On the theory of the katabatic slope wind

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ABSTRACT

The known Prandtl slope wind problem is considered for the case when the turbulent field in the surface layer is specified in accordance with Monin–Obukhov similarity theory. The solution of such a problem turns out to be a particular case of the solution for another more general problem published recently. On the basis of the former solution, simple analytical and graphical relationships have been obtained between external parameters of the problem, and those characteristics of the katabatic slope wind which seem to be most available from the observations. The paper strives to encourage the carrying out of special slope wind observations.

1. Introduction

The wind which sometimes develops over a heated or cooled mountain slope, when there are weak external pressure gradients, is known as slope wind (SW). The cause of this phenomenon was discussed by Prandtl (1942). He formulated and solved the steady-state problem for a laminar SW over a warmed slope which was an infinite, thermally homogeneous, inclined plane, assuming that external wind was absent. Prandtl's model has been developed by other researchers, who have taken into account the influence of time-derivatives, the Coriolis force, curvature of the slope, and some other factors. Reviews of the SW theory can be found in papers by Defant (1951), Smith (1979) and in the author's monograph (Gutman, 1972).

The aforementioned researchers have shown that almost all the simplifying assumptions of Prandtl do not make the problem less realistic. In fact, it turns out that if the slope is not very gentle, time-derivatives in the equations are negligible in comparison with other terms (the process is quasi-steady-state), and buoyancy forces dominate the Coriolis force. Consequently, the Coriolis force does not play an important role. Prandtl's constraint, concerning the shape of the slope, indicates that the solution should be applied only to vast plane parts of mountain slopes. It seems that

Prandtl's most substantial constraint is the assumption of flow laminarity, since turbulence is the principal mechanism of energy transfer from the underlying surface into the atmosphere, in the problem considered.

To the best of our knowledge, the papers dealing with this problem either have used too simplified a description of turbulence or, in using a more accurate model of turbulence, have been based on numerical methods. In the first case, the authors probably could not have counted on good agreement between theory and observation data. In the second case they presented only some particular examples of computations, for which the authors probably did not hope to find corresponding observations. As a result, published papers contain almost no comparisons of theory and observation. Understandably, such a situation has obviously not favored the carrying out of special observations on SW. It is therefore, perhaps for this reason that the presence of experimental data on SW in the literature is very scanty.

The recent paper by Gutman and Melgarejo (1981) (hereafter denoted by GM) deals with a somewhat different and more general problem. There, the authors were interested only in relationships between internal and external parameters of the boundary layer. Actually, by delving more deeply into this work, one can find the solution of the problem presently under consideration. It

should be noted that the paper employs a relatively reliable model of turbulence, enabling us to expect a tolerable agreement with observational data.

The present paper is based on the results of GM. Our major objective is to obtain simple analytical and graphical relationships between the external parameters of the problem and those characteristics of SW which seem most accessible to observation.

It is important to note that SW differs essentially in its structure, depending on whether the slope is heated or cooled, due to different stability conditions in the boundary layer. In the present paper we limit ourselves to the stable case, in which the slope surface is cooled and the wind direction is downslope (Katabatic wind). This case is, probably, comparable to some types of night mountain winds or glacier winds observed in nature.

2. The model

Consider a mesoscale problem of a turbulent SW, occurring in a stable stratified atmosphere, at rest over a homogeneously cooled, slightly inclined plane underlying surface.

Then, according to Prandtl (1942) the problem can be described by the following ordinary equations:

$$\frac{d}{dz} K_D \frac{du}{dz} - \beta \theta' \sin \psi = 0. \tag{1}$$

$$\frac{d}{dz} K_H \left(\frac{d\theta'}{dz} + \gamma \right) + \gamma u \sin \psi = 0. \tag{2}$$

with the boundary conditions

$$\text{at} \quad z = z_0, \quad u = 0, \quad \theta' - T_0 < 0. \tag{3}$$

$$\text{at} \quad z = \infty, \quad u = \theta' = 0, \tag{4}$$

where z is the coordinate, normal to the slope, oriented upwards (the x-axis is assumed to be oriented downwards along the slope), u is the wind component along the x-axis, θ' is the potential temperature deviation so that $\theta' = \theta - \Theta_0 - \gamma z'$, θ is the potential temperature of air, $\gamma = $ const is the vertical gradient of potential temperature above the boundary layer, $\Theta_0 = $ const is a mean value of the potential temperature, z' is the vertical coordinate, K_D and K_H are vertical eddy diffusion coefficients for momentum and heat, $\beta = $ const and $z_0 = $ const

are the buoyancy and the roughness parameters respectively, and, finally, ψ is the angle of inclination of the underlying surface to the horizontal.

We assume that ψ, a constant, is of the order of 10^{-1} (~6°), that $-T_0$ is of the order of 5–10 °C and, moreover, that T_0 is given and remains constant along the slope. We shall choose a model of turbulence which describes the influence of the underlying surface on the adjacent atmospheric layer with the accuracy which is necessary for our aims. The second criterion in our choice of a turbulence model is the possibility of obtaining an analytical solution to the problem.

Somewhat simplifying the formulation from GM we make use of the following interpolation formula:

$$K = K_D = K_H = \begin{cases} \kappa u_* z & (z_0 \leqslant z \leqslant z_s) \\ \kappa u_* z_s & (z \geqslant z_s) \end{cases}, \tag{5}$$

where $z_s = L/\alpha$ will be interpreted as a characteristic height of the surface layer, $L = u_*^2/\kappa^2 \beta T_*$ is Monin–Obukhov length, α is an empirical constant, κ is the Karman constant, u_* and T_* are friction velocity and temperature, respectively.

Obviously, eq. (5) in the surface layer corresponds to the Monin–Obukhov similarity theory. The fact that the underlying surface is slightly inclined does not violate the similarity laws. Unlike GM we assume that K_D is equal to K_H, although in reality they differ slightly. The degree of accuracy of the problem under consideration allows us to neglect this slight difference. In the layers above the surface layer, eq. (5) is probably less associated with the theory, or with the actual distribution of K_D and K_H with height. However, in the present problem this should not be of great importance, since perturbations of the meteorological fields are small in the upper part of the boundary layer. Furthermore, we note that u_* and T_* are unknown, and have to be found. Therefore, the following relationships should be satisfied

$$\text{at} \quad z = z_0, \quad z \frac{du}{dz} = \frac{u_*}{\kappa}, \quad z \left(\frac{d\theta'}{dz} + \gamma \right) = T_*. \tag{6}$$

The latter can be considered as a definition of u_* and T_*. Thus, the problem becomes a closed one. In conclusion, we note that, as known, $\gamma \ll d\theta'/dz$ in the surface layer. In that part of the boundary layer where $z > z_s$, the term $d(\gamma K)/dz$ equals zero, because of the assumption $K = $ const. Therefore, without bringing additional constraints into the

problem, one can neglect γ in the brackets in eqs. (2) and (6).

3. Calculation formulae

One can show that the problem (1)–(6) corresponds to case II from GM, where the solution was obtained in the following complex form:

$$iu + \sqrt{\frac{\beta}{\gamma}} \cdot \theta'$$

$$= \begin{cases} \dfrac{u_*}{\kappa}(\eta + i)\left(B + iA + \ln \dfrac{\kappa z}{\lambda_s}\right) & (z_0 \leqslant z \leqslant z_s), \quad (7) \\[3mm] \dfrac{u_*}{\kappa}(\eta + i)\left(B + iA + \ln \dfrac{\kappa}{\alpha\mu_s}\right)\exp\left(\sqrt{\dfrac{i}{\alpha\mu_s}}\right. \\[3mm] \qquad\qquad \left. -\sqrt{i\alpha\mu_s} \cdot \dfrac{z}{\lambda_s}\right) & (z > z_s). \quad (8) \end{cases}$$

This solution is valid, if

$$\frac{z_s}{\lambda_s} = \frac{1}{\alpha\mu_s} \ll 1. \tag{9}$$

This inequality, as will be shown later, is indeed satisfied.

The relationships between the internal parameters u_* and T_* and the given external parameters of the problem are the following:

$$\tilde{u}_* = \frac{u_*}{\kappa(-T_0)\sqrt{\beta/\gamma}} = \frac{1}{(1 + \eta^2)A},$$

$$\tilde{T}_* = \frac{T_*}{-T_0} = \eta\tilde{u}_*.$$

$$\ln R_0 = B + \eta A + \ln\left(\frac{\sin\psi}{\kappa\tilde{u}_*}\right)\left(R_0' = \frac{-T_0}{\gamma z_0}\right). \tag{10}$$

In eqs (7)–(10) we denoted

$$\eta = \kappa\frac{T_*}{u_*}\sqrt{\frac{\beta}{\gamma}} = \frac{f}{\kappa^2\sqrt{\beta\gamma}}\mu, \quad \left(\mu = \frac{\kappa^3\beta T_*}{fu_*}\right). \tag{11}$$

where f is the Coriolis parameter and μ is the stratification parameter, known from planetary boundary layer (PBL) theory. Furthermore,

$$\mu_s = \frac{\lambda_s}{L} = \frac{\kappa^3 T_*}{u_* \sin\psi}\sqrt{\frac{\beta}{\gamma}} = \frac{\kappa^2}{\sin\psi}\eta$$

$$\left(\lambda_s = \frac{\kappa u_*}{\beta\gamma\sin\psi}\right). \tag{12}$$

In this problem λ_s, μ_s and R_0' are analogous to the known characteristic height of the PBL $\lambda = \kappa u_*/f$, stratification parameter μ and surface Rossby parameter $R_0' = G/fz_0$, respectively, where G is the geostrophic wind. The approximate presentation of A and B which figure in (7), (10) is the following:

$$A \equiv A(\mu_s) = \sqrt{\frac{\alpha\mu_s}{2}} + \sqrt{\frac{1}{2\alpha\mu_s}},$$

$$B \equiv B(\mu_s) = \ln\left(\frac{\alpha\mu_s}{\kappa}\right) - \sqrt{\frac{\alpha\mu_s}{2}} + 1 + \sqrt{\frac{1}{2\alpha\mu_s}}. \tag{13}$$

In GM it was shown that the functions $A(\mu_s)$ and $B(\mu_s)$ have the same significance, that of the known universal functions $A(\mu)$ and $B(\mu)$, figured in many theoretical and experimental works on the PBL, and which differ from them only by their argument. We make use of this circumstance in order to specify the value of the parameter α, which is important for further computations. Experimental researches concerning the PBL suggest different values for α in the range of 2 to 9.8 (Pruitt et al., 1972; Businger et al., 1971; Hicks, 1976; Zilitinkevich, 1970). We have not succeeded in finding values for α agreeing with SW observations. Therefore we have chosen α so that differences between the functions computed by the formulae (13) and the functions $A(\mu)$, $B(\mu)$ obtained experimentally are a minimum. If one takes $A(\mu)$ and $B(\mu)$ suggested by Arya (1975), it becomes possible to set $\alpha \simeq 7$. The degree of accuracy of the problem suggests that it is not necessary to have α more accurately. The curves $A(\mu_s)$ and $B(\mu_s)$, computed by (13) for $\alpha = 7$ and $A(\mu)$, $B(\mu)$ obtained from Arya's formulae are plotted in Fig. 1. As can be seen, theoretical (solid line) and experimental (dotted line) curves are rather close for the relatively large variation of the argument. The differences, which occur when the argument is small, can be explained by the fact that the turbulence model employed becomes less accurate when stratification is close to neutral.

In order to calculate the internal parameters, it is convenient to make use of the non-dimensional internal parameters $\tilde{u}_*$ and $\tilde{T}_*$, which are intro-
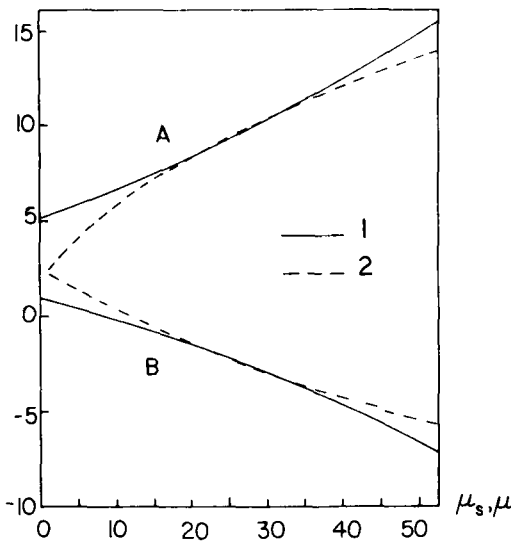
Fig. 1. Dependences of the universal functions A and B on μ or μ_s based on experimental data (1) and on theory (2).
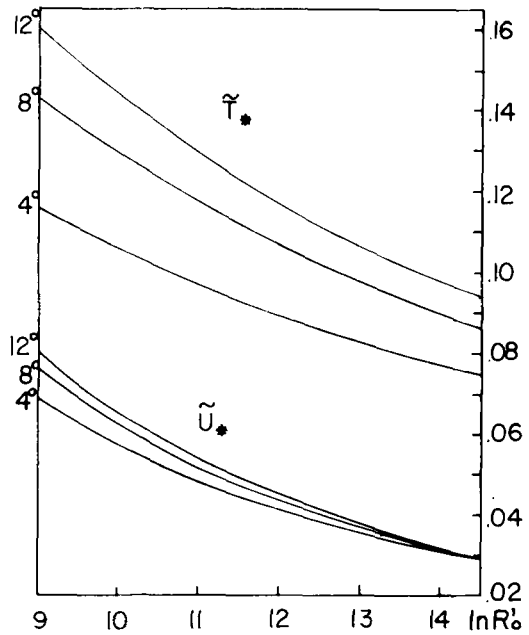

Fig. 2. Dependence of the non-dimensional internal parameters $\tilde{u}_*$ and $\tilde{T}_*$ on R_0' for $\psi = 4°, 8°, 12°$.

duced by (10). Setting $\kappa = 0.4$ and $\alpha = 7$ and assigning η different values from 1 to 4.2, we calculated $\tilde{u}_*$, $\tilde{T}_*$ and R_0' for three values of $\psi = 4°$, $8°$, $12°$. As a result, the curves, which are shown in Fig. 2, were constructed. The range of variation of R_0' was chosen to encompass the possible variations of R_0' in nature. The same computations made it possible to plot the curves η versus $\ln R_0'$ for the same values of ψ in Fig. 3, which allow us to calculate μ and μ_s, using (11) and (12), if the external parameters of the problem are known. Having specified μ_s for the smallest values of η and for the greatest values of ψ, we realize that μ_s cannot become significantly less than 1. This proves the validity of the inequality (13).

Making use of the second line in eq. (5), and the expression

$$z_s = \frac{-T_0}{\alpha \gamma} \frac{\tilde{u}_*}{\eta}, \qquad (14)$$

which is a consequence of eqs. (10)–(12), we can estimate the turbulent coefficient K. With the help of Figs. 2 and 3, we find that K varies from values less than 1 m^2 s^{-1} to a few square meters per second. The results of this estimation seems realistic.

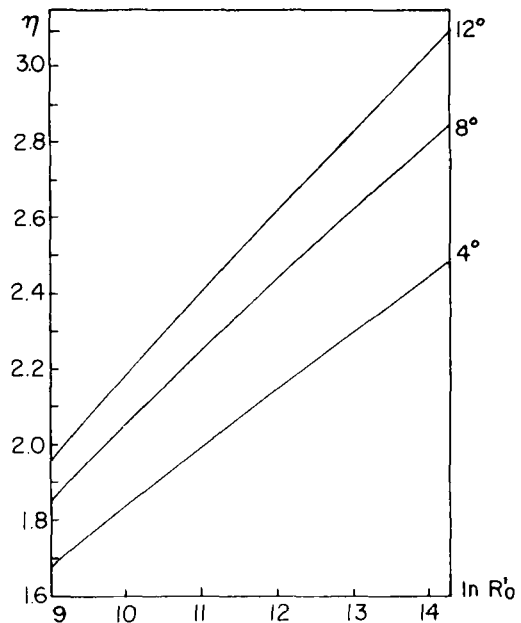Fig. 3. Dependence of the non-dimensional parameter η on R_0' for $\psi = 4°, 8°, 12°$.

Concerning the derivation of formulae for the vertical profiles of u and θ' we can consider u_* and T_* to be known. Partition of the real and imaginary parts in (12) and an elementary transformation, taking into account (10), yields

$$
u = \begin{cases} \dfrac{u_*}{\kappa} \ln \dfrac{z}{z_0}, & (z_0 \leqslant z \leqslant z_s), \\[2mm] \dfrac{u_*}{\kappa} \sqrt{c_1^2 + c_2^2} \sin (\xi_0 + \xi)\, e^{-\xi} & (z > z_s), \end{cases} \tag{15}
$$

$$
\theta' = -T_0 \cdot \begin{cases} -1 + \tilde{T}_* \ln \dfrac{z}{z_0}, & (z_0 \leqslant z \leqslant z_s), \\[2mm] -u_* \sqrt{c_1^2 + c_2^2} \cos (\xi_0 + \xi)\, e^{-\xi}, & \\ & (z \geqslant z_s), \end{cases} \tag{16}
$$

where

$$
c_1 = \ln \frac{\tilde{u}_* R_0}{a\eta}, \quad c_2 = \frac{1}{\tilde{u}_*} - \eta c_1,
$$
$$
\xi = \frac{z - z_s}{z_s 2a\mu_s}, \quad \xi_0 = \arctan \frac{c_1}{c_2}. \tag{17}
$$

The fact that we have obtained logarithmic profiles for wind and potential temperature is not surprising. On the contrary, this could have been anticipated, since turbulent terms dominate in the surface layer equations of the SW as well as in those of the PBL. Hence, the slope inclination influences the meteorological fields in the surface layer by changing u_* and T_*.

Making use of eqs. (15)–(17) and the curves in Figs. 2 and 3, it is easy to calculate u and θ' as functions of z, for given R_0' and ψ.

With the help of eqs. (15) and (17), we obtain expressions for maximum wind height z_m, and for the height z_r at which wind has returned to zero:

$$
z_m = z_s \cdot |1 + (\tfrac{1}{4}\pi - \xi_0) \sqrt{2a\mu_s}|,
$$
$$
z_r = z_s \cdot |1 + (\pi - \xi_0) \sqrt{2a\mu_s}|. \tag{18}
$$

The dependance of the non-dimensional parameters $\tilde{z}_m = z_m \cdot a\gamma/(-T_0)$ and $\tilde{z}_r = z_r a\gamma/(-T_0)$ upon R_0' and ψ, calculated with the help of (18), is presented in Fig. 4.

Setting $\xi = \tfrac{1}{4}\pi - \xi_0$ in eq. (15), one can obtain an expression for the maximum wind velocity u_m. The computations show that the dependance of the non-dimensional value $\tilde{u}_m = u_m/(-T_0)\sqrt{\beta/\gamma}$ on ψ is weak (relative variations of $\tilde{u}_m$ are less than
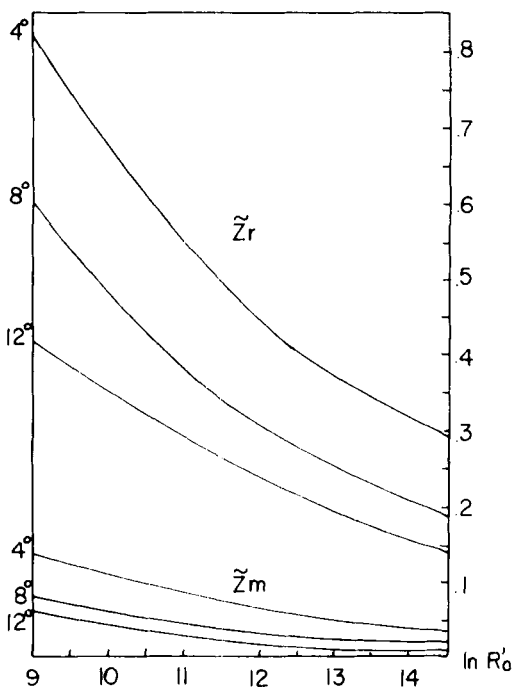

Fig. 4. Dependence of flow parameters $\tilde{z}_r$ and $\tilde{z}_m$ on R_0' for $\psi = 4°, 8°, 12°$.

10%). This permits construction of the following approximate formula:

$$
\tilde{u}_m = 0.41 - 0.01 \ln R_0'. \tag{19}
$$

It is worth mentioning that Prandtl (1942) noticed that, for both laminar as well as turbulent flow, the maximum SW should not depend on ψ, and he gave a physical explanation for this fact.

It is known from observations (Defant, 1951) that the maximum SW does not exceed a few meters per second. Since $\beta = 3 \cdot 10^{-2}$ m s^{-2} deg^{-1}, $\gamma = 3 \cdot 10^{-3}$ deg m^{-1}, we find that $-T_0$ should be of the order of a few degrees. With the help of Fig. 4, one can conclude that SW observational data (Defant, 1951), correspond in general to the results of the theory presented in this paper ($z_m \approx 30$ m, $z_r \approx 100$ m). Unfortunately, we did not succeed in finding any experimental data for which we could have carried out concrete computations.

It should be noted that according to eq. (15), the maximum wind velocity above the reversal point (which is directed upslope) is about $\tfrac{1}{23}$ of the maximum wind velocity in the main flow, which is

rather weak, as we mentioned above. Therefore, it is hardly possible to detect an opposite flow by observations; one can consider z_r as an upper boundary of the SW flow. From eq. (15) it is easy to find the expression for the integrated down-slope flux of air:

$$\int_{z_0}^{\infty} u \, dz = -\frac{H}{\rho c_p \gamma \sin \psi} \quad (H = -\rho c_p \kappa u_* T_*), \quad (20)$$

where H is the heat flux into the slope (this fact may be of some interest in itself), ρ is the air density, and c_p is the specific heat at constant pressure.

In conclusion, we would like to express our hope that this paper will encourage the carrying out of special experiments on SW.

4. Acknowledgements

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