NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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In the work of V. Z. Vlasov (ref. 1), equations of the equilibrium of shallow shells are given, with account taken of the torsional stress state. These results are of great importance for practical computations.

Vlasov employed a coordinate system that coincided with the lines of curvature of the middle surface of the shell. The survey article by A. L. Goldenveizer, and A. I. Lurye (ref. 2, p. 579) shows that this circumstance does not always assure the applicability of the equations of shallow shells. A sphere referred to the geographic system of coordinates may be used as an example. The coefficients of the first quadratic form, in this case, are $A = R$ and $B = R \sin \theta$.

The expression in the preceding case of the tangential forces in terms of the stress function given by Vlasov will approximately satisfy the first two equations of the momentless theory of shallow shells. The error, as can easily be verified, will be of the order of $\sin \theta$ compared with unity and, therefore, depends on the choice of the coordinate system. The error is found to be less the farther the pole of the geographical system of coordinates is removed from the part of the shell under consideration. This circumstance may sometimes be met in computations.

The present report does not employ the lines of curvature as the coordinate system, but employs "almost cartesian coordinates" (the coordinates obtained by cutting the surface into two mutually orthogonal systems of parallel planes). This choice of coordinates will, in certain cases, be more natural for the problem under consideration, and will also be free from the previously mentioned fault.

1. Let the middle surface of the shell be given by

$$z = \lambda f(x_1,x_2) \tag{1.1}$$

where λ is a certain nondimensional parameter, and $f(x_1,x_2)$ is a function having partial derivatives with respect to the arguments x_1 and

*"K teorii tonkikh pologikh obolochek." Prik. Mat. i Mekh., vol. XIII, 1949, pp. 547-550.

x_2 up to and including the third order. In deriving the general relations of the theory of shallow shells, we shall neglect, by comparison with unity, terms containing the products of a certain nondimensional factor by the square and higher powers of the parameter λ.

The components of the tensor of the first differential form are

$$g_{11} = 1 + (\partial_1 z)^2 \qquad g_{12} = \partial_1 z \partial_2 z \qquad g_{22} = 1 + (\partial_2 z)^2 \tag{1.2}$$

where, as shown in the following paragraphs, the index of ∂ denotes differentiation with respect to the variable of the corresponding index. Rejecting terms containing squares of the parameter λ in expressions (1.2), we obtain

$$g_{11} = 1 \qquad g_{12} = 0 \qquad g_{22} = 1 \tag{1.3}$$

With the same accuracy, the relations are obtained for the covariant, contravariant, and mixed components of the same tensor, so that

$$g^{11} = g^{22} = 1 \qquad g_1^2 = g^{12} = 0 \qquad g_1^1 = g_2^2 = 1 \tag{1.4}$$

The decomposition along the axes of the vector normal to the middle surface has the form

$$n = -i\partial_1 z - j\partial_2 z + k \tag{1.5}$$

The components of the tensor of the second differential form of the middle surface of the shell are

$$b_{11} = \frac{\partial_{11} z}{B} \qquad b_{12} = \frac{\partial_{12} z}{B} \qquad b_{22} = \frac{\partial_{22} z}{B} \qquad \left[B = \sqrt{1 + (\partial z_1)^2 + (\partial z_2)^2} \right] \tag{1.6}$$

Therefore, considering the assumed degree of accuracy, we have

$$\left.\begin{aligned} & b_{11} = \lambda r \qquad b_{12} = \lambda s \qquad b_{22} = \lambda t \\ & \text{where} \\ & r = \partial_{11} f \qquad s = \partial_{12} f \qquad t = \partial_{22} f \end{aligned}\right\} \tag{1.7}$$

For the covariant, contravariant, and mixed components of this tensor we have

$$b^{11} = b_1^1 = \lambda r \qquad b_1^2 = b^{12} = \lambda s \qquad b_2^2 = b^{22} = \lambda t \tag{1.8}$$

The Christoffel symbols $\Gamma^{\gamma}_{\alpha\beta}$ and $\Gamma_{\alpha\beta\omega}$ are connected by the relations

$$\Gamma^{\gamma}_{\alpha\beta} = g^{\gamma\omega}\Gamma_{\alpha\beta\omega} = \frac{1}{2} g^{\gamma\omega}(\partial_\alpha g_{\beta\omega} + \partial_\beta g_{\alpha\omega} - \partial_\omega g_{\alpha\beta}) \tag{1.9}$$

On the basis of equations (1.3),

$$\Gamma^{\gamma}_{\alpha\beta} = \Gamma_{\alpha\beta\omega} = 0 \tag{1.10}$$

Hence, the covariant and contravariant derivatives of the vector will, with the assumed degree of accuracy, coincide with the usual derivatives

$$\nabla_x r_\alpha = \partial_x r_\alpha \qquad \nabla_x r^\alpha = \partial_x r^\alpha \tag{1.11}$$

The second equation of equations (1.3) justifies the conclusion that with the assumed degree of accuracy, the lines x_1 = constant and x_2 = constant on the shallow shell will be orthogonal; however, in the general case, they do not coincide with the lines of curvature. Hence, it is not possible here to directly apply the known formulas of the theory of shells and the coordinate method of deriving the required formulas, and formulas similar to them become unsuitable. For this reason, following the findings of A. I. Lurye and A. L. Goldenveizer, we shall apply the more general mathematical apparatus of the theory of shells based on the methods of tensor analysis.

2. The position of an arbitrary point of the shell is determined by the radius vector

$$r = p + nz \tag{2.1}$$

where p is the radius vector of the foot of the normal drawn through the point to the middle surface. On the surfaces bounding the shell,

$$r^+ = p + \frac{1}{2} hn \qquad r^- = p - \frac{1}{2} hn \tag{2.2}$$

The principal vectors of the surface have the form

$$r_k = p_k - zb_k^s p_s \qquad r_3 = n \tag{2.3}$$

The components of the fundamental metric tensor of the shell are

$$a_{ik} = g_{ik} - 2zb_{ik} + b_{\omega i}b_k^{\omega}z^2 \qquad (2.4)$$

In our case, with the degree of accuracy assumed, we have

$$a_{11} = 1 - 2\lambda zr \qquad a_{12} = -\,2\lambda zs \qquad a_{22} = 1 - 2\lambda zt \qquad (2.5)$$

$$a = 1 - 2z\lambda(r + t) \qquad (2.6)$$

$$a^{11} = 1 + 2\lambda zr \qquad a^{22} = 2\lambda zs \qquad a^{22} = 1 + 2\lambda zt \qquad (2.7)$$

3. Let v^1, w, and v^2 be the components of the vector of small displacement, v of the points of the middle surface along the lines x_1 = constant and x_2 = constant, and n along the normal. Then

$$V = v^k p_k + wn \qquad (3.1)$$

The radius vector of the deformed surface is

$$'p = p + v^k p_k + wn \qquad (3.2)$$

The principal vectors on the deformed surface are

$$'p_\alpha = p_\alpha + (\nabla_\alpha v^k - wb_\alpha^k)\, p_k + (w_\alpha + b_{k\alpha}v^k)n \qquad (3.3)$$

where ∇ is the symbol of covariant differentiation, in our case, $\nabla = \partial$. Then, the six magnitudes $\varepsilon_{ik}\beta_{ik}$ determining the changes of the coefficients of the first and second differential forms of the middle surface of a shallow shell, on the basis of formulas (2.1.5), and (2.1.11) given by A. I. Lurye (ref. 4), are presented in the form

$$\varepsilon_{11} = \partial_1 v^1 - \lambda rw \qquad \varepsilon_{12} = \frac{1}{2}\left(\partial_1 v^2 + \partial_2 v^1\right) + \lambda sw \qquad \varepsilon_{22} = \partial_2 v^2 - \lambda tw$$

$$\beta_{ik} = \partial_k v^\alpha b_{i\alpha} + \partial_{ik}w + \partial_i v^\alpha b_{k\alpha} + v^\alpha \partial_i b_{k\alpha} \qquad (3.4)$$

In the problems under consideration, and with the bending of the shell, an essential part is also played by the deformation of elongation, the first being comparable in magnitude with the second. On the basis of this assumption, it is possible, in the last three equations of equations (3.4) characterizing the changes in curvature of the surface, to neglect the terms containing the displacement components v^1 and v^2 in comparison with $\partial_{ik}w$. We then have

$$\beta_{11} = \partial_{11}w \qquad \beta_{12} = \partial_{12}w \qquad \beta_{22} = \partial_{22}w \qquad (3.5)$$

4. The expressions connecting the forces and moments with the deformations of the middle surface are reduced to the form

$$\left.\begin{array}{lll} T_1 = B(\varepsilon_{11} + \nu\varepsilon_{22}) & T_2 = B(\varepsilon_{22} + \nu\varepsilon_{11}) & S = B(1 - \nu)\varepsilon_{12} \\ G_1 = - D(\beta_{11} + \nu\beta_{22}) & G_2 = - D(\beta_{22} + \nu\beta_{11}) & H = - D(1 - \nu)\beta_{12} \end{array}\right\} \quad (4.1)$$

where

$$B = \frac{Eh}{1 - \nu^2} \qquad D = \frac{Eh^3}{12(1 - \nu^2)}$$

The static equations of an element of the shell, as derived by A. I. Lurye (ref. 3) can, for our case, be written in the form

$$\left.\begin{array}{l} \partial_1 T_1 + \partial_2 S - N_1\lambda r - N_2\lambda s + E^1 = 0 \\ \partial_1 S + \partial_2 T_2 - N_2\lambda t - N_1\lambda s + E^2 = 0 \\ \partial_1 N_1 + \partial_2 N_2 + T_1\lambda r + T_2\lambda t + 2S\lambda s + E^3 = 0 \\ \partial_1 H + \partial_2 G_2 - N_2 + M_1 = 0 \\ \partial G_1 + \partial_2 H - N_1 + M_2 = 0 \end{array}\right\} \quad (4.2)$$

In the following paragraphs we shall, in the first two equations of equations (4.2), neglect the terms $N_1\lambda r$, $N_2\lambda s$, $N_2\lambda t$, and $N_1\lambda s$ by comparison with the others (this is the usual hypothesis which is assumed even in the theory of large deflections). The equations (4.2) will then become

$$\left.\begin{array}{l} \partial_1 T_1 + \partial_2 S + E^1 = 0 \\ \partial_1 S + \partial_2 T_2 + E^2 = 0 \\ \partial_1 N_1 + \partial_2 N_2 + T_1\lambda r + T_2\lambda t + 2S\lambda s + E^3 = 0 \\ \partial_1 H + \partial_2 G_2 - N_2 + M_1 = 0 \\ \partial_1 G_1 + \partial_2 H - N_1 + M_2 = 0 \end{array}\right\} \quad (4.3)$$

In equations (4.3) as in equations (4.2), E^1, E^2, E^3, M_1, and M_2, the components of the principal vector and principal moment of the external forces are applied to a point of the surface of an element of the shell along its principal directions.

In the third equation of this system, we substitue the values of the forces N_1 and N_2 found from the last two equations of the system, and making use of equations (4.1) easily transform the remaining equations (4.3) into three equations in the components of the variables, so that

$$\left.\begin{aligned}
&\partial_{11}v^1 + \frac{1-\nu}{2}\partial_{22}v^1 + \frac{1+\nu}{2}\partial_{12}v^2 - \lambda\partial_1[w(r+\nu t)] - \lambda(1-\nu)\partial_2(sw) + \frac{E^1}{B} = 0 \\
&\partial_{22}v^2 + \frac{1-\nu}{2}\partial_{11}v^2 + \frac{1+\nu}{2}\partial_{12}v^1 - \lambda\partial_2[w(r\nu+t)] - \lambda(1-\nu)\partial_1(sw) + \frac{E^2}{B} = 0 \\
&\nabla^2\nabla^2 w + \frac{B}{D}\lambda^2 w(r^2 + 2\nu rt + t^2 + s^2) - \frac{B}{D}\lambda[r(\partial_1 v^1 + \nu\partial_2 v^2) + \\
&\qquad t(\partial_2 v^2 + \nu\partial_1 v^1) + s(1-\nu)(\partial_1 v^2 + \partial_2 v^1)] - \frac{E^3}{D} = 0
\end{aligned}\right\} \quad (4.4)$$

Further, it is easy to show that the static equations of the shell (eqs. (4.3)), can be reduced to two differential equations for the two functions. For simplicity of computation, we shall consider the case of the equilibrium of a shallow shell where, in equations (4.3), the magnitudes $E^1 = E^2 = M_1 = M_2 = 0$. We shall choose the function F so that

$$T_1 = h\partial_{22}F \qquad S = -h\partial_{12}F \qquad T_2 = h\partial_{11}F \quad (4.5)$$

Then, the two first equations of the system (4.3) will be identically satisfied. We shall call the function F (by analogy with the Airy function in the plane problem of the theory of elasticity), the force function. Further, on the basis of formulas (4.1) and (3.4), we find

$$\left.\begin{aligned}
T_1 &= B[\partial_1 v^1 + \nu\partial_2 v^2 - \lambda w(r + \nu t)] \\
T_2 &= B[\partial_2 v^2 + \nu\partial_1 v^1 - \lambda w(t + \nu r)] \\
S &= \frac{B(1-\nu)}{2}(\partial_1 v^2 + \partial_2 v^1 - 2\lambda sw)
\end{aligned}\right\} \quad (4.6)$$

Whence, the following relations are easily obtained:

$$\left.\begin{aligned}
T_1 - \nu T_2 &= Eh(\partial_1 v^1 - \lambda rw) \\
T_2 - \nu T_1 &= Eh(\partial_2 v^2 - \lambda tw) \\
2(1+\nu)S &= (\partial_1 v^2 + \partial_2 v^1 - 2\lambda sw)Eh
\end{aligned}\right\} \quad (4.7)$$

Differentiating the first equation of equations (4.7) twice, with respect to x_2, the second twice with respect to x_1, and the third with respect to x_1 and x_2, and subtracting the third from the sum of the first two, we obtain

$$\partial_{22}(T_1-\nu T_2)+\partial_{11}(T_2-\nu T_1)-2(1+\nu)\partial_{12}S = -\lambda E(r\partial_{22}w-2s\partial_{12}w+t\partial_{11}w) \tag{4.8}$$

Substituting the forces T_1, T_2, and S in place of their values from equations (4.5), we obtain an equation connecting the functions F and w:

$$\nabla^2\nabla^2 F = - \lambda E(r\partial_{22}w - 2s\partial_{12}w + t\partial_{11}w) \tag{4.9}$$

The second equation of these functions is obtained by eliminating the forces N_1 and N_2 with the aid of the last three equations of equations (4.3), and successively replacing the forces and moments by the functions F and w in the first of these three equations. This equation has the form

$$\nabla^2\nabla^2 w = \frac{1}{D}\left[E^3 + \lambda h(r\partial_{22}F - 2s\partial_{12}F + t\partial_{11}F)\right] \tag{4.10}$$

REFERENCES

1. Vlasov, V. Z.: PMM, vol. VIII, no. 2, 1944.

2. Goldenveizer, A. L., and Lurye, A. I.: PMM, vol. XI, no. 5, 1947.

3. Lurye, A. I.: PMM, vol. IV, no. 2, 1940.

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