pulse vanishingly small. Suppose a_1 is a δ-function such that

$$\int a_1\, d\tau = w.$$

There is then an instantaneous increase of thickness for all x of amount w. Equation (43) shows that the subsequent development of this perturbation brings h_1 at the datum snout to a maximum value of $2w$ before it diminishes to zero. Thus, an initially uniform perturbation of thickness is just doubled at the datum snout—in contrast to the diffusionless case where it grows by a factor $1/4\delta$, say 40. The whipcrack simile of paper I, although qualitatively still correct, is less appropriate than it seemed at first.

The factor 2 in equation (43) is a consequence of the special model we are using. The same factor appeared in equation (29), from which we deduced that the maximum rate of advance after a sudden permanent change, A, was twice the initial rate of advance. In real glaciers we may suppose that the maximum response to a pulse which is short compared with δ^{-1} ($\sim$ 1 000 yr) will be approximately given by (43), and will be independent of the precise shape of the pulse, the correct factor being ~ 2.

The conclusions about pulses of greatest practical importance may thus be summarized as follows:

(a) for a pulse which is short compared with about 1 000 yr, but of any shape, the maximum increase in thickness at the datum position of the snout is roughly twice the total thickness of additional accumulation (or total deficiency of ablation). Thus the effectiveness of a rectangular pulse of given strength in producing an advance of the glacier increases at first almost in proportion to its duration, the proportionality breaking down for $t_0 \sim$ 1 000 yr.

(b) the flood maximum due to a rectangular pulse occurs after a delay of a few natural time units if the pulse is very short. For longer pulses the maximum occurs sooner after the end of the pulse, or at the end of the pulse itself.

(c) it is only for pulses which last for long times ($\sim$ 1 000 yr) that the very large advances of the glacier, with $h_1 \sim A/\delta$, can occur.

(d) there is a limit ($h_1 = A/\delta$) to the advance that can be caused by a pulse of given strength, however long it lasts.

(iii) *Frequency response*

An alternative way of examining the connexion between $a_1(t)$ and $h_1(x, t)$ is to calculate the response for a harmonic variation of a_1, putting

$$a_1 = A\,e^{i\omega t},$$

where A is a real constant. This could be one Fourier component of the actual variation $a_1(t)$ that occurs. The response of the glacier is readily found by using equation (33) as

$$h_1(x, t) = H(x)\,e^{i\omega t},$$

where the complex response amplitude $H(x)$ is given by

$$H = A\{2(x+\delta)+i\omega\}(1+i\omega)^{-1}(2\delta+i\omega)^{-1}. \tag{44}$$

At the datum position of the snout, $x = 1-\delta$,

$$H = A(2+i\omega)(1+i\omega)^{-1}(2\delta+i\omega)^{-1}, \tag{45}$$

or, if $H = |H|e^{-i\phi}$, so that ϕ is the phase lag of H on A,

$$|H| = A\left\{\frac{4+\omega^2}{(1+\omega^2)(4\delta^2+\omega^2)}\right\}^{\frac{1}{2}}, \qquad \tan\phi = \frac{\omega(2+2\delta+\omega^2)}{4\delta-\omega^2(1-2\delta)},$$

$0 \leqslant \phi < \pi$. $|H|$ and ϕ are shown in Figure 5 plotted against the period $\tau = 2\pi/\omega$, on a logarithmic scale, taking $\delta = 0{\cdot}006$. For periods less than about 0·5 time units, say less than 3 yr, the phase lag is effectively 90°, and the amplitude

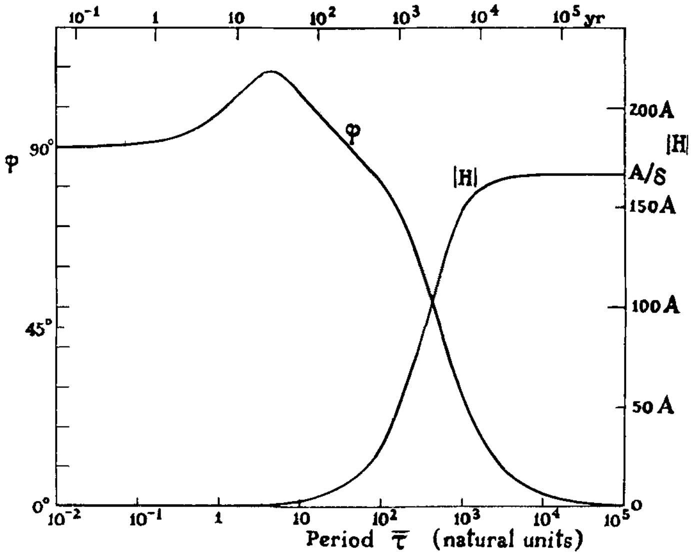

FIG. 5.—Response of the snout to a harmonic variation of accumulation rate. The amplitude $|H|$ and phase lag ϕ of h_1 are plotted against the period $\bar{\tau}$ on a logarithmic scale. Natural units, $l = 1$, $\sigma = 1$, are used, and δ is taken as 0·006. The illustrative scale of years at the top is for $\sigma = 6$ yr.

$|H| = \tau A_1/2\pi$, which is comparatively small. This range includes the winter-summer changes, where the maximum thickness naturally occurs a quarter-period after the maximum rate of accumulation. For longer periods ϕ increases to a maximum of $\tan^{-1}(-2\sqrt{2}) = 109°$ at $\bar{\tau} = \pi\sqrt{2} = 4{\cdot}4$; it remains above 90° until $\bar{\tau} \simeq \pi\delta^{-\frac{1}{2}} = 40$, and then decreases very slowly to zero. Meanwhile $|H|$ increases to approach $A/\delta = 167A$ at large $\bar{\tau}$, as one would expect. Notice that it is only for periods of several thousand years or more that the very large responses with $|H| \sim A/\delta$ can occur, in agreement with the similar result for rectangular pulses. If the time unit $\sigma = 6$ yr, the curve for ϕ shows that, for all periods less than 1 000 yr, the phase lag is within the range 90° ± 20°.

The numerical problem of calculating the frequency response of an actual glacier from field data is considered in III; it is found that the curves computed for South Cascade Glacier are in fact very similar in general form to those of Figure 5.

7. The inverse problem—calculation of climate from variations of the snout

Equation (34) gives the variation in thickness at the datum snout $h_1(t)$ which results from an arbitrary variation of rate of accumulation $a_1(t)$. We now wish to find the inverse relation, to express $a_1(t)$ in terms of $h_1(t)$ at the datum snout.

Differentiating (34) with respect to t gives (36). A further differentation gives

$$\ddot{h}_1 = \dot{a}_1 + (1-2\delta)a_1 - \frac{1}{1-2\delta}\bar{a}_1 + \frac{4\delta(1-\delta)}{1-2\delta}\bar{\bar{a}}_1. \tag{46}$$

$\bar{a}_1$ and $\bar{\bar{a}}_1$ may now be eliminated between (34), (36) and (46) to give

$$\dot{a}_1 + 2a_1 = \ddot{h}_1 + (1+2\delta)\dot{h}_1 + 2\delta h_1 = f(t), \text{ say,}$$

which is a differential equation for $a_1(t)$ whose solution with initial condition $a_1 = 0$ at $t = -T$, is

$$a_1(t) = e^{-2t}\int_{-T}^{t} e^{2\tau} f(\tau)\, d\tau.$$

Substituting for $f(\tau)$ and integrating by parts,

$$a_1 = \dot{h}_1 - (1-2\delta)h_1 + (1-\delta)\bar{h}_1, \tag{47}$$

where we have put $\dot{h}_1 = h_1 = 0$ at $t = -T$, and where $\bar{h}_1$ is defined by

$$\bar{h}_1 \int_{-T}^{t} e^{2\tau}\, d\tau = \int_{-T}^{t} h_1 e^{2\tau}\, d\tau$$

or, since T is very large,

$$\dot{h}_1 = 2\int_{-T}^{t} h_1 e^{-2(t-\tau)}\, d\tau. \tag{48}$$

$\bar{h}_1$ is thus an average of h_1 over the recent past with a time scale of $\frac{1}{2}$. Since $\delta \ll 1$, equation (47) may be written approximately

$$a_1 \simeq \dot{h}_1 - h_1 + \bar{h}_1. \tag{49}$$

Thus $a_1(t)$ is expressed in terms of $h_1(t)$ at the datum snout in a remarkably simple way. The term $\dot{h}_1$ represents the immediate effect of a_1, which is to increase h_1 at the rate a_1. The terms in h_1 and $\bar{h}_1$ show that, in order to deduce $a_1(t)$ from $h_1(t)$, we need to know also the current value of h_1 and its average over the recent past. Notice, however, that equation (47) contains no long-time average of h_1, $\bar{\bar{h}}_1$ say, with time scale of order δ^{-1}. To deduce $h_1(t)$ or $\dot{h}_1(t)$ from $a_1(t)$ it is necessary to know $\bar{\bar{a}}_1$, but to deduce $a_1(t)$ from $h_1(t)$ it is not necessary to know $\bar{\bar{h}}_1$. In other words, knowledge of the behaviour of h_1 at a time $\sim \delta^{-1}$ in the past is not necessary for a calculation of the value of a_1 now. This unsymmetrical relationship between a_1 and h_1 is perhaps the most interesting result of the paper. It is made crudely understandable if we remember that h_1 is the result of a type of integration of a_1 over past time. Therefore, correspondingly, a_1 must be a type of time derivative of h_1. Now to find the time derivative of a function at time t we only need to know the function in the immediate neighbourhood of t—on the other hand, to find the integral of a function up to time t we need to know the whole function back to $-\infty$. Hence it is not surprising that the distant past of a_1 (which is "integrated") enters the problem, while the distant past of h_1 (which is "differentiated") does not. The point is of considerable practical importance for the following reason.

The data normally available are the positions of the snout of a glacier over some limited period of time. If historical records are used, this period may be 300 years in a very favourable case, but 50 or 100 years is a more common figure. The function $h_1(t)$ at the datum snout may then be deduced (with an arbitrary zero). $\bar{h}_1$ will also be deducible over the whole period, but excluding an interval of duration $\sim \frac{1}{2}$, say 3 yr, at the beginning. Then, by using (49), $a_1(t)$ would be deducible over the same period as $\bar{h}_1$. We have seen that $1/2\delta$ may be about 500 yr. If a time average of h_1 over periods of this magnitude appeared in equation (49) it would be impossible to use historical records of the variations of glaciers to deduce climate. From this point of view the fact that equation (49) does not contain $\bar{\bar{h}}_1$ is crucial.

The same point may be expressed in another way. Suppose $a_1(t)$ were known from time $t = 0$ onwards. In order to solve equation (10) the initial function $h_1(x, 0)$ would also have to be known. But suppose it was not known, and was replaced by a false initial condition, $h_1(x, 0) = 0$, for example. The difference between the true and the false initial condition would cause a transient lasting for a time $\sim 1/2\delta$, and therefore during this time the solution would not be correct. Consequently, in order to calculate h_1 at the present time from the record of $a_1(t)$, without knowing an initial condition, the record must go back into the past at least a time $\sim 1/2\delta$.

On the other hand, to proceed the other way round, and to try to go from a knowledge of $h_1(t)$ at the snout to the function $a_1(t)$ which has caused it, is an entirely different type of problem; there seems to be no simple general way of seeing how far back into the past the record of h_1 must go before the present value of a_1 can be calculated. However, the answer to this question given by the special model is quite specific, namely a time of order $\frac{1}{2}$ of a natural unit.

8. Further developments

How to extend these results to the general case, where $c_0(x)$ and $D_0(x)$ do not have the special forms we have assigned them but are given graphically, is discussed in detail in III, but a few remarks on the problem may be made here.

In view of the simplicity of the results for the special model in terms of time averages into the past, like $\bar{h}_1$, $\bar{a}_1$ and $\bar{\bar{a}}_1$, it is natural to search for a similar result in the general case. But, because no finite set of time constants generally exists, no progress on these lines has been possible. The more fruitful approach has been to devise a general method of calculating the frequency response curve. This is done for low frequencies by finding series expansions of H/A and of A/H in powers of $i\omega$. For the datum snout of the special model the expansions are readily obtained from (45) as

$$H/A = \delta^{-1} - \tfrac{1}{2}\delta^{-2}(1+\delta)i\omega + \tfrac{1}{4}\delta^{-3}(1+\delta+2\delta^2)(i\omega)^2 - \dots \quad (\omega < 2\delta),$$

and

$$A/H = \delta + (1+\delta)(\tfrac{1}{2}i\omega) + (1-\delta)\{(\tfrac{1}{2}i\omega)^2 - (\tfrac{1}{2}i\omega)^3 + (\tfrac{1}{2}i\omega)^4 - \dots\} \quad (\omega < 2).$$

Since we are dealing with the single Fourier component $a_1 = A \exp i\omega t$, the two series may also be written, after multiplying by $\exp i\omega t$, as

$$h_1 = \delta^{-1}a_1 - \tfrac{1}{2}\delta^{-2}(1+\delta)\dot{a}_1 + \tfrac{1}{4}\delta^{-3}(1+\delta+2\delta^2)\ddot{a}_1 - \dots \quad (\omega < 2\delta), \tag{50}$$

and

$$a_1 = \delta h_1 + \tfrac{1}{2}(1+\delta)\dot{h}_1 + (1-\delta)(\tfrac{1}{4}\ddot{h}_1 - \tfrac{1}{8}\dddot{h}_1 + \dots) \quad (\omega < 2), \tag{51}$$

or, since $\delta \ll 1$,

$$h_1 \simeq \delta^{-1}a_1 - \tfrac{1}{2}\delta^{-2}\dot{a}_1 + \tfrac{1}{4}\delta^{-3}\ddot{a}_1 - \dots \quad (\omega < 2\delta) \tag{52}$$

and

$$a_1 \simeq \delta h_1 + \tfrac{1}{2}\dot{h}_1 + \tfrac{1}{4}\ddot{h}_1 - \tfrac{1}{8}\dddot{h}_1 + \dots \quad (\omega < 2). \tag{53}$$

These expansions are valid when h_1 and a_1 are single Fourier components, but, since the equations are linear, we can add the solutions for different ω. If then the actual function $a_1(t)$ is first filtered to remove all Fourier components with $\omega > 2\delta$, (52) will hold for the filtered function. Similarly (53) holds for a function $h_1(t)$ filtered to remove all frequencies $\omega > 2$. Equation (52) could thus be used to some extent for inferring $h_1(t)$ from $a_1(t)$, whereas (53) is appropriate for the reverse problem. The great difference in the radii of convergence (in terms of ω) of the two series is yet another manifestation of the asymmetry between these two problems encountered earlier. It means that in practice a series of type (52) is virtually useless, whereas one like (53) is useful. The two analogous series computed in III for South Cascade Glacier show exactly the same asymmetry.

As a check of consistency equation (51) may be obtained directly from (47), for, by repeated integration by parts of (48), $\bar{h}_1(t)$ may be expressed in terms of $h_1, \dot{h}_1, \ddot{h}_1 \ldots$ all evaluated at t:

$$\bar{h}_1 = h_1 - \tfrac{1}{2}\dot{h}_1 + \tfrac{1}{4}\ddot{h}_1 - \ldots .$$

Substituting this into (47) leads at once to (51).

The second and third terms of (53) may be combined if we write

$$a_1(t) \simeq \delta h_1(t) + \tfrac{1}{2}\dot{h}_1(t+\tfrac{1}{2}) - \tfrac{3}{16}\dddot{h}_1(t) + \ldots .$$

Thus, if the datum state is such that $\delta h_1(t)$ is small, and if $\dddot{h}_1(t)$ etc. can be neglected, turning points of h_1, that is maximum advances and retreats of the glacier, will occur half a time unit after zeros of a_1. Further development of this type of argument is given in III.

Once the frequency response curve for a glacier is known there still remains the question of the most efficient numerical methods to use in applying it to a record of $l_1(t)$, or $h_1(t)$, but discussion of this is best carried out in the context of an actual record.

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Appendix

To show the appearance of a long time constant when $E \neq 1$

We use equation (9), and first make the substitution

$$q^* = q_1 - Ax$$

to give

$$\frac{\partial q^*}{\partial t} = -c_0 \frac{\partial q^*}{\partial x} + D_0 \frac{\partial^2 q^*}{\partial x^2}.$$

If we put $q^* = X(x)\phi(t)$, the variables separate to give

$$\dot{\phi} = \lambda\phi \quad \text{i.e. } \phi(t) = e^{\lambda t}$$

and

$$D_0 X'' - c_0 X' - \lambda X = 0, \tag{54}$$

where λ is a constant. The ordinary differential equation (54) is solved by putting

$$X(x) = x^n(1 + px), \tag{55}$$

where n and p are constants. Substituting into equation (54), with c_0 and D_0 given by equations (21) and (22), and equating coefficients of x^{n+2}, we find

$$n = -1 \quad \text{or} \quad n = E^{-1} \quad \text{or} \quad p = 0.$$

If $n = -1$, $X \to \infty$ at $x = 0$, and we discard this possibility.

Equating coefficients of x^n gives

$$\lambda = E(1-\delta)n(n-1) - n, \tag{56}$$

and hence, for $n = E^{-1}$, $\lambda = \lambda_1$ say, where

$$\lambda_1 = -1 + \delta(1 - E^{-1}) \simeq 1 \quad \text{for} \quad \delta \ll 1.$$

Equating coefficients of x^{n+1} and putting $n = E^{-1}$ gives $p = -1/(1-2\delta)$. Thus a solution is

$$q^* = x^{1/E}\left(1 - \frac{x}{1-2\delta}\right)e^{\lambda_1 t}. \tag{57}$$

The alternative $p = 0$ gives, from the coefficient of x^{n+1}, $n = 0$ or $n = 1 + E^{-1}$. The case $n = 0$ gives a finite value of X (and therefore of q_1) at $x = 0$, and we therefore discard it. The case $n = 1 + E^{-1}$ gives, from equation (56), $\lambda = \lambda_2$ say, where

$$\lambda_2 = -(1 + E^{-1})\delta,$$

and hence the solution

$$q^* = x^{1+1/E} e^{\lambda_2 t}. \tag{58}$$

When $E = 1$, $\lambda_2 = -2\delta$, and thus λ_1 and λ_2 correspond respectively to the short and long time constants previously found. The linear combination of solutions (57) and (58) which corresponds to the solution for $E = 1$ already found, equation (27), is, changing the variable back to q_1,

$$\frac{q_1}{A} = x - x^{1/E} e^{\lambda_1 t} + \frac{1}{1-2\delta}(e^{\lambda_1 t} - e^{\lambda_2 t})x^{1+1/E}. \tag{59}$$

At $t = 0$, $q_1/A = x - x^{1/E}$, and so the initial condition $q_1(x, 0) = 0$ is only satisfied if $E = 1$. Thus, if E is close but not equal to 1, equation (59) is an exact solution of the differential equation for an initial condition which differs only slightly from $q_1 = 0$. The salient point is that the solution still contains a long time constant of order $1/2\delta$. We therefore conclude that a time scale of this order is not an exception which only appears when $E = 1$, but that it has a general significance.